# MODIFIED REFLECTIVE SYMMETRIES FOR IMPROVING THE EFFICIENCY OF THE MMP CODES

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*ABSTRACT. A more general concept for the treatment of symmetry in the 3D MMP codes allows considerable gains in usability. Up to now, symmetry decomposition of functions has been made with functions that are continuous across the symmetry plane. Now discontinuous symmetrizations are added. Possible continuity conditions in the symmetry plane can be enforced explicitly with matching points. Not only are problem sizes reduced, but the creation of good models in the MMP sense is considerably simplified in some common applications. The improved approach also illustrates some of the "classical" methods for calculating obstacles in waveguides.*

## 1. INTRODUCTION

The MMP codes (Multiple Multi Pole) are an implementation of the Generalized Multipole Technique (GMT) for computation of time harmonic fields in piecewise homogeneous, linear and isotropic regions. In each domain the field is approximated by a linear combination of solutions of the Helmholtz equation, mostly multipoles. The field expansions are matched in points on the boundary between the domains, leading the field problem back to the solution of an overdetermined, linear system of equations. The incident field $f^{inc}$ (excitation) as part of the total field in the solution

$$f^{tot} = f^{inc} + f^{sc}$$

is included in the equation system with the only fixed parameter. All other parameters of the expansions modeling the scattered field $f^{sc}$ will be determined in the solution related to the excitation. For a more detailed description see Reference [1].

## 2. BOUNDARY VALUE PROBLEMS AND SYMMETRY

The geometry of boundary value problems is often *symmetric*, i.e. invariant to certain symmetry transformations in space. Exploiting these symmetries pays off in a considerable reduction of computational time and memory requirements and, furthermore, brings numerical advantages. Symmetries are also an easy way of introducing perfectly conducting infinite planes into a model.

The mathematical tool for treatment of symmetries is group theory. A complete and understandable description of its application to boundary value and eigenvalue problems is given in [2, 3, 4, 5, 6]; in this section only the consequences are outlined.

A linear symmetry transformation in $n$-dimensional space is represented by an $n$ by $n$ matrix $D$. The various symmetry transformations which are applicable to a given geometry can be seen as representations of the elements $s$ of a group $G$. They are consequently written as $D(s)$. A matrix $A$ *has the symmetry of the group G* if

$$D(s)A = AD(s) \qquad \forall s \in G. \tag{1}$$

For the matrix equation in a boundary value problem

$$Ac = b$$

Equation (1) means that a symmetry transformation of the boundary values $b$ is equivalent to the same transformation of the result $c$.

Group theory shows that a transformation $T$ can be found which transforms $A$ into a block diagonal matrix $\widetilde{A}$ according to

$$\widetilde{A} = T^{-1}AT. \tag{2}$$

The number and the size of the blocks is determined by the group $G$. The transformations $T$ are tabulated for the most common symmetry groups.

As a consequence, the boundary value problem splits up into $K$ smaller *symmetry adapted* ones, which can be solved separately at a lower overall expense. Note that it is not necessary for the boundary conditions to be symmetric.

$$Ac = b \quad \rightarrow \quad A_k c_k = b_k \quad (k = 1, \ldots, K) \tag{3}$$

The $A_k$ usually have a better condition number and are in any case numerically better due to their lower dimension. An additional reduction of problem size takes place if some symmetry components of the excitation $b_k$ of $b$ are equal to zero. Therefore the corresponding part of the problem has only the trivial solution $c_k = 0$.

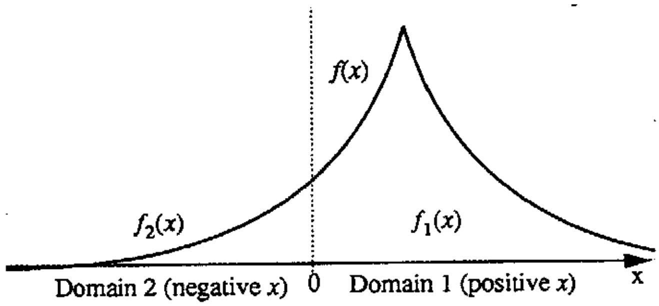


Figure 1: The scalar function $f(x)$ is defined for positive $x$ as $f_1(x)$ and for negative $x$ as $f_2(x)$.

In homogeneous boundary value problems (eigenvalue problems) splitting the problem into symmetry adapted ones helps to separate modes and, in particular, degenerate solutions.

