Finite element modelling schemes for the design and analysis of electrical machines

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Abstract— **Electrical machines are complex objects. We have found that a variety of numerical techniques are required in order to model them using finite elements. We concentrate here on the different formulations which are useful in modelling such devices. Examples of modelling some of these machines using the MEGA package are described.**

Introduction

Many electrical machines and other electromagnetic devices can be difficult to model using finite elements. They can contain features such as magnetic nonlinearity, movement, geometric complexity and connection to an external circuit. Here we describe some features of a general purpose finite element package MEGA which allows some of the less pathological problems to be treated. First we review the formulations used in the finite element models.

Finite Element Formulations

The non conducting and conducting regions are modelled using the magnetic scalar potential, ψ, and the magnetic vector potential, $\mathbf{A}$, respectively. This approach leads to an economic description of the field problem.

Non Conducting Regions

Non conducting regions are modelled using magnetic scalar potentials, either the total scalar ψ ,defined as $\mathbf{H}_T = -\nabla\psi$, or the reduced scalar ϕ, defined as $\mathbf{H}_T = -\nabla\phi + \mathbf{H}_S$. Here $\mathbf{H}_T$ is the total magnetic field intensity and $\mathbf{H}_S$ is the field defined as $\nabla\times\mathbf{H}_S = \mathbf{J}_S$, where $\mathbf{J}_S$ is the source current density. The basic method outlined in [1] has been extended to allow voltage forced conditions [2], and to produce cuts for solving multiply connected problems. Both scalars give rise to a Laplacian type equation which has to be solved:

$$\nabla\cdot\mu\nabla\psi = 0 \tag{1}$$

Conducting Regions Including the Minkowski Transformation

Fields in conductors can be modelled using $\mathbf{A}$, the magnetic vector potential, and V, the electric scalar potential.

Using $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla V + \mathbf{u}\times\nabla\times\mathbf{A}$, where $\mathbf{u}$ is the material velocity, we obtain:

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{A} = \sigma\left(-\frac{\partial\mathbf{A}}{\partial t} + \mathbf{u}\times\nabla\times\mathbf{A} - \nabla V\right) \tag{2}$$

$$\nabla\cdot\sigma\left(\frac{\partial\mathbf{A}}{\partial t} - \mathbf{u}\times\nabla\times\mathbf{A} + \nabla V\right) = 0 \tag{3}$$

The term involving $\mathbf{u}\times\nabla\times\mathbf{A}$ in the above arises from the Minkowski transformation and is only valid if the moving media cross section normal to the direction of motion is invariant. Where appropriate [3], it is economical to dispense with V from the above set of equations. If we substitute $V = \mathbf{A}\cdot\mathbf{u}$ in the above formulation we have:

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{A} = \sigma\left(-\frac{\partial\mathbf{A}}{\partial t} - (\mathbf{u}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{u} - \mathbf{A}\times(\nabla\times\mathbf{u})\right) \tag{4}$$

Now a solution of (4) involving only $\mathbf{A}$ is required.

The uniqueness of $\mathbf{A}$ is fixed by using a penalty term to specify the divergence of $\mathbf{A}$ and forcing the normal component of $\mathbf{A}$ to be zero on the inside surface of conductors. The last two terms in (4) are non zero in the case of rotational velocity [4] but are zero in the case of translational velocity [5]. Equations (1) and (4) are solved using a Galerkin finite element scheme [6]. If nonlinear, the equations are solved using a Newton-Raphson scheme. This scheme can be very unstable if used directly. A modified scheme which uses line searches has been found to be necessary in the general case [7]. The terms involving $\mathbf{u}$ in (4) lead to numerical instability. This is alleviated by using an upwinding scheme [8]. The scheme here allows for a conductor moving at a velocity $\mathbf{u}$, and leads to an asymmetric global matrix which has to be solved. The non moving case is of course symmetric.

Electrostatic Problems

Electrostatics problems may sometimes be formulated in terms of the electric scalar potential V, so that

$$\mathbf{E} = -\nabla V \tag{5}$$

Since

$$\nabla \cdot D = \rho, \quad (6)$$

we obtain the Laplacian in V:

$$-\nabla \cdot \epsilon \nabla V = \rho \quad (7)$$

As usual this may be solved using a Galerkin technique and either 2D or 3D finite elements. After some manipulation, this results in, for a 3D system:

$$\int_{\Omega} \nabla N \cdot \epsilon \nabla V \, d\Omega - \int N \epsilon \frac{\partial V}{\partial n} \, dS = \int_{\Omega} N \rho \, d\Omega \quad (8)$$

Note that this formulation results in an exact enforcement of the $\mathbf{E} \times \mathbf{n}$ continuous condition and that the $\mathbf{D} \cdot \mathbf{n}$ condition is weakly correct.

