A Summary Review on 25 Years of Progress and Future Challenges in FDTD and FETD Techniques

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***Abstract*—**The finite-difference time-domain (FDTD) method has established itself among the most popular methods for the numerical solution of Maxwell equations. Reasons for its popularity include its versatility, matrix-free characteristic, ease for parallelization, and low computational complexity. In recent years, the finite-element time-domain (FETD) has also become another very popular algorithm for solving time-domain Maxwell equations due to its geometrical flexibility and the steady growth in hardware computing power. In this review, we succinctly recollect some of the milestones in the development of FDTD and FETD over the last 25 years, and briefly discuss some challenges for the future development of these two algorithms.

Index terms— finite-difference time-domain, finite-element time-domain, Maxwell equations.

I. INTRODUCTION

In its basic form as introduced by Yee [1] and pioneered by Taflove [2], the finite-difference time-domain (FDTD) method is a conceptually very simple algorithm for solving Maxwell equations. FDTD basically relies on the approximation of the space-derivatives in the (Ampere's and Faraday's) first-order curl equations by central-differences on a staggered Cartesian (rectangular or hexahedral) grid and on a time-discretization following a "leap-frog" update. This leads to an algorithm that is second-order accurate in both space and time, i.e., which converges with the second power both on the spatial cell size Δs and the time-step size Δt. The conceptual simplicity of FDTD should not belittle its power. Because FDTD is a matrix-free algorithm (i.e., it requires no linear algebra), its memory requirements scale only linearly with the number of unknowns. This, added to the fact that FDTD is massively parallelizable, makes it well suited for next-generation petascale machines and beyond.

Another popular algorithm for solving Maxwell equations is the finite-element time-domain (FETD) method. There are two basic popular approaches for constructing FETD methods for Maxwell equations. The first one is based on the discretization of the second-order vector Helmholtz wave equation for either the electric or magnetic field (after elimination of the other field) through an expansion of the unknown field in terms of vector basis functions [3], [4]. The second FETD approach is based on the discretization of the first-order coupled Maxwell curl equations (i.e., Faraday's and Ampere's laws) by expanding the electric and magnetic fields in terms of mixed elements—most often edge elements for the electric field and face elements for the magnetic flux density [5]. Because of their efficiency and versatility, FDTD and FETD have enjoyed widespread use by the computational electromagnetics (CEM) community.

Figure 1 purports to show the steady growth in the popularity of FDTD, as exemplified by the yearly number of papers in the 1986-2007 period obtained from a search under title/topic fields "finite-difference time-domain" or "FDTD" in the *ISI Web of Science*™ database, as of earlier 2009 (this plot is not intended to indicate the total number of FDTD-related papers, which is much higher).

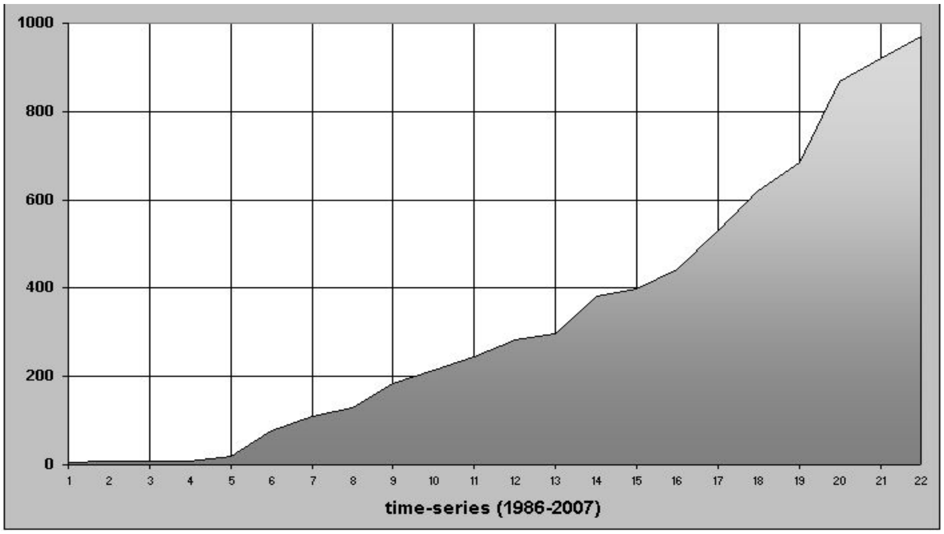

Fig. 1. Evolution on the popularity of FDTD exemplified by the yearly number of papers (1986-2007 period) obtained by a search under title/topic fields "finite-difference time-domain" or "FDTD" in the *ISI Web of Science*™ database, as of earlier 2009.

In this summary review, we recollect some of milestones in the development and applications of FDTD and FETD for Maxwell's equations during the last 25 years, and discuss some its future challenges. The list of references included here is relatively brief and by no means representative of the full extension of the volume of research efforts in this 25-year period.

A good source of references on FDTD is the book by Taflove and Hagness [2]. On FETD, a good reference source is the book by Jin [4]. A quite comprehensive list of catalogued references on FDTD in the period up to 1995 is available in [6]. A recent review on FDTD and FETD algorithms for complex (i.e., dispersive, anisotropic, inhomogeneous, nonlinear) media can be found in [7].

II. 25 YEARS OF PROGRESS IN FDTD: A BIRD'S EYE VIEW

Despite its introduction by Yee 18 years earlier (1966), FDTD was still a relatively incipient method 25 years ago (1984). This can be explained by the fact FDTD is a volumetric method and the computer memory resources for solving practical engineering problems were well beyond the reach of the average user at that time. The numerical method of choice in those years was the (frequency-domain) method of moments. Early pioneers in FDTD algorithmic developments in the 1970s were Taflove [8], Holland [9], and Kunz [10] in the U.S. The acronym "FDTD" was actually not present in the 1966 Yee's paper, and was coined by Taflove only in 1980 [11]. In Europe, Weiland independently developed a twin discretization methodology dubbed finite-integration-technique (FIT) [12]. The latter is based on the *integral* representation of Maxwell equations akin to a finite volume approach that, in a Cartesian grid, reduces to a set of equations identical to FDTD. We will not delve here into the FIT method and its extensions.