Instead of explicitly performing the transformation $T$ on $A$, it is also possible to set up the matrices $A_k$ directly using symmetry adapted expansion functions.

## 3. REFLECTIVE SYMMETRIES IN THE 3D MMP CODE

In the 3D MMP code one or more reflective symmetries about the coordinate planes $X = 0$, $Y = 0$ and $Z = 0$ of the global coordinate system are implemented. Due to the orthogonality of these planes, the consideration of symmetries can be made quite intuitively for one plane of symmetry at a time.

With respect to one symmetry plane a scalar field $f$ can be split up into an *even* and an *odd* component $f^+$ and $f^-$ from which the complete function can be reconstituted as

$$f = f^+ + f^-. \tag{4}$$

For actually decomposing a function, the symmetrizations

$$f^+(x) = \frac{1}{2}\left[f(+x) + f(-x)\right] \tag{5a}$$

$$f^-(x) = \frac{1}{2}\left[f(+x) - f(-x)\right]. \tag{5b}$$

are usually used, where + and − denote the even and odd components, $+x$ and $-x$ denote the coordinates perpendicular to the plane. The advantage of the relations (5) is that they yield components $f^+$ and $f^-$ which are *continuous* at the symmetry plane at $x = 0$.

It is also possible to obtain the even or odd field in the reflected domain by extending a function defined only in the principal domain into the reflected domain by

$$f^+(x) = \begin{cases} \frac{1}{2}f(x), & \text{for } x > 0 \\ \frac{1}{2}f(-x), & \text{for } x < 0 \end{cases} \tag{6a}$$

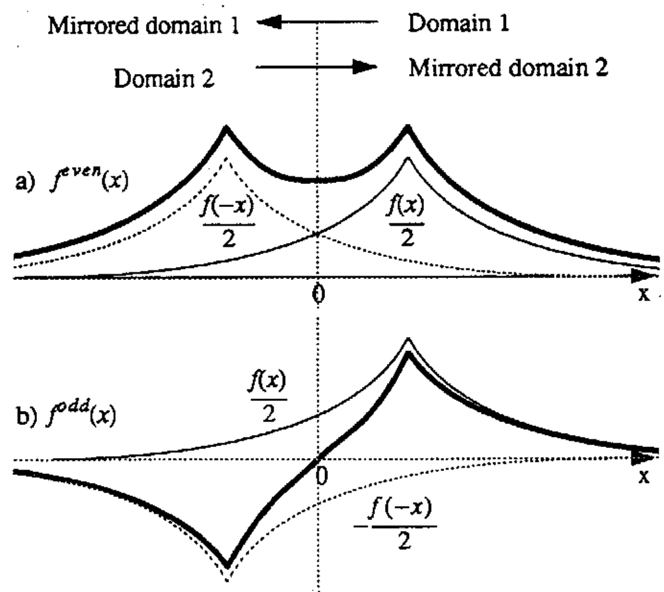


Figure 2: a) and b) show the even and odd symmetric functions decomposed from $f(x)$ according to Equation 5.

$$f^-(x) = \begin{cases} \frac{1}{2}f(x), & \text{for } x > 0 \\ \frac{1}{2}f(-x), & \text{for } x < 0 \end{cases}. \tag{6b}$$

These components are *not* continuous at $x = 0$. This, however, is of minor importance for domains which are not intersected by the symmetry plane. The effect of the decomposition according to 6 is shown in Figure 3.

In the 3D MMP code [1] no difference is made between domains which are intersected or not, therefore (5) is always used.

For electromagnetic vector fields, additional dependencies between the components have to be taken into account. Because Maxwell's equations are invariant with respect to reflections about a plane and due to the dualitz of the electric and magnetic field, the components must transform in one of the two ways:

$$\begin{aligned} E_\perp(+x) &= -E_\perp(-x) \\ H_\perp(+x) &= H_\perp(-x) \\ E_\parallel(+x) &= E_\parallel(-x) \\ H_\parallel(+x) &= -H_\parallel(-x) \end{aligned} \tag{7}$$

or like

$$\begin{aligned} E_\perp(+x) &= E_\perp(-x) \\ H_\perp(+x) &= -H_\perp(-x) \\ E_\parallel(+x) &= -E_\parallel(-x) \\ H_\parallel(+x) &= H_\parallel(-x) \end{aligned} \tag{8}$$