Combining the field and circuit equations

Electrical machines are almost always connected to a fixed voltage supply or an electrical circuit such as an inverter. Unfortunately many finite element packages only allow constant current sources. Often a finite element model is used to derive a simpler equivalent circuit which may be connected to the external circuit.

In some situations it is not possible to use the finite element field model to derive an equivalent circuit. For example if the field equations are nonlinear then the equivalent circuit would have to be identified for all possible field states, which is not practical. It is possible in some circumstances to use a separate field and circuit model and iterate until the interface conditions are met. This may work if the field and circuit are loosely coupled but for the tightly coupled case a combined solution is attractive. Another advantage of the tightly coupled finite element/circuit model is that the user interface is easier to deal with, as the complexity of the situation is handled in the software. This means that as far as the designer is concerned, the computer model is conceptually very easy to visualise and hopefully very similar to just wiring up an experiment in the laboratory.

The general problem is to combine the field equations in terms of potentials with circuit node equations. In our scheme we use various field formulations for 2D and 3D. In the circuit problem we solve for nodal voltages using Kirchoff's current equations at each node. To couple the two models we must identify the voltage and current within the field equations. These can then be used directly in Kirchoff's circuit equations.

The field equations fall into two forms depending on the formulation,

- Current is the source term:

$$\begin{bmatrix} K & Q \\ Q^T & R \end{bmatrix} \begin{pmatrix} X_{field} \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} \quad (9)$$

- Voltage is the source term:

$$\begin{bmatrix} K & W \\ W^T & L \end{bmatrix} \begin{pmatrix} X_{field} \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ V \end{pmatrix} \quad (10)$$

Example of a 2D motor

Consider a 2D model with coils of a given turns density.

- Each wound coil has a known current distribution (but unknown value I_c).
- The *turns density* **t**defines the distribution of current.
- Then current density is

$$\mathbf{J} = I_{coil}\mathbf{t} \quad (11)$$

- The voltage across the terminals is found by integrating the back e.m.f.

$$V = l \int \mathbf{t} \cdot \dot{\mathbf{A}} \, dS \quad (12)$$

The equations to be solved are,

$$-\nabla \cdot \frac{1}{\mu} \nabla A_z + \sigma \frac{\partial A_z}{\partial t} - \mathbf{t} \mathbf{I}_{coil} = 0 \quad (13)$$

$$\int \sigma \mathbf{t} \frac{\partial A_z}{\partial t} \, dS - V = 0 \quad (14)$$

After applying the usual Galerkin procedure we get a set of equations that can be expressed in matrix form,

$$\begin{bmatrix} K & W \\ W^T & 0 \end{bmatrix} \begin{pmatrix} \mathbf{A} \\ I_c \end{pmatrix} = \begin{pmatrix} 0 \\ V \end{pmatrix} = 0 \quad (15)$$

If this is connected to ports A and B, we have:

$$\begin{bmatrix} K & W & 0 & 0 \\ W^T & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{A} \\ I_c \\ V_A \\ V_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_A \\ I_B \end{pmatrix} = 0 \quad (16)$$

3D model of a car alternator

The car alternator can be difficult to model because of its complex shape. Figure 1 shows a 3D finite element model of a typical claw-pole type car alternator. It has 12 rotor poles and 36 stator slots. It is difficult to model because the features of its rotor and stator are so different both circumferentially and axially that the finite element meshes at the interface between these objects will be totally incompatible. As a result, creating a sensible mesh in the air gap will be difficult.

To overcome this meshing difficulty, we create separate meshes for the stator and rotor and then bring them together to touch in the middle of the air gap. The two meshes are then coupled together by linking their potential variables on the interface using Lagrange multipliers. The main advantage of this approach is that the nodes of the two meshes do not need to be 'matched' on the interface. As a result, the individual meshes can be made to be as well-formed as possible.

This method is used to solve the FE model shown in Fig. 1. Due to symmetry, only one-twelveth of the whole alternator is modelled. The resulting vector plot of $\mathbf{B}$ near the tip of one of the rotor pole is shown in Fig. 2.