The early 1980s witnessed a surge in the development of absorbing boundary conditions (ABCs) for FDTD, including Mur and Liao ABCs that allowed for accurate simulations of open-space problems [13-16]. At that time, the first electromagnetic scattering FDTD models computing radar cross-section structures were developed [17,18]. The late 1980s were the period when FDTD applications to waveguides [19], microstrip circuits [20], and biological media [21] became feasible under (then) milder computational resources. This was facilitated not only by the continual growth of computational power, but also by concurrent algorithmic developments such as contour-path conformal modeling techniques [22-26] to reduce staircasing

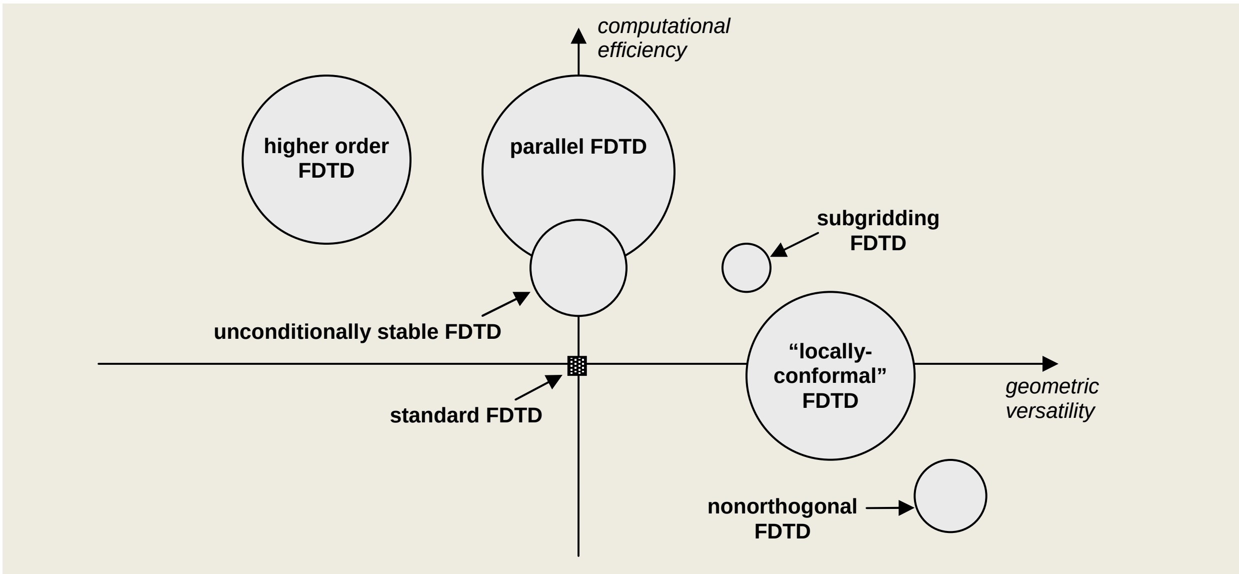

Fig. 2. Diagram illustrating various (non-inclusive) extensions of the “FDTD family” of methods developed towards improving geometrical versatility and/or computational efficiency of the standard FDTD. A negative correlation (trade-off) is apparent between these two objectives. The radii of the circles are approximately proportional to the number of entries for Google™ searches of each extension name in conjunction with the “FDTD” acronym. The radii serve as a rough indicator of the relative “historical popularity” of each approach.

error and increase geometrical flexibility, and by lumped equivalent circuit models to model sub-cell features and sources [27]. These techniques later played an important role in improving the suitability of commercial FDTD packages to RF, microwave, and antenna problems.

It was in the early 1990s that FDTD applications to the modeling of realistic circuits, antennas, and radiation problems [28-33] and of optical devices [34-36] began to appear very frequently in the literature. Also around that time, the extension of FDTD to frequency-dispersive media by means of recursive convolution approaches and later by auxiliary differential equation techniques [37, 38] provided further impetus for FDTD applications to complex media problems [7]. Also in the 1990s, new FDTD schemes were introduced for the efficient analysis of periodic structures [39-42].

With the increase of the electric size of the problems being tackled, the challenge of grid (numerical) dispersion error came to the forefront in the late 1980s and early 1990s. As a result, a series of high-order FDTD algorithms were developed to mitigate grid dispersion based on the use of a larger number of terms in the Fourier expansion to approximate the spatial (and time) derivatives leading to enlarged finite-difference stencils [2]. This effort remains an area of active research interest to this day, with the development of ever more sophisticated higher order FDTD algorithms that include pre-asymptotic higher-order algorithms providing optimized (tailored) numerical dispersion curves in a particular frequency band and/or grid size [43,44]. Of note also is the development of pseudo-spectral time-domain (PSTD) methods with low dispersion error even for discretization scales near the Nyquist limit [45].

The introduction of the perfectly matched layer (PML) by Berenger in the mid 1990s [46-50] provided a major improvement on the dynamical range of open-domain FDTD simulations, which under mild computational costs could then reach 80 dB. It also allowed for the use of better use of FDTD in open domains with dispersive media such as in earth media [51].

The development of unconditionally stable algorithms for FDTD in the late 1990s and early years of the present decade— starting with alternating-direction-implicit (ADI) schemes [52,53] and later with split-step schemes such as the locally-one-dimensional (LOD) scheme [54]— represented another major milestone in the

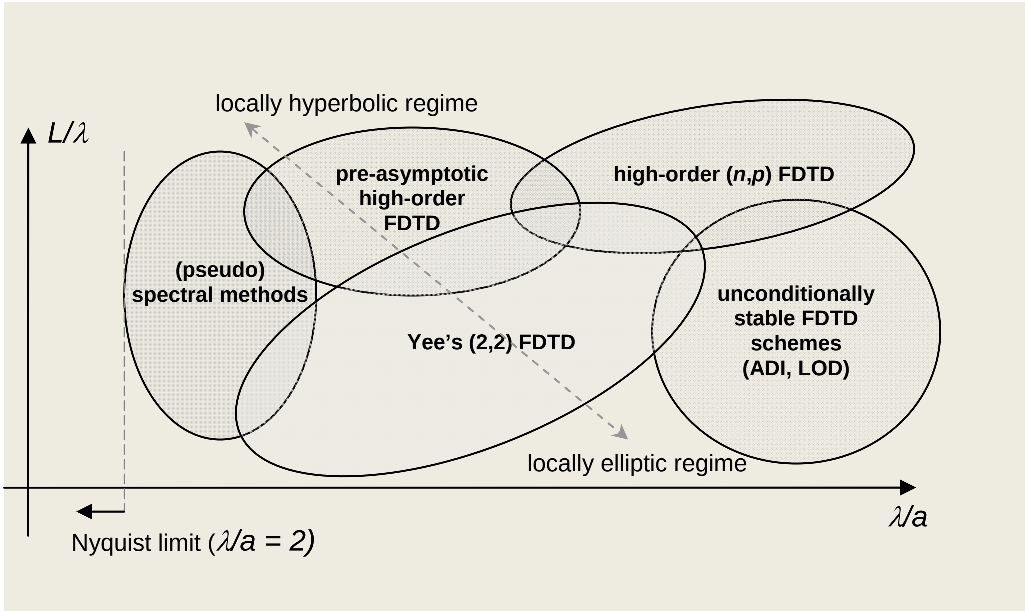

Fig. 3. Diagram illustrating the range of applicability of various FDTD-related algorithms vs. problem size and discretization scale. *L* denotes the domain size, λ is the central wavelength, and *a* is the spatial step (cell size). Indicative values are not provided as they would be implementation- and machine-dependent.

progress of FDTD because it lifted the Courant stability limit. Under these FDTD extensions, the time step is not bounded anymore by the stability criterion, but by accuracy considerations only. This has allowed the more efficient use of FDTD in problems that necessitate the use of highly refined grids (say, with an excess of 500 grid points per wavelength). The main challenges for unconditionally stable FDTD schemes are the introduction of new error mechanisms (more prominently, splitting errors) and the ever present numerical dispersion, which has different behavior than conventional FDTD [55].

