Comparison of Convergence Rates of Edge and Nodal Finite Elements for Magnetostatic Problems

M. L. y Gonzalez[1], R. C. Mesquita[1], J. P. A. Bastos[2]
[1]Universidade Federal de Minas Gerais - Belo Horizonte - MG - Brazil.
[2]Universidade Federal de Santa Catarina - Florianópolis - SC - Brazil.
E-mail : manuel@novell.cpdee.ufmg.br

Abstract – **Mathematical theory is used to obtain convergence estimations for magnetostatic formulations using nodal and edge elements. Numerical results of two closed boundary problems that confirm the theory are presented.**

I. Introduction

Much research has been done recently with nodal and edge elements in electromagnetic problems. However, it has not been discussed how an approximate solution, obtained by the finite element method, converges to the exact solution of the problem, when the mesh is successively refined. Recent papers [1, 2] show the characteristics of those elements and their convergence rates, when the field variable interpolated is the magnetic field **H**. In [3] and [4], convergence rates for regular meshes of nodal finite elements are presented, when the field variable interpolated is the magnetic vector potential **A**. In approximately closed boundary magnetostatic problems, the obtained results confirm the theoretical convergence rates [3,4].

Nevertheless, the authors do not know any convergence study that compares the edge and nodal elements when the magnetic vector potential **A** is being used as the field variable.

In this paper, convergence rates for nodal and edge elements with one degree of freedom per edge (also named Nédélec elements or Mixed elements), applying regular meshes, are established. Convergence rates of closed boundary problems are also presented, applying nodal and Nédélec elements.

II. Description of the Elements

Nodal elements represent vector functions with continuous components and show good results in homogeneous domains. Edge elements guarantee the continuity of the tangential components of these functions across the element's interfaces, allowing the discontinuity of the normal component. They can be applied in homogeneous or inhomogeneous domains, and they are better than the nodal elements in the treatment of the singularity of re-entrant corners [1,5].

The nodal elements in 3-D have 3 degrees of freedom per node, while the Nédélec elements have 1 degree of freedom per edge. Some results seem to indicate that the Nédélec elements are more efficient computationally than the nodal elements [1]. If the following assumptions are made: (i) there exists a large mesh with hexahedral finite elements; (ii) the reduction of the number of unknowns due to the boundary conditions can be ignored; (iii) no gauge is applied in the problem; then it is found that the number of nonzero elements in the global matrix is equal to 236 Ne for nodal elements, and 99 Ne for edge elements, where Ne is the number of elements in the mesh [6]. So, it seems that, in terms of memory occupation, edge elements are better than nodal elements.

III. Mathematical Formulation

In the computation of magnetostatic fields using the magnetic vector potential, the following equation is used:

$$\nabla x (\upsilon \nabla x \mathbf{A}) = \mathbf{J} , \qquad (1)$$

where υ is the magnetic reluctivity and **J** the current density in the problem domain Ω. This equation is a partial differential equation of second order that has the weak form

$$\mathbf{D}(\mathbf{A}, w) = \mathbf{f}(\mathbf{J}, w) \qquad \forall w \in \mathcal{V}, \qquad (2)$$

where **D** and **f** are symmetric bilinear forms that consist of integrals over Ω, and $\mathcal{V}$ is the space of admissible functions. This weak form has only derivatives of first order.

IV. Error Estimates and Convergence Rates

A. Nodal Elements

Let $\mathbf{A}_h$ be an approximate solution found by the finite element method to the magnetic vector potential, and $\mathbf{A}$ the exact solution of the problem. Let $\mathbf{e} = \mathbf{A} - \mathbf{A}_h$ be the error between $\mathbf{A}$ and $\mathbf{A}_h$, then we define the following Sobolev norms:

$$\|e\|_{H^m(\Omega)} = \left[\int_\Omega \sum_{\alpha=0}^{m} \left(e^\alpha\right)^2 dx \right]^{1/2} \tag{3}$$

and

$$\|\mathbf{A}\|_{H^{k+1}(\Omega)} = \left[\int_\Omega \sum_{\alpha=0}^{k+1} \left(\mathbf{A}^\alpha\right)^2 dx \right]^{1/2}, \tag{4}$$