. The index $\perp$ stands for the component of a vector perpendicular to the plane, $\parallel$ for the one parallel to the plane. For the 3D MMP code, an electromagnetic field

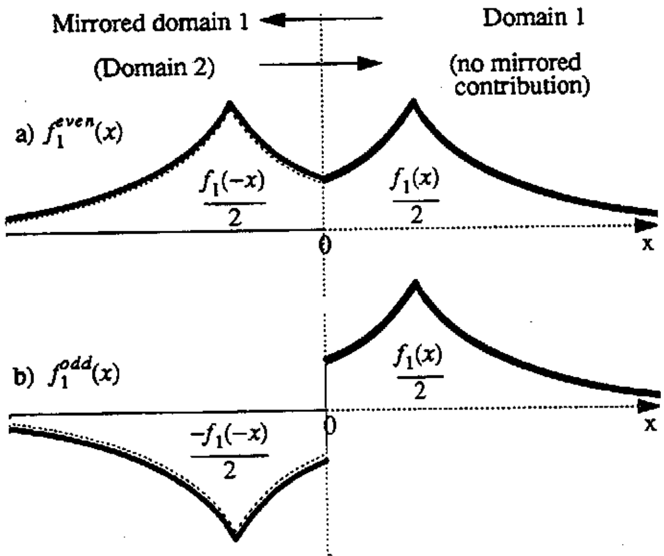


Figure 3: a) and b) are the even and odd symmetric functions decomposed from $f(x)$ according to Equation 6. Only the part $f_1$ defined in domain 1 is shown here. For $f_2$ in domain 2, an equivalent construction is made.

is defined to be even about a plane if (7) is true and odd if (8) holds. On the symmetry plane itself for an even field

$$E_{\perp} = 0 \quad \text{and} \quad H_{\parallel} = 0 \tag{9}$$

are valid, and for an odd field

$$E_{\parallel} = 0 \quad \text{and} \quad H_{\perp} = 0 \tag{10}$$

are valid.

As a result, the problem

$$Ac = b$$

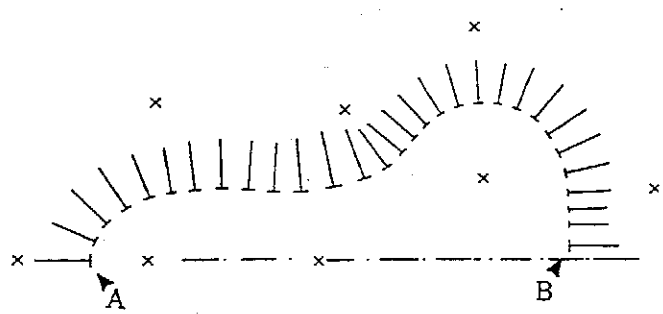


Figure 4: MMP model of a symmetric boundary value problem with one plane of symmetry. Note the different discretization in A and B. With a decomposition according to 5 it is never necessary to put matching points onto the symmetry plane.

with respect to one reflective plane splits up into an even and an odd problem

$$A^+c^+ = b^+ \quad \text{and} \quad A^-c^- = b^-.$$

$A^+$ and $A^-$ can be set up directly and each has only approximately half the number of parameters and equations of $A$. The field is evaluated by summing the symmetry components

$$f = \sum_{i=1}^{N^+ +1} c_i^+ f_i^+ + \sum_{i=1}^{N^- +1} c_i^- f_i^- . \tag{11}$$

The above considerations can be superposed for three orthogonal planes of symmetry. Both the inhomogeneity of the problem and the solution split up into eight symmetry adapted components

$$\begin{aligned} f = & f^{+++} + f^{++-} + f^{+-+} + f^{+--} + \\ & f^{-++} + f^{-+-} + f^{--+} + f^{---} \end{aligned} \tag{12}$$

$$\begin{aligned} b = & b^{+++} + b^{++-} + b^{+-+} + b^{+--} + \\ & b^{-++} + b^{-+-} + b^{--+} + b^{---} \end{aligned} \tag{13}$$

and the whole problem divides into the eight parts

$$\begin{aligned} A^{+++}c^{+++} &= b^{+++} \\ A^{++-}c^{++-} &= b^{++-} \\ A^{+-+}c^{+-+} &= b^{+-+} \\ A^{+--}c^{+--} &= b^{+--} \\ A^{-++}c^{-++} &= b^{-++} \\ A^{-+-}c^{-+-} &= b^{-+-} \\ A^{--+}c^{--+} &= b^{--+} \\ A^{---}c^{---} &= b^{---}. \end{aligned} \tag{14}$$

Each of these symmetry adapted matrices has only one eighth of the number of rows and columns of the original problem.