Since the early days of FDTD applications, it was promptly recognized that one of its main limitations is the staircasing error [2]. In addition to contour-path conformal modeling techniques stated above, so-called "subgridding" techniques—a type of structured *h*-refinement whereby different grid cell sizes are used in different portions of the FDTD domain and connect though interfaces with hanging nodes—provide one possible approach to mitigate staircasing. Subgridding was first developed in the late 80's with major impetus occurring in the 90's [56-61]. Research continues to this day to develop stable subgridding algorithms that can provide low spurious reflections at the fine-coarse grid interfaces and small aliasing error from the necessary interpolation/decimation operations [62-64]. *Nonorthogonal* FDTD algorithms—relying on the use covariant and contravariant field components in a non-orthogonal coordinate system—are also used introduced to mitigate staircasing. They were first introduced in the early 1980s, but early versions of nonorthogonal FDTD algorithms were prone to numerical instabilities due to subtle inconsistencies in the spatial discretization. Nonorthogonal FDTD algorithms with (conditional) stability were developed only in the late 1990s [65,66]. These developments for the most part still relied upon nonorthogonal, but still *structured* grids. The use of *unstructured* grids for FDTD is not as natural because the traditional derivation of finite-differences becomes somewhat contrived. In this case, FETD becomes a more natural choice for the spatial discretization. Figure 2 illustrated various extensions of the basic FDTD method toward improving its accuracy and/or geometrical flexibility. In general, there is a mild trade-off between these two objectives in FDTD, leading to the "negative correlation" illustrated in this Figure.

It should be pointed out FDTD is also quite suited for solving Maxwell equations in complex media with dispersive, anisotropic and/or nonlinear properties. A discussion on these extensions and applications is beyond the objectives of this paper, but a detailed review can be found in [7].

The late part of the present decade has witnessed a wide popularization of a number of user-friendly, commercial software that feature the FDTD as their main "solver engine". Commercial PC-based codes have become pervasive in the RF, microwave, antenna, and optical communities and have also influenced the direction of research efforts. Furthermore, it has become apparent that certain research areas—such as device design—have become increasingly less reliant on "in-house" development of analysis (in particular numerical) tools. If this tendency continues, it is expected that this will lead to an increased "niching" of efforts by the CEM community and perhaps closer alignment with the computational physics and applied mathematics community, and perhaps less with the microwave and antenna engineering community at-large.

One important limitation present in commercial codes is related to the optimal choice in the "FDTD family" of depends on the nature and size of the problem, as illustrated in Figure 3. Commercial codes do not (yet) incorporate capabilities that would necessarily lead to the optimal choice of method for a given problem.

III. 25 YEARS OF PROGRESS IN FETD: A BIRD'S EYE VIEW

FETD is a relatively less mature than FDTD. As mentioned before, the main motivation for the development of FETD has been to increase the geometrical flexibility. This is because FETD is naturally based upon irregular (unstructured) grids, and thus capable of better adapting to curved or slanted geometries than a Cartesian FDTD. Compared to FDTD, the two major drawbacks for FETD are (1) the need for a pre-processing mesh generation step and (2) the need for a (sparse) linear solve at each time step.

The early FETD approaches for solving Maxwell equations were developed in the mid 1980s and were based on a point-matched approach combined with nodal basis functions for each field component [3]. Although successfully for some problems especially in two-dimensions, this "nodal" approach was prone to spurious or ghost modes (also known as "spectral pollution"). This was a problem not restricted to FETD per se, but it is also present in other algorithms based on irregular grids, including the frequency-domain FE. For many years, the problem of spurious modes evaded a fundamental solution. Only *ad hoc* approaches such as inclusion of penalty terms seemed to work in suppressing spurious modes. It was only with the development of edge elements (also known as Whitney or curl-conforming elements) that the problem of spurious modes was finally overcome in FE (and FETD) [67].

Most often, the various extensions developed for the basic FETD algorithm mirror the progress observed in FDTD with a time lag of a few years. The application of PML absorbing boundary condition to FETD was first seen, for example, in the late 1990s and it is currently still under active development. Differently from FDTD, FETD naturally allows for a conformal PML implementation over curved grid boundaries [68-70], which permits a more compact (i.e., with less buffer space) grid, especially for scattering problems. Another approach to truncate the grid boundaries in FE has been to use FE-BI (boundary integral) formulations [4]. Due to difficulties caused by causality requirements and stability issues, FE-BI approaches are relatively less developed in the time-domain than in frequency-domain. Extensions of FETD to complex media were first developed in the 1990s and continue to this day, more recently pushed by technological advances in remote sensing and metamaterials, for example [7]. Similarly to FDTD, FETD is also prone to numerical dispersion error and higher-order versions of FETD do exist to combat this problem. However, the dispersion error in FETD irregular grids manifests itself in a *quasi-isotropic* fashion, as opposed to the *anisotropic* dispersion observed in the conventional (Cartesian) FDTD grid. This is because the irregular grid "averages out" the cumulative dispersion error along the various directions.

As stated, FETD methods provide more accurate geometric representation than FDTD. Moreover, FETD methods are much more amenable to high-order accuracy (*p*-refinement) in general geometries by means of higher order basis functions, as opposed to enlarged stencils in FDTD. In particular, hierarchical higher order functions are particularly advantageous for *p*-adaptation because they can be implemented "in succession" and elements of different order can be coexist in the same mesh [71,72]. Also, *h-p* adaptive refinement methods are more suited for FETD methods [73,74].

Since FETD requires sparse linear algebra, an important associated area of research is on efficient linear solvers for large sparse linear systems. FETD requires iterative solvers and good preconditioners for large problems. For smaller problems direct solution methods often suffice (and are typically preferred since they avoid convergence issues). The nature of the matrix solvers in FETD depend on the particular time-discretization scheme being employed. Broadly speaking, time-discretization schemes utilized in FETD fall into two classes: The first class (I, sometimes referred to "implicit FETD" [4] although in a different sense from FDTD) necessitates the inversion of a system matrix that is a combination of stiffness and mass matrices, whereas the second class (II, sometimes referred to as "explicit FETD" [4] again in different sense from FDTD) necessitates the inversion of the mass matrix only [75]. The mass matrix is (when appropriately constructed) symmetric positive-definite, while the stiffness matrix is singular, hence linear systems resulting from class II are more benign. The ensuing trade-off is that Class II algorithms lead to conditionally stable algorithms, while Class I can produce unconditionally stable algorithms with no stability bound on the time-step.