where k is the order of the polynomials used in the local basic functions, m the order of derivatives that appear in (2), e^α and $\mathbf{A}^\alpha$ all the derivatives of order α of e and $\mathbf{A}$, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z_+^n$ (Set of all ordered n-tuples of non-negative integers), and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Considering that the finite element mesh is regular, and using the Aubin-Nitsche theorem [8, 9], the following estimate for the error norm is obtained:

$$\|e\|_{H^s(\Omega)} \le C_0 h^\mu \|\mathbf{A}\|_{H^{k+1}(\Omega)}, \tag{5}$$

for $0 \le s \le m$ and $\mu = \min\,[k+1-s,\, 2(k+1-m)]$. h is the parameter of the mesh, defined by the diameter of the smallest circle that contains the largest element of the mesh, and C_0 is a constant independent of $\mathbf{A}$ and h.

If our interest is in $\|e\|_{H^0(\Omega)}$, and we know that $\|\mathbf{A}\|_{H^{k+1}(\Omega)}$ is constant, we can set $C_1 = C_0 \|\mathbf{A}\|_{H^{k+1}(\Omega)}$, obtaining

$$\|e\|_{H^0(\Omega)} \le C_1 h^\mu . \tag{6}$$

But, $m = 1$, then $\mu = k+1$ since $2k \ge (k+1)$. In this way, the convergence rate is of order $O(h^{k+1})$ for the magnetic vector potential $\mathbf{A}$. As $\mathbf{B}$, the magnetic flux density, is computed from $\mathbf{A}$ by building the curl, and the convergence rate decreases with the derivatives [8], the convergence rate for $\mathbf{B}$ is of order $O(h^k)$.

B. Edge Elements

Let Γ be the boundary of the problem, η the unit normal vector in Γ, then we can define the following Hilbert spaces:

$$H(\mathrm{curl},\Omega) = \left\{ \mathbf{A} \in \left(L_2(\Omega)\right)^3 : \nabla x\mathbf{A} \in \left(L_2(\Omega)\right)^3 \right\} \tag{7}$$

and

$$H_0(\mathrm{curl},\Omega) = \left\{ \mathbf{A} \in H(\mathrm{curl},\Omega) : \mathbf{A}x\eta = \mathbf{0} \text{ in } \Gamma \right\} \tag{8}$$

Let P_k be the polynomial space of degree $\le k$ and $\tilde{P}_k$ the homogeneous polynomial space of degree k. Then we can also define the space

$$S_k = \left\{ \mathbf{p}(\mathbf{x}) \in \left(\tilde{P}(\mathbf{x})\right)^3 : \mathbf{p}(\mathbf{x}).\mathbf{x} = 0 \right\}, \tag{9}$$

where $\mathbf{p}(\mathbf{x}).\mathbf{x}$ is the inner product between $\mathbf{p}(\mathbf{x})$ and $\mathbf{x}$, that is, $\mathbf{p}(\mathbf{x})$ and $\mathbf{x}$ are orthogonal. Let U_h be the space of approximate function, $A_h \in U_h$, and the projection operator $\pi_h \mathbf{A} \in U_h$, so that the following inner products are zero:

$$\left(\nabla x(\mathbf{A} - \pi_h \mathbf{A}), \nabla x\mathbf{V}\right) = \mathbf{0}, \qquad \forall \mathbf{V} \in U_h \tag{10}$$

and

$$\left(\mathbf{A} - \pi_h \mathbf{A}, \nabla \mathbf{p}(\mathbf{x})\right) = \mathbf{0}, \qquad \forall \mathbf{p}(\mathbf{x}) \in S_k . \tag{11}$$

Then, we have the following theorem [7]:

Consider a regular mesh and assume that $\mathbf{A} \in H_0(\mathrm{curl},\Omega)$ and $\pi_h \mathbf{A}$ is defined as above. Then

(i) if $\mathbf{A} \in \left(H^{k+1}(\Omega)\right)^3$, then

$$\|\mathbf{A} - \pi_h \mathbf{A}\|_{H^0(\Omega)} \le C h^k \|\mathbf{A}\|_{H^{k+1}(\Omega)} . \tag{12}$$