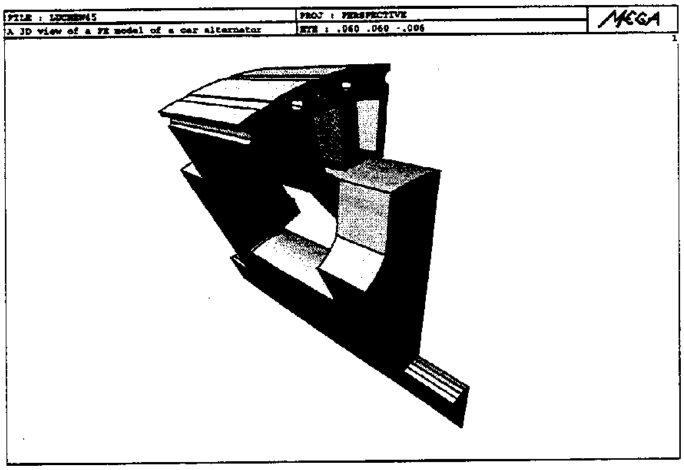

Fig. 1. A 3D view of a FE model of a car alternator

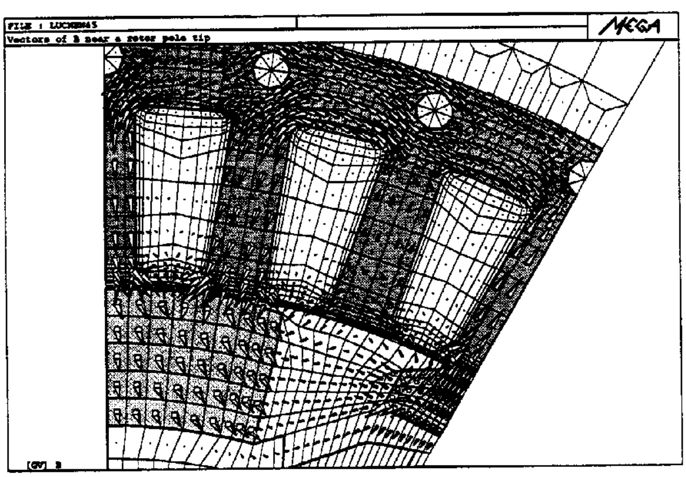

Fig. 2. Field vectors near the rotor pole tip

An Electrostatic Micro Machine

Often electrostatic devices move or rotate and finite element solutions would be required at many positions. This may be achieved using Lagrange multipliers in much the same way as for magnetics formulations [9]. If a region is split up into two meshes which have some common interface which will allow relative movement, such as a cylinder or a flat plane, equations such as (8) may be used in each mesh. These meshes would still be disconnected and the natural boundary condition would prevail on the common interface. The meshes can be joined using Lagrange multipliers. The condition that V is continuous may be enforced at the common interface using Lagrange multipliers. The Lagrange multipliers may be identified with $\frac{\partial V}{\partial n}$ so that, as before, the $\mathbf{D} \cdot \mathbf{n}$ continuous condition is weakly satisfied and the continuity of V and therefore $\mathbf{E} \times \mathbf{n}$ is correct in an average sense. The two meshes need not have the same mesh at the common interface, nor the same number of nodes.

Figure 3 shows a small electrostatic machine, diameter $15 \times 10^{-5} m$. The torque versus position curve is of some interest to designers of such machines. The Lagrange sliding interface method is used to solve the problem with the rotor in 12 different positions, as shown in Fig. 4. This is achieved with very little user effort.

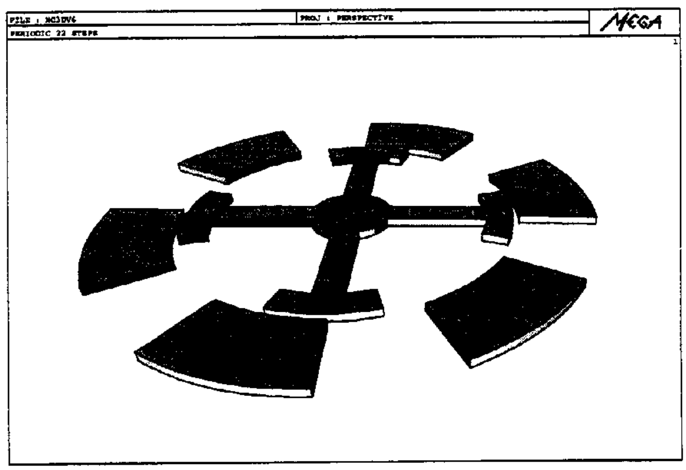

Fig. 3. A micro machine

Periodic boundary conditions

Often symmetry may be used to reduce the size of a finite element model. This section deals with another type of feature, periodicity. This is a function of the shape of the device and the state of the fields within it.