Table 1: Comparison of some basic FDTD and FETD properties.

	FDTD	FETD
staircasing error	yes	no*
linear algebra	none	real and sparse
numerical dispersion	anisotropic	isotropic**
higher-order	larger stencils	*p*-refinement
mesh generation (pre-processing) step	absent	present**

(*)—linear facets (**)—for irregular grids

As mentioned above, FETD algorithms have been traditionally based on the discretization of the second-order wave equation using edges elements for the electric *or* magnetic field, as opposed to using the two first-order Maxwell equations. The solution space of the former is larger compared to the latter, admitting (spurious) gradient fields with linear growth as solutions. Normally, if the initial conditions are properly set (divergence-free), these solutions are not excited. For long-time simulations however, numerical round-off error introduced by the linear solver can lead to the excitation of such modes. More recently, mixed FETD formulations directly based upon the first-order Maxwell curl equations have become increasingly popular [5,74-76]. In this case, two different sets of basis functions are used (hence the name mixed), most often edge elements for the electric field and face elements for the magnetic field. This choice is informed by using the language of differential forms for Maxwell equations—as opposed to vector fields— where the electric field is a one-form and the magnetic flux density is a two-form [77]. This application of mixed basis functions satisfies a discrete version of the de Rham diagram, and thus it avoids spurious modes (see Section IV). Using a leap-frog scheme for the time-discretization, mixed *E-B* FETD produces conditionally stable algorithms with no secular growth modes [5]. Furthermore, under proper choice of edge and face basis function in a Cartesian grid and after mass lumping, the mixed *E-B* FETD recovers Yee's FDTD (see Section IV). It also suggests a consistent way to extend Yee's FDTD to slanted/curved interfaces and to construct hybrid FDTD/FETD algorithms.

IV. BRIDGING THE GAP BETWEEN FDTD AND FETD

Both FDTD and FETD are partial-differential-equation (PDE) based algorithms and—as considered here—applied to the same set of equations. Hence, it is only reasonable to expect, at some fundamental level, some major congruence between these two algorithms. Indeed, it can be shown that the FDTD is equivalent to (or it can be viewed as a special case of) FETD under the following choices for the FE discretization:

i) *Regular* quadrilateral (2-D) or hexahedral (3-D) grid.

ii) *Mixed basis functions* to expand the electric and magnetic fields (i.e., edge elements for ***E*** and face elements for ***B***) along with Galerkin testing or construction of Galerkin Hodge operators [70,74-76],

iii) *Mass lumping* applied to the mass matrix to approximate it as a diagonal matrix, and

iv) *Leap-frog* update for the time discretization.

A key reference on mass lumping schemes for Maxwell equations is [78]. The geometric underpinning for all these choices in FETD becomes apparent when Maxwell's equations are cast in terms of exterior differential forms [67, 77]. In this representation, the electric field intensity vector ***E*** is the proxy of a one-from *E*, whereas the magnetic flux density vector ***B*** is the proxy of a two-form, *B*. More generally, *p*-forms are objects that can be associated at the discrete level with "*p*-cells" of the mesh (p=0: nodes, p=1: edges, p=2: faces, p=3: volumes) and admit a natural discrete representation (cochains [79]) in terms of the so-called Whitney *p*-forms [67]. The latter recover edge elements for p=1 and face elements for p=2, for example. Moreover, the reason for staggered grids in FDTD is geometrically motivated by the fact that objects on the primal grid possess internal orientation (i.e., are "ordinary" differential forms) such as *E* and *B*, whereas objects on the dual grid possess external orientation (i.e., they are "twisted" differential forms) such as *H* and *D*, see illustration in Fig. 4 (left) [77]. These two kinds of discrete differential forms are defined on two grids (cell complexes), each inheriting one type of orientation (primal and dual grid, or ordinary and twisted complex, see Fig. 4).

When using differential forms, all vector differential operators such as div, curl, and grad are unified and become reduced to different incarnations of the *exterior derivative operator d* [77,80]. The exterior derivative *d* admits a trivial implementation on an arbitrary mesh in terms of its adjoint: the *boundary operator* ∂ [80]. The boundary operator carries the intuitive meaning, i.e., it maps an edge into its (two) boundary nodes; it maps a face into its (three, in the case or a triangular or tetrahedral mesh, or four, in the case of a rectangular or hexahedral mesh) boundary edges; and so on, as illustrated in Fig. 4 (right) [77,80,81]. Note that ∂^2=0 is verified for any mesh element (i.e., the boundary of a boundary is zero). This identity is simply a generalization of the vector calculus identities div curl=0 and curl grad =0.

In relation with these identities, it should also be pointed out that any FDTD or FETD implementation should obey a discrete version of the so-called *de Rham diagram* [67,82], which is illustrated in Fig. 5. Essentially, the de Rham diagram implies that (in a simply connected domain) the space of discrete zero-curl fields is isomorphic (i.e., one-to-one) to the space of discrete gradient fields; the space of discrete zero-divergence fields is isomorphic to the space of discrete curl fields; and so on, mirroring the properties of the respective continuum spaces. Conformity to the de Rham diagram is a key property to avoid appearance of spurious modes during the discretization process [67,82]. In the conventional (Yee's) FDTD scheme, the de Rham diagram is trivially verified. However, this is not true for subgridded FDTD, contour-path FDTD, or hybrid FDTD/FETD implementations, for example. In those cases, care should be exercised to make sure the resulting formulation follows the discrete de Rham diagram.

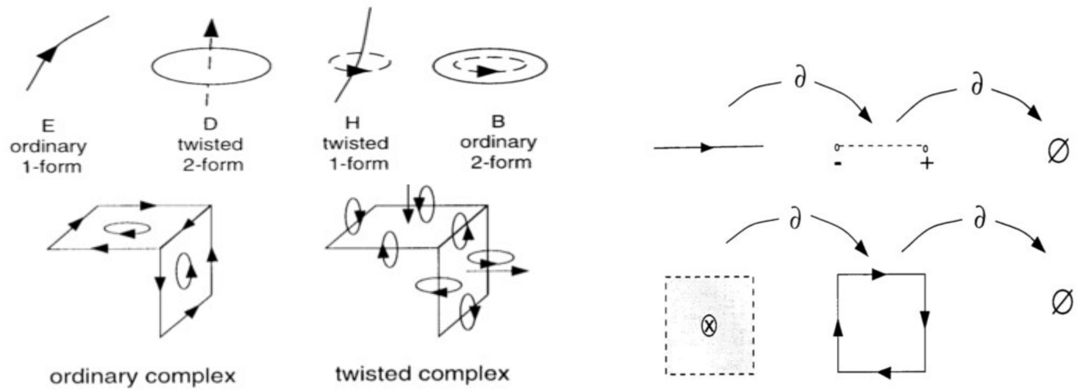

Fig. 4. *Left*—Internal (ordinary forms) and external (twisted forms) orientations for electromagnetic fields. This leads naturally to dual staggered grids (cell complexes). For simplicity, we depict a regular, hexahedral mesh. *Right*—Representation of the boundary operator ∂ acting on mesh elements (edge and cell/face). Note that the boundary of a boundary is always zero: ∂^2=0, which generalizes the identities curl grad =0, and div curl =0 distilled from their metric structure (Reprinted with permission from F. L. Teixeira and W. C. Chew, *J. Math. Phys.,* vol. 40, no. 1, pp. 169–187, 1999. © 1999, American Institute of Physics).