To discuss the impact of the symmetry on the problem we assume a symmetric MMP model (Figure 4).

A nonsymmetric *expansion* can be symmetrized with (5a) and (5b). An expansion and its reflected counterpart produce the same symmetrized functions, so only one of them has to be considered. An expansion with origin on the symmetry plane needs special treatment. If it is properly oriented, it may already be even or odd about the plane and consequently adapted to one of the symmetries. In this case it is important that it enters only either $A^+$ or $A^-$ respectively. In the wrong matrix it will produce a zero column, and the matrix will become singular. Each plane of symmetry in which a multipole or normal expansion lies leads to a reduction of the number of its parameters by a factor of 2. But if the expansion is not properly oriented, all functions are symmetrized, and the expansion will enter both $A^+$ and $A^-$ with the full number of coefficients. Because normal expansions must not be used multiply, their origin has always to be on the intersection of all symmetry planes.

The *matching points* can be treated by analogy. In a point and its reflected counterpart, the values of a symmetric field are identical except for a possible change of sign. This results in the same equations for both points and consequently only one has to be considered. In a point on a symmetry plane several of the components are zero because of (9) and (10), and the symmetrization of the remaining components is simplified. Note that it is usually not necessary to put a matching point on a symmetry plane (cf. Figure 4); the exception will be discussed below. Although zero rows in an overdetermined system of equations do not cause much harm, they waste computing time during updating. 3D MMP therefore correctly eliminates these trivial equations from the system of equations.

*Field points* are in contrast to the matching points often lying on symmetry planes. The reason is that the field is easier to interpret there due to the reduced number of non-zero components. However, the consideration of these symmetries does not save much computational time, as evaluation of the field is not very costly.

The choice of the *principal domain*, i.e., the side on which the expansions and matching points are considered, is not unique. In the MMP programs it is generally assumed to be the intersection of the positive half-spaces (i.e., the side on which the coordinates are positive) of the symmetry planes involved. Having expansions on the other side of a symmetry plane can result in a change of sign of the corresponding matrix columns and, as a consequence, of the parameters. Having matching points on the other side of the symmetry plane can affect only the sign of the corresponding rows and therefore does not influence the solution.

## 4. LIMITATIONS

The treatment of symmetries with the decomposition (5), however, fails to work in some important cases. Examples are the cases of waveguides with discontinuities or openings in infinite walls. This does not only lead to an increased problem size, i.e. increased number of rows and equations in the system matrix, but also to important restrictions in modeling due to the rules for pole setting. The models for the two examples are shown in Figures 5 and 6, respectively.

The main problem is that if a symmetry plane is introduced then applying Equation (5) causes the symmetrized functions of the excitation $A$ and of a scattered wave $T$ on the other side of the symmetry plane to become the same. Therefore, the scattering problem is

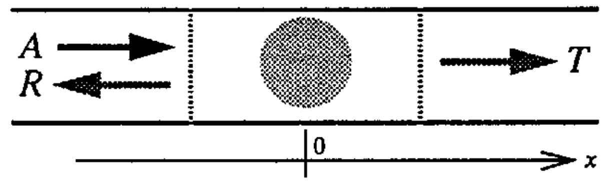


Figure 5: This waveguide with a symmetric obstacle in the center is divided into four regions: the two waveguide ends with their modes (Excitation $A$, Reflection $R$ and Transmission $T$) and the center region with expansions for the scattered field in and around the scatterer.

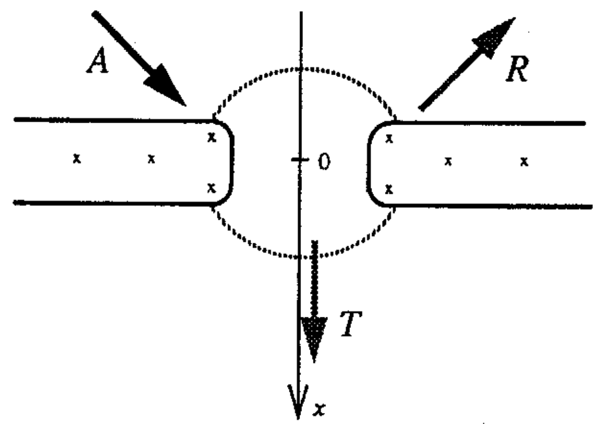


Figure 6: Simplified model of an opening in an infinite plane without consideration of symmetries. The scattered field $T$ contains only scattered components. The relevant multipole expansions around the edges have to be positioned within the infinite wall (indicated with x). Using symmetries, the incident and reflected waves $A$ and $R$ are mirrored to the positive $x$ side below the opening, but must vanish when superposed. Details of the modeling of apertures to get good results are presented in Reference [7].