One of the earliest references to periodic boundary conditions may be found in [10], so the concept is well established. However, all of the published work up to now (as far as the authors know) deals with scalar variables. When solving 3D eddy current problems, vector variables are required, at least in conducting regions. These are slightly more complex and are described here.

If periodic boundary conditions exist on some parts of a device, a relationship between some potentials on boundaries is implied, of the following form:

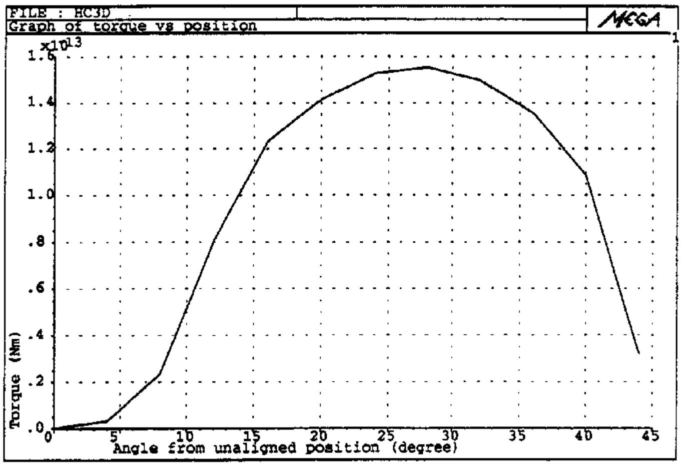

Fig. 4. Torque versus rotor angle for a micro machine

$$slavevariable = P * mastervariable \tag{17}$$

In this case the slave variables depend on the master variables. Only the master variables appear as degrees of freedom in the final set of equations. In the general case, for a vector variable at a periodic node, we can relate the slave variable to the master variable using a transformation matrix and a periodic parameter.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_{slave} = P \begin{bmatrix} l_{xx} & l_{xy} & l_{xz} \\ l_{yx} & l_{yy} & l_{yz} \\ l_{zx} & l_{zy} & l_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_{master} \tag{18}$$

$$\psi_{slave} = P\psi_{master} \tag{19}$$

The transformation matrix shown in (18) 'rotates' the slave vector so that the components of slave and master are aligned. This is more fully explained in [11]. The P in the above is the periodic parameter.

In general, the information required for establishing the periodic constraints in a typical electrical machine model are the location of the axis of rotational symmetry, a defined master-slave boundary, and the degrees of mechanical (M) and electrical (E) rotation. In the case of time transient or magnetostatic problems (P) the periodic parameter would normally be equal to cos E, where E would be 0 or 180 degrees. For linear time harmonic problems where complex numbers are used, P could be complex.

If the axis of mechanical symmetry is, for example, the z-axis, the potentials at the slave nodes are related to those of the master nodes in the following way:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_{slave} = \cos E \begin{bmatrix} \cos M & -\sin M & 0 \\ \sin M & \cos M & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_{master} \tag{20}$$

$$\psi_{slave} = (\cos E)\psi_{master} \tag{21}$$

A PERIODIC TEST PROBLEM

The device to be modelled is a purpose built test rig, having the basic configuration of a switched reluctance machine. All iron parts are solid so that we have a 3D nonlinear eddy current problem. A full model of the test rig is shown in Fig. 5. A detailed description is presented in [7]. The purpose of the experiment is to use the finite element method to simulate the behaviour of the time-transient torque when a step voltage is applied to the system, and to validate these predicted results against measurement.

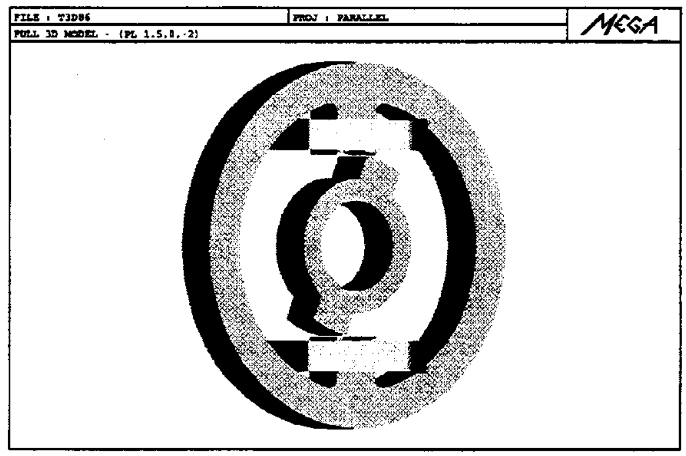

Fig. 5. Full model of the test rig

When using a full model of the test rig, which does not take advantage of the periodic boundary conditions, satisfactory correlation can be obtained between predicted and measured results. The discrepancy is less than 2% at a near-stabilized current. However, this problem is computationally expensive to solve, in both time and space. The file containing the results at each time step requires approximately 1.5 GBytes of disk space. In order to reduce the computational demands, periodicity constraints are implemented in the finite element software package, MEGA, and a new model, half the size, is constructed.