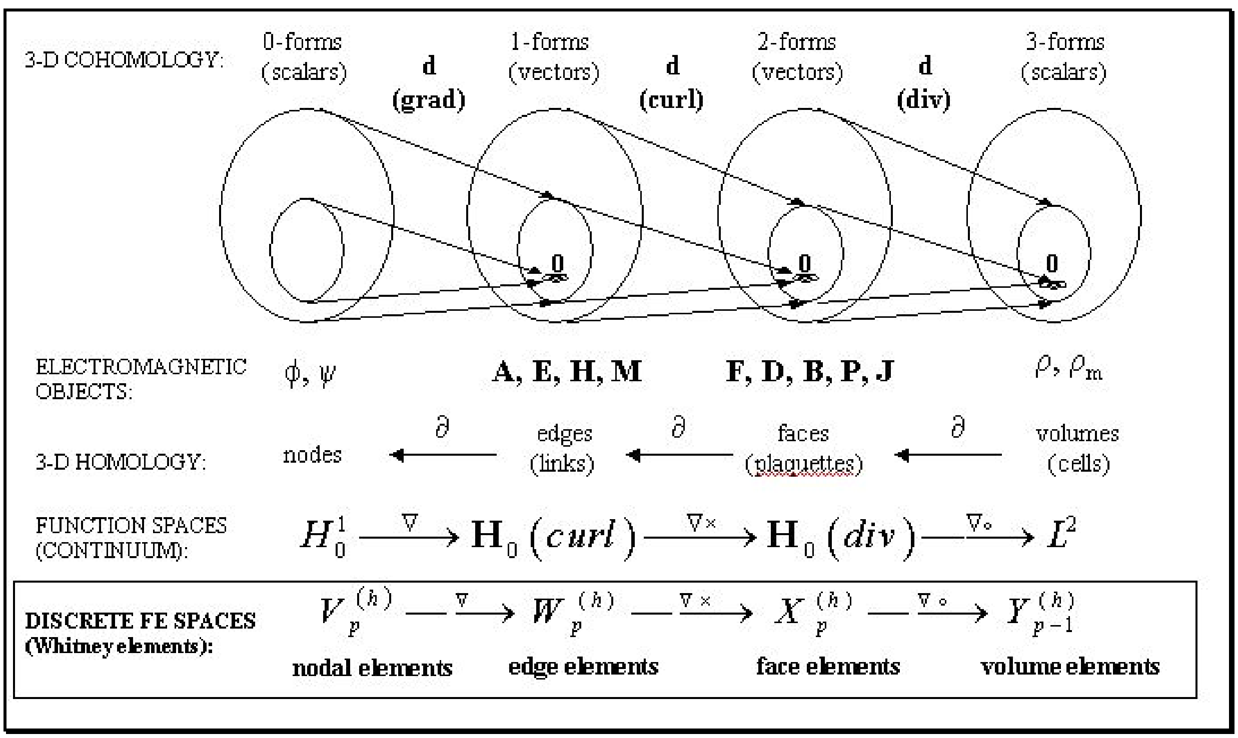

Fig. 5. Schematic illustration of the de Rham diagram (in a simply connected domain) and the relation among the various finite element spaces [67,82], differential forms [67,77], mesh components [80,81], and discrete operator maps [77]. The exterior derivative operator *d* (upper row) plays the role of the grad, curl, and div operators (fourth row) when acting on 0-, 1-, and 2-forms, respectively [77,81]. Both (consistent) FDTD and FETD algorithms should obey this diagram to avoid spurious modes.

V. FUTURE CHALLENGES

Prediction of future trends is always speculative and prone to miss “black swan” type of events. With this caveat in mind, we list below some of the future challenge areas for FDTD and FETD.

5.1 Parallelization and hardware developments— Since both FDTD and FETD algorithms are highly parallelizable, the full exploitation of parallel architectures and multicore processors with improved throughput and latency remains an important task. For FETD, linear solvers that explicitly exploit parallel architectures are likely to gain more importance. Coupled developments in FPGA and software-configurable microprocessors design are also likely to enhance the overall performance of FDTD and FETD codes. Of major importance also is the exploitation of graphics processing units (GPUs) and their highly parallel structure for acceleration of both FDTD and sparse linear solvers (the latter with obvious impact on FETD) [83].

5.2 Grid dispersion error control— For very large scale problems, minimization of grid-dispersion error is a critical issue. Ideally, this should be done with minimal impact on the underlying sparsity of the methods. In FDTD, pre-asymptotic high-order schemes have come a long way towards this objective, but similar progress remains to be achieved in FETD.

5.3 Adaptation— Further development in *a priori* and *a posteriori* error indicators in time-domain will certainly benefit the development of fully adaptive meshing techniques, either based on structured (for FDTD, such as subgridding techniques) or unstructured meshes for FETD, and using either static or dynamically adaptation [84].

5.4 Multi-domain approaches and domain-decomposition—Development of domain-decomposition (DD) techniques as a “divide-and-conquer” methodology to reduce the CPU requirements and most importantly, memory requirements in FETD, is another future challenge. Much progress has been done in recent years in frequency-domain DD-FE techniques, but application to the time-domain remains a challenge. In FDTD, the (possibly adaptive) use of Huygens’ boxes to minimize the solution space remains an area for future developments especially for problems with highly disparate geometrical sizes, and for applications such as antenna-platform and antenna-antenna interaction problems [85].

5.5 Hybridization with integral-equation and asymptotic methods—The seamless integration (hybridization) of FETD and FETD with either full-wave TD-IE (integral equation) and/or high-frequency asymptotic approaches employing, for example, Gaussian beams or TD-UTD (uniform theory of diffraction), in dynamically adaptive schemes remains to be achieved.

5.6 Discrete differential forms—Application of differential forms is of particular interest to provide “design principles” of new FDTD and FETD compatible discretization schemes for more arbitrary mesh elements (polyhedral, concave) and in composite/heterogeneous grids [86].

5.7 Asynchronous time-stepping—Time-stepping is a relatively primitive and costly approach to enforce causality in time-domain methods. Possibly, the exploitation of discrete-event simulation approaches whereby dynamical states are updated asynchronously on demand (i.e., only when necessary) [87]—instead of synchronously—will certainly be an important development to extend the applicability of FDTD and FETD to, for example, problems with disparate time-scales (multiscale).