immediately solved for both symmetry parts by exactly cancelling the excitation with the decomposed scattered wave, resulting in the trivial solution with zero overall field.

## 5. IMPROVED SYMMETRIES

This problem can be solved, however, by symmetrizing the field functions according to Equation (6). This is never a problem within domains which are not intersected by the symmetry plane; the question arises, however, as how to proceed if this should be the case. The solution is to introduce matching points *on* the symmetry plane in order to match the discontinuous components of both sides and thus render them continuous, i.e., enforce

(9) and (10). There is no restriction on the remaining components of the field; A continuous field will result, although the derivative will not be continuous for single components across the symmetry plane. As these new matching points are treated in the same way as the other matching points, a residual error on the symmetry plane is allowed. In essence, the procedure is equivalent to introducing a fictitious boundary on the symmetry plane and — strictly speaking — cutting the symmetry plane out of the considered domain, thus removing the problem. The residual error of matching points on the symmetry plane can be reduced with respect to the error on ordinary matching points by weighting the respective equations accordingly.

Not only is the problem reduced to two smaller, symmetric problems, but in addition the modeling becomes considerably easier for a number of problems, e.g., openings in screens or obstacles in waveguides. However, the symmetric problem parts are now in principle larger than half of the unsymmetric problem, since matching points (and therefore equations) are necessary in the symmetry plane. The problem therefore becomes a ***truly one-sided problem***, the result of which is "mirrored" on the other side. Multipoles, for example, can now be placed without the restriction that the mirror image of the expansion also has to be kept out of the domain. They can be put into the mirrored domain.

In the case of ordinary symmetries with continuous functions across the symmetry plane an expansion outside the symmetry planes is complemented by another expansion in the mirrored position. For a multipole close to the symmetry plane the heuristic rules for dependence between the multipole and its mirrored image have to be obeyed, i.e. the multipole must not be placed too close to the symmetry plane. Alternatively, it may be placed on the symmetry plane itself, where the pole and its image become equivalent and only the symmetry conforming components may be used.

On the other hand, symmetrized functions defined according to Equation (6) are evaluated in the principal domain and just reflected to the other side; there is no further interaction between the two sides (Figure 7).

The physical interpretation of the boundary conditions which have to be met on the symmetry plane is quite interesting: In the case of even symmetry, they are equivalent to a ***perfectly conducting magnetic wall***, in the case of odd symmetry to a ***perfectly conducting electric wall***. This provides a connection with "classical" methods for computing obstacles in waveguides.

## 6. IMPLEMENTATION

The basis for the following discussion is the program [1].

To accommodate the additional treatment of symmetries, the implementation of the symmetry decomposition within the 3D MMP main program has to be slightly changed.

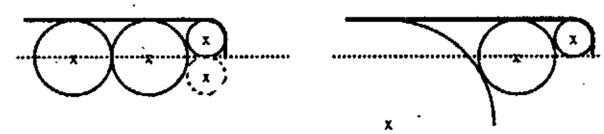

Figure 7: Dependence between multipole and its mirror image, and non-dependence for multipoles symmetrized according to Equation 6. The "forbidden" regions are indicated by the spheres. The positions of the poles are no longer limited to the inside of the conducting plane.

It is necessary to add symmetry data to domains or expansions (or both), in order to decide which of the symmetry decompositions (5) or (6) shall be used across each symmetry plane.

The symmetry decomposition (5) was originally obtained by adding up the field components of a fixed expansion in all ***mirror images of the field point***, unless the expansion was known to be symmetry conforming. This is in principle equivalent to adding up the contributions of all the ***mirror images of the expansion*** in a fixed point (Figure 10). Also, as has been mentioned above, it has up to now not been important whether an expansion or field point in the model were actually located in the principal domain or at one of the mirror images, as all the mirror images were