(ii) if $\nabla x\mathbf{A} \in \left(H^k(\Omega)\right)^3$, and $\pi_h \mathbf{A}$ is well defined as above, then

$$\|\nabla x(\mathbf{A} - \pi_h \mathbf{A})\|_{H^0(\Omega)} \le C h^k \|\nabla x\mathbf{A}\|_{H^k(\Omega)} . \tag{13}$$

Let $\mathbf{A}_h = \pi_h \mathbf{A}$ and $\mathbf{B}_h = \nabla \mathrm{x} \mathbf{A}_h$. Then we notice that the convergence rates for **A** and **B** are of order $\mathbf{O}(h^k)$.

V. Results

Using 3-D electromagnetic field computation programs developed by our research groups, the error estimates for the following problems have been determined:

(i) Infinite square coaxial nonmagnetic cable in air.
(ii) Infinite rectangular magnetic busbar in air.

The problem (i) was chosen because it behaves as a closed boundary problem (null magnetic field in the proximity of the cable), when the current that flows in the outward conductor is chosen to cancel the magnetic field in the proximity of the cable, created by the current that flows in the inner conductor. In this way, it is possible to represent the boundary conditions of the problem in exact manner, and one avoids that the boundary condition errors could introduce errors in the computation of the convergence rates.

The problem (ii) was chosen because it has two permeabilities and behaves as a closed boundary problem when the permeability of the busbar is much greater than the air permeability.

In the following section all those problems are presented with their convergence rates. The analytic solutions are found in [10].

Infinite Square Coaxial Nonmagnetic Cable in Air

A cable with an inner conductor of 10 cm x 10 cm, and an outward conductor of 50 cm x 50 cm with a thickness of 10 cm was considered, as shown in Fig. 1. A current of 10,000 A was assumed in both conductors. The current flows axially in opposite senses in the inner and in the outer conductors. The boundary condition **A** = **0** was considered to be 10 cm outside of outward conductor.

Let us substitute **B** in (5) and (6). Then, L_2 error for **B** is

$$\mathrm{L2Berr} = \left[\int_{\Omega} (\mathbf{B} - \mathbf{B}_h)^2 \, \mathrm{dx} \right]^{1/2} . \qquad (14)$$

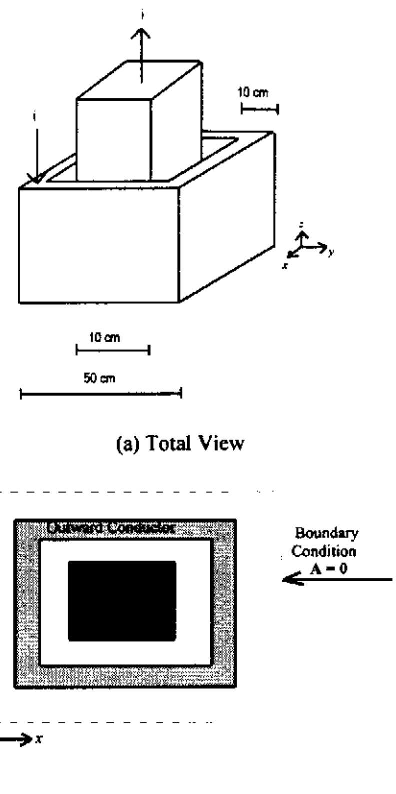
The results obtained for this error norm in the nodal and Nédélec Elements, when the finite element mesh is refined (-log h is increased), are shown in Fig. 2 and 3 respectively.

Using linear regression, we have found a convergence rate of μ = 0,961 for the nodal elements [4] and μ = 0,959 for the edge elements. As these computations were executed with interpolating polynomials of 1st degree (k = 1), the rates are near 1, and confirm the theory.

Infinite Rectangular Magnetic Busbar

it was considered a busbar of 10 cm x 20 cm with magnetic permeability μ_r = 1000. The current of 400 A flows axially in the busbar, and the boundary condition **A** = **0** was assumed at a distance of 15 cm from the busbar center, as shown in Fig. 4.

The computation of the values of L2Berr is shown in Figs. 5 and 6. Using also linear regressions, we obtained a convergence rate of μ = 0.991 [3] for the nodal elements and μ = 0,987 for the edge elements.