5.8 Hybrid FDTD/FETD —Since FETD provides better geometrical flexibility and FDTD better memory scalability, it is only natural to seek an hybridization of these two methods—using FETD in regions with high geometrical complexity and FDTD elsewhere [88]. Earlier hybrid FDTD/FETD schemes were often plagued by numerical instabilities and spurious modes [89]. Recently, consistent hybrid FDTD/FETD methods based on vector elements and free from instabilities were put forth [90]. These methods obey the consistency rules discussed in Section IV. It is expected that further development of hybrid FDTD/FETD, such as integration of higher order and extension to complex media, will make it a method of choice for many electromagnetic problems [91].

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REFERENCES

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell´s equations in isotropic media," *IEEE Trans. Antennas Propag.*, vol. 14, pp. 302-307, 1966.

[2] A. Taflove and S. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 3rd ed., Artech House, 2005.

[3] J. F. Lee, R. Lee, and A. C. Cangellaris, "Time domain finite element methods," *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 430-442, 1997.

[4] J. M. Jin, *The Finite Element Method in Electromagnetics*, 2nd ed., John Wiley, 2003.

[5] B. Donderici and F. L. Teixeira, "Mixed finite-element time-domain method for transient Maxwell equations in doubly dispersive media," *IEEE Trans. Microwave Theory Tech.*, vol. 56, no. 1, pp. 113-120, 2008.

[6] K. L. Shlager and J. B. Schneider, "A selective survey of the finite-difference time-domain literature," *IEEE Antennas Propag. Mag.*, vol. 37, no. 4, pp. 39-56, 1995.

[7] F. L. Teixeira, "Time-domain finite-difference and finite-element methods for Maxwell equations in complex media," *IEEE Trans. Antennas Propag.*, vol. 56, no. 8, pp. 2150-2166, 2008.

[8] A. Taflove and M. E. Brodwin, "Numerical solution of steady-state electromagnetic scattering problems using the time-dependent Maxwell's equations," *IEEE Trans. Microwave Theory Tech.,* vol. 23, pp. 623–630, 1975.

[9] R. Holland, "Threde: A free-field EMP coupling and scattering code," *IEEE Trans. Nuclear Sci.,* vol. 24, pp. 2416–2421, 1977.

[10] K. S. Kunz and K. M. Lee, "A three-dimensional finite-difference solution of the external response of an aircraft to a complex transient EM environment," *IEEE Trans. Electromagn. Compat.,* vol. 20, pp. 328–341, 1978.

[11] A. Taflove, "Application of the finite-difference time-domain method to sinusoidal steady state electromagnetic penetration problems," *IEEE Trans. on Electromagn. Compat.,* vol. 22, pp. 191–202, 1980.

[12] T. Weiland, "A discretization method for the solution of Maxwell's equations for six-component fields," *Electronics and Communications AEU,* vol. 31, no.3, pp. 116–120, 1977.

[13] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time-domain electromagnetic field equations," *IEEE Trans. Electromagn. Compat.,* vol. 23, pp. 377–382., 1981.

[14] Z. P. Liao, H. L. Wong, B. P. Yang, and Y. F. Yuan, "A transmitting boundary for transient wave analysis," *Scientia Sinica A* vol. 27, pp. 1063–1076, 1984.

[15] G. A. Kriegsmann, A. Taflove, and K. R. Umashankar, "A new formulation of electromagnetic wave scattering using an on-surface radiation boundary condition approach," *IEEE Trans. Antennas Propag.,* vol. 35, pp. 153–161, 1987.

[16] T. G. Moore, J. G. Blaschak, A. Taflove, and G. A. Kriegsmann, "Theory and application of radiation boundary operators," *IEEE Trans. Antennas Propag.,* vol. 36, pp. 1797–1812, 1988.

[17] K. R. Umashankar and A. Taflove, "A novel method to analyze electromagnetic scattering of complex objects," *IEEE Trans. Electromagn. Compat.,* vol. 24, pp. 397–405, 1982.

[18] A. Taflove and K. R. Umashankar, “Radar cross section of general three-dimensional scatterers,” *IEEE Trans. Electromagn. Compat.,* vol. 25, pp. 433–440, 1983.

[19] D. H. Choi and W. J. Hoefer, “The finite-difference time-domain method and its application to eigenvalue problems,” *IEEE Trans. Microwave Theory Tech.,* vol. 34, pp. 1464–1470, 1986.

[20] X. Zhang, J. Fang, K. K. Mei, and Y. Liu, “Calculation of the dispersive characteristics of microstrips by the time-domain finite-difference method,” *IEEE Trans. Microwave Theory Tech.,* vol. 36, pp. 263–267, 1988.

[21] D. M. Sullivan, O. P. Gandhi, and A. Taflove, “Use of the finite-difference time-domain method in calculating EM absorption in man models,” *IEEE Trans. Biomed. Eng.,* vol. 35, pp. 179–186, 1988.

[22] K. R. Umashankar, A. Taflove, and B. Beker, “Calculation and experimental validation of induced currents on coupled wires in an arbitrary shaped cavity,” *IEEE Trans. Antennas Propag.,* vol. 35, pp. 1248–1257, 1987.

[23] A. Taflove, K. R. Umashankar, B. Beker, F. A. Harfoush, and K. S. Yee, “Detailed FDTD analysis of electromagnetic fields penetrating narrow slots and lapped joints in thick conducting screens,” *IEEE Trans. Antennas Propag.,* vol. 36, pp. 247–257, 1988.

[24] C. J. Railton, I. J. Craddock, and J. B. Schneider, "An improved locally distorted CPFDTD algorithm with provable stability," *Electron. Lett.*, vol. 31, no. 18, pp. 1585-1586, 1995.

[25] S. Dey and R. Mittra, “A locally conformal finite-difference time-domain (FDTD) algorithm for modeling three-dimensional perfectly conducting objects,” *IEEE Microwave Guided Wave Lett*., vol. 7, pp. 273-275, 1997.

[26] C. J. Railton and J. B. Schneider, "An analytical and numerical analysis of several locally-conformal FDTD Schemes," *IEEE Trans. Microwave Theory Tech.*, vol. 47, no. 1, pp. 56-66, 1999

[27] W. Gwarek, “Analysis of an arbitrarily shaped planar circuit — A time-domain approach,” *IEEE Trans. Microwave Theory Tech.,* vol. 33, pp. 1067–1072, 1985.

[28] G. Maloney, G. S. Smith, and W. R. Scott, Jr., “Accurate computation of the radiation from simple antennas using the finite-difference time-domain method,” *IEEE Trans. Antennas Propag.,* vol. 38, pp. 1059–1065., 1990

[29] D. S. Katz, A. Taflove, M. J. Piket-May, and K. R. Umashankar, “FDTD analysis of electromagnetic wave radiation from systems containing horn antennas,” *IEEE Trans. Antennas Propag.,* vol. 39, pp. 1203–1212, 1991.