Only one half of the device width need be modelled for reason of symmetry. Further symmetry simplification is not possible because of the unaligned position of the rotor with respect to the stator pole. However a periodic model, shown in Fig. 6, can be constructed, containing only one of the coils and a periodic boundary.

The rig is excited from a constant voltage supply. A step voltage of 23.14V is applied to the coils which have a total resistance of 3.09Ω [7]. The coil currents are therefore unknowns in the system and must be calculated. This is carried out using the techniques described in the first section.

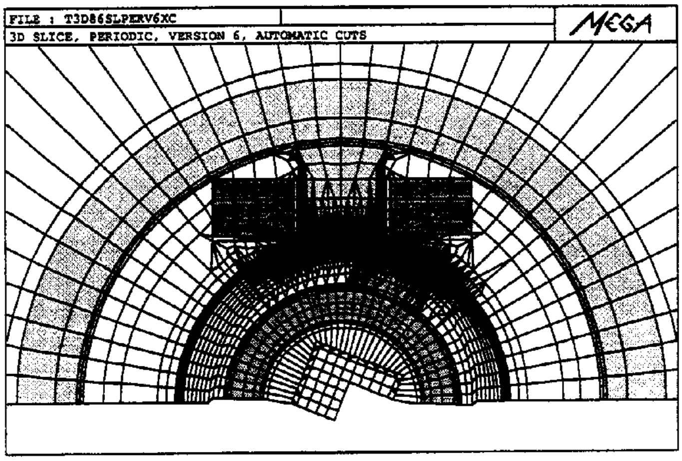

Fig. 6. A view of the periodic model mesh

Results

A preliminary validation of the periodic model is carried out using a 1mm-thick slice of the original model. The periodic and non periodic 3D slice models yield identical torque curves, as may be observed in Fig. 7. However, since the periodic model contains only half the number of nodes compared to the non periodic one, the answer file containing all the time-step results is also decreased by the same factor (35 MBytes vs 70MBytes). The solving time on a DEC ALPHA model 3000 workstation has reduced from 6 hours to 2.4 hours, a saving of approximately 60%.

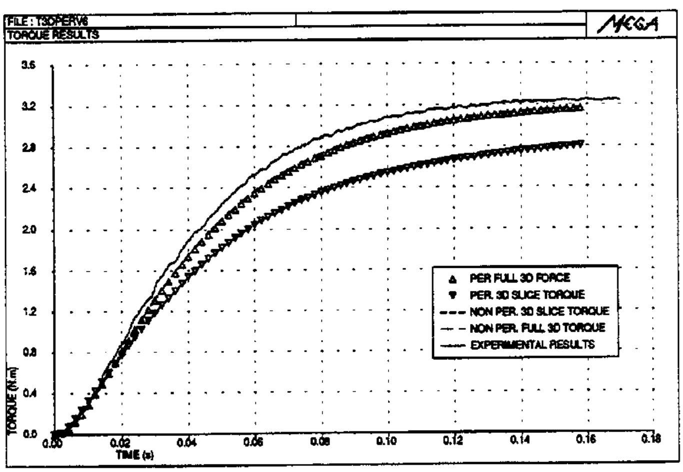

Fig. 7. Comparison of measured and calculated torque

Full periodic and nonperiodic models were then constructed. Both of these torque curves agree very well with each other, as is shown in Fig. 7. The agreement with experimental results is also quite good. At 0.16s, the last computed time shown on the curve, the full 3D model predictions agree with experimental results to within 3%. Results from the slice models, which are essentially 2D, are of course less accurate. Here the agreement with measurement is approximately 14% at 0.16s.

Conclusions

Some of the techniques which can help in modelling electrical machines using finite elements have been presented. The sliding interface technique allows a machine to rotate in a realistic manner, while connected to an external circuit. The use of periodic boundary conditions can sometimes yield a more economic solution. Despite many recent developments around the world, electrical machines still present some difficult challenges for the modeller.

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