(a) Total View

(b) Plant View

Fig. 1 - View of Coaxial Cable

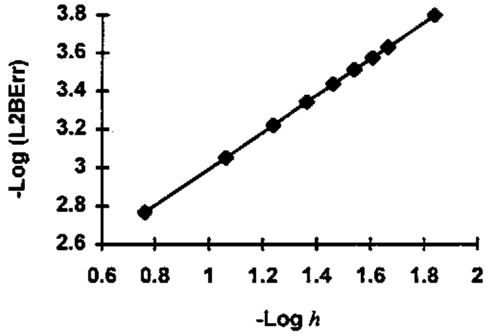


Fig. 2 - Results of Coaxial Cable with Nodal elements

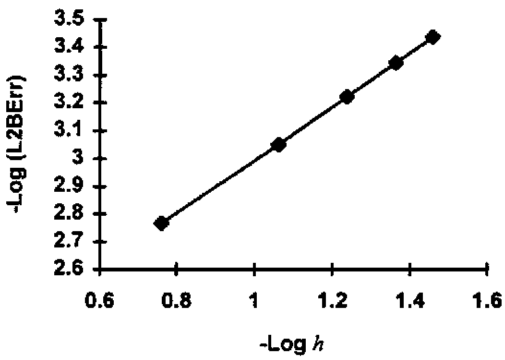


Fig. 3 - Results of Coaxial Cable with Edge Elements

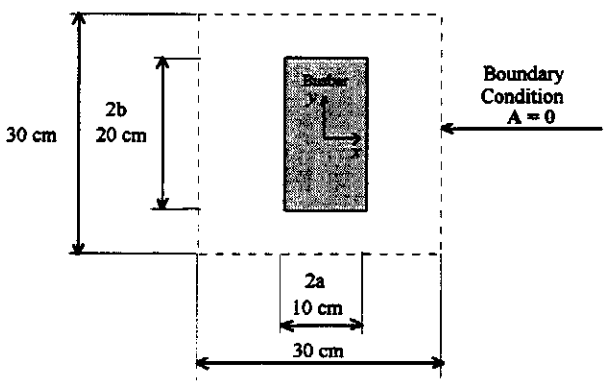


Fig. 4 - View of Magnetic Busbar

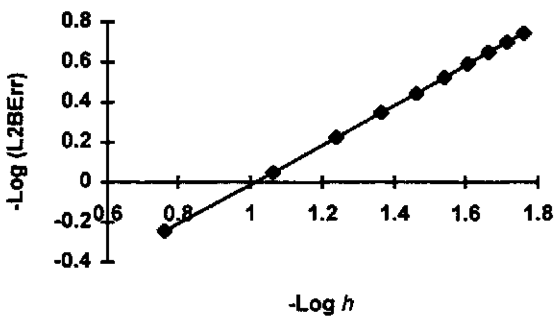


Fig. 5 - Results for Magnetic Busbar with Nodal Elements

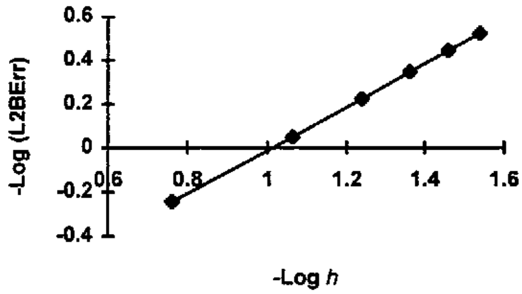


Fig. 6 - Results for Magnetic Busbar with Edge Elements

VI. CONCLUSIONS

The convergence rates for magnetic vector potential formulations using nodal and edge finite elements with one degree of freedom per edge (Nédélec Elements) and regular meshes were presented.

Theoretically, when nodal elements are used, the convergence rates are of order $\mathbf{O}(h^{k+1})$ for the magnetic vector potential **A** and $\mathbf{O}(h^{k})$ for the magnetic flux density **B**. When edge elements are used, the convergence rate is of order $\mathbf{O}(h^{k})$ for **A** and **B**.

Finally, some computational examples that confirm the theoretical results are presented.

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