[30] K. Li, C. F. Lee, S. Y. Poh, R. T. Shin, and J. A. Kong, “Application of FDTD method to analysis of electromagnetic radiation from VLSI heatsink configurations,” *IEEE Trans. Electromagn. Compat*., vol. 35, no. 2, pp. 204 – 214, 1993.

[31] R. Lee and T.T. Chia, “Analysis of electromagnetic scattering from a cavity with a complex termination by means of a hybrid ray-FDTD method,” *IEEE Trans. Antennas Propag.*, vol. 41, pp. 1560-1569, 1993.

[32] V. A. Thomas, M. E. Jones, M. J. Piket-May, A. Taflove, and E. Harrigan, “The use of SPICE lumped circuits as sub-grid models for FDTD high-speed electronic circuit design,” *IEEE Microwave Guided Wave Lett.,* vol. 4, pp. 141–143, 1994.

[33] A. Bataineh, R. Lee, and F. Ozguner, “Electrical characterization of high-speed interconnects with a parallel three-dimensional finite difference time-domain algorithm,” *Simulation*, vol. 64, pp. 289-295, 1995.

[34] E. Sano and T. Shibata, “Full-wave analysis of picosecond photoconductive switches,” *IEEE J. Quantum Electron.,* vol. 26, pp. 372–377, 1990.

[35] P. M. Goorjian and A. Taflove, “Direct time integration of Maxwell’s equations in nonlinear dispersive media for propagation and scattering of femtosecond electromagnetic solitons,” *Opt. Lett.*, vol. 17, pp. 180–182, 1992.

[36] R. W. Ziolkowski and J. B. Judkins, “Full-wave vector Maxwell’s equations modeling

of self-focusing of ultra-short optical pulses in a nonlinear Kerr medium exhibiting a finite response time," *J. Opt. Soc. Am. B,* vol. 10, pp. 186–198, 1983.

[37] R. Luebbers, F. Hunsberger, K. Kunz, R. Standler, and M. Schneider, "A frequency-dependent finite-difference time-domain formulation for dispersive materials," *IEEE Trans. Electromagn. Compat.,* vol. 32, pp. 222–227, 1990.

[38] R. M. Joseph, S. C. Hagness, and A. Taflove, "Direct time integration of Maxwell's equations in linear dispersive media with absorption for scattering and propagation of femtosecond electromagnetic pulses," *Opt. Lett.,* vol. 16, pp. 1412–1414, 1991.

[39] A. C. Cangellaris, M. Gribbons, and G. Sohos, "A hybrid spectral/FDTD method for the electromagnetic analysis of guided waves in periodic structures," *IEEE Microwave Guided Wave Lett.*, vol. 3, no. 10, pp. 375–377, 1993.

[40] P. Harms, R. Mittra, and W. Ko, "Implementation of the periodic boundary condition in the finite-difference time-domain algorithm for FSS structures," *IEEE Trans. Antennas Propag.*, vol. 42, no. 9, pp.1317–1324, 1994.

[41] A. Taflove and S. Hagness, "Analysis of Periodic Structures," in *Computational Electrodynamics: The Finite Difference Time Domain Method*. Boston, MA: Artech House, 1995, ch. 13.

[42] M. Celuch-Marcysiak and W. K. Gwarek, "Spatially looped algorithms for time-domain analysis of periodic structures," *IEEE Trans. Microwave Theory Tech.*, vol. 43, no. 4, pp. 860–865, 1995.

[43] S. Wang and F. L. Teixeira, "Dispersion-relation-preserving FDTD schemes of large-scale three-dimensional problems," *IEEE Trans. Antennas Propag.*, vol. 51, no. 8, pp. 1818-1828, 2003.

[44] B. Finkelstein and R. Kastner, "A comprehensive new methodology for formulating FDTD schemes with controlled order of accuracy and dispersion," *IEEE Trans. Antennas Propag.*, vol. 56, no. 11, pp. 3516-3525, 2008.

[45] Q.H. Liu, "The PSTD algorithm: A time-domain method requiring only two cells per wavelength," *Microwave Opt. Technol. Lett.*, vol. 15, no. 3, pp. 357-361, 1997.

[46] J. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys.*, vol. 114, pp. 185–200, 1994.

[47] D. S. Katz, E. T. Thiele, and A. Taflove, "Validation and extension to three dimensions of the Berenger PML absorbing boundary condition for FDTD meshes," *IEEE Microwave Guided Wave Lett.,* vol. 4, pp. 268–270, 1994.

[48] Z. S. Sacks, D. M. Kingsland, R. Lee, and J. F. Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," *IEEE Trans. Antennas Propag.,* vol. 43, pp. 1460–1463, 1995.

[49] F. L. Teixeira and W. C. Chew, "Complex space approach to perfectly matched layers: A review and some new developments," *Int. J. Num. Model.,* vol. 13, no. 5, pp. 441-455, 2000.

[50] F. L. Teixeira, K. P. Hwang, W. C. Chew, and J. M. Jin, "Conformal PML-FDTD schemes for electromagnetic field simulations: A dynamic stability study," *IEEE Trans. Antennas Propag.*, vol. 49, no. 6, pp. 902-907, 2001.

[51] C. D. Moss, F. L. Teixeira, Y.E. Yang, and J. A. Kong, "Finite-difference time-domain simulation of scattering from objects in continuous random media," *IEEE Trans. Geosci. Remote Sens.*, vol. 40, no. 1, pp. 178-186, 2002.

[52] T. Namiki, "A new FDTD algorithm based on alternating-direction implicit method," *IEEE Trans. Microwave Theory Tech.*, vol. 47, no. 10, pp. 2003–2007, Oct. 1999.

[53] F. Zheng, Z. Chen, and J. Zhang, "Toward the development of a three-dimensional unconditionally stable finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 48, no. 9, pp. 1550–1558, Sep. 2000.

[54] V. E. do Nascimento, B.-H. V. Borges, and F. L. Teixeira, "Split-field PML implementations for the unconditionally stable LOD-FDTD method," *IEEE*

Microwave Wireless Components Lett., vol. 16, no. 7, pp. 398-400, 2006.

[55] K.-Y. Jung and F. L. Teixeira, “An iterative unconditionally stable LOD-FDTD method,” *IEEE Microwave Wireless Components Lett.*, vol. 18, no. 2, pp. 76-78, 2008.

[56] S. S. Zivanovic, K. S.Yee, and K. K. Mei, “A subgridding method for the time-domain finite-difference method to solve Maxwell’s equations,” *IEEE Trans. Microwave Theory Tech.*, vol. 39, no. 3, pp. 471–479, 1991.

[57] S. Kapoor, “Sub-cellular technique for finite-difference time-domain method,” *IEEE Trans. Microwave Theory Tech.*, vol. 45, no. 5, pp. 673–677, 1997.

[58] M. Okoniewski, E. Okoniewska, and M. A. Stuchly, “Three-dimensional subgridding algorithm for FDTD,” *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 422–429, 1997.

[59] K. M. Krishnaiah and C. J. Railton, ‘‘Passive equivalent circuit of FDTD: an application to subgridding,’’ *Electron. Lett.*, vol. 33, 1277–1278, 1997.

[60] P. Thoma and T. Weiland, ‘‘A consistent subgridding scheme for the finite difference time domain method,’’ *Int. J. Num. Model.* **9**, 359–374, 1996.

[61] K. M. Krishnaiah and C. J. Railton, "A stable subgridding algorithm and its application to eigenvalue problems," *IEEE Trans. Microwave Theory Tech*., vol. 47, pp. 620-628, 1999.

[62] S. Wang, F. L. Teixeira, R. Lee, and J.-F. Lee, `Optimization of subgridding schemes for FDTD,' *IEEE Microwave Wireless Components Lett.*, vol. 12, no. 6, pp. 223-225, 2002.

[63] B. Donderici and F. L. Teixeira, “Improved FDTD subgridding algorithms via digital filtering and domain overriding,” *IEEE Trans. Antennas Propag.*, vol. 53, no. 9, pp. 2938-2951, 2005.

[64] J. P. Berenger, “A Huygens subgridding for the FDTD method,” *IEEE Trans. Antennas Propag.*, vol. 54, no. 12, pp. 3797 – 3804, 2006.

[65] R. Schuhmann and T. Weiland, “A stable interpolation technique for FDTD on non-orthogonal grids,” *Int. J. Num. Model.*, vol. 11, no. 6, pp. 299 – 306, 1999.

[66] S. D. Gedney and J. A. Roden, “Numerical stability of nonorthogonal FDTD methods,” *IEEE Trans. Antennas Propag.*, vol. 48, issue 2, pp. 231-239, 2000.

[67] A. Bossavit, “Whitney forms: A new class of finite elements for three-dimensional computations in electromagnetics,” *IEE Proc. A* , vol. 135, pp. 493–500, 1988.

[68] F. L. Teixeira and W. C. Chew, “Analytical derivation of a conformal perfectly matched absorber for electromagnetic waves,” *Microwave and Optical Technology Letters*, vol. 17, no. 4, pp. 231-236, 1998.

[69] T. Rylander and J.-M. Jin, “Conformal perfectly matched layers for the time domain finite element method,” in *IEEE AP-S Int. Symp. Digest*, vol. 1, pp. 698–701, 2003.

[70] B. Donderici and F. L. Teixeira, “Conformal perfectly matched layer for the mixed finite-element time-domain method,” *IEEE Trans. Antennas Propag.*, vol. 56, no. 4, pp. 1017-1026, 2008.

[71] M. M. Ilic and B. M. Notaros, “Higher order hierarchical curved hexahedral vector finite elements for electromagnetic modeling,” *IEEE Trans. Microwave Theory Tech.*, vol. 51, no. 3, pp. 1026-103, 2003.

[72] R. N. Rieben , G. H. Rodrigue , and D. A. White, “A high order mixed vector finite element method for solving the time dependent Maxwell equations on unstructured grids,” *J. Comp. Phys.*, vol. 204, no. 2, pp. 490-519, 2005.

[73] G. Cohen, *Higher Order Numerical Methods for Transient Wave Equations*, Springer-Verlag, 2001.

[74] N. Marais and D. B. Davidson, “Numerical evaluation of high-order finite element time domain formulations in electromagnetics,” *IEEE Trans. Antennas Propag.*, vol. 56, no. 12, pp. 3743-3751, 2008.

[75] B. He and F. L. Teixeira, “A sparse and explicit FETD via approximate inverse Hodge (mass) matrix,” *IEEE Microwave Wireless Components Lett.*, vol. 16, no. 6, pp. 348-350, 2006.

[76] M. Wong, O. Picon, and V. F. Hanna, “A finite element method based on Whitney

forms to solve Maxwell equations in the time domain," *IEEE Trans. Magn.*, vol. 31, no. 3, pp. 1618–1621, 1995.

[77] F. L. Teixeira and W. C. Chew, "Lattice electromagnetic theory from a topological viewpoint," *J. Math. Phys.,* vol. 40, no. 1, pp. 169–187, 1999.

[78] G. Cohen and P. Monk, "Gauss point mass lumping schemes for Maxwell's equations," *Numer. Meth. Partial Diff. Eq.*, vol. 14 , pp. 63–88, 1998.

[79] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Reading, Mass., 1984.

[80] F. L. Teixeira (ed.), *Geometrical Methods for Computational Electromagnetics*, Progress In Electromagnetics Research Series 32, EMW Publishing, Cambridge, Mass., 2001.

[81] B. He and F. L. Teixeira, "On the degrees of freedom of lattice electrodynamics," *Phys. Lett. A,* vol. 336, pp. 1-7, 2005.

[82] P. Monk, Finite *Element Methods for Maxwell's Equations*, Oxford Univ. Press, 2003.

[83] S. Adams, J. Payne, and R. Boppana, "Finite Difference Time Domain (FDTD) Simulations Using Graphics Processors," *Proc. 2007 DoD High Performance Computing Modernization Program Users Group Conf.*, pp. 334-338, 2007.

[84] Y. Liu and C.D. Sarris, "Fast Time-Domain Simulation of Optical Waveguide Structures with a Multilevel Dynamically Adaptive Mesh Refinement FDTD Approach," *J. Lightwave Technol.*, vol. 24, no. 8, pp. 3235-3247, 2006.

[85] R. A. Abd-Alhameed and P. S. Excell, Complex computational electromagnetic using hybridization techniques," in *Advances in Information Technologies for Electromagnetics* (L. Tarricone, A. Espostio, eds.), Springer-Verlag, pp.69-145, 2006.

[86] B. Donderici and F. L. Teixeira, "Accurate interfacing of heterogeneous structured FDTD grid components," *IEEE Trans. Antennas Propag.*, vol. 54, no. 6. pp. 1826-1835, 2006.

[87] A. Lew, A., J.E. Mardsen, M. Ortiz, and M. West, "Asynchronous variational integrators," *Arch. Rational Mech. Anal.*, vol. 167, p. 85, 2003.

[88] D. J. Riley and C. D. Turner, "VOLMAX: a solid-model-based, transient volumetric Maxwell solver using hybrid grids," *IEEE Antennas Propag. Mag.*, vol. 39, no. 1, pp. 20-33, 1997.

[89] C.-T. Hwang and R.-B. Wu, "Treating late-time instability of hybrid finite-element/finite-difference time-domain method," *IEEE Trans. Antennas Propag.*, vol. 47, no. 2. pp. 227-232, 1999.

[90] T. Rylander and A. Bondeson, "Stable FDTD-FEM hybrid method for Maxwell's equations," *Comp. Phys. Comm.,* vol. 125, pp. 75–82, 2000.

[91] J. Jin and D. J. Riley, *Finite Element Analysis of Antennas and Arrays*, Wiley-IEEE Press, 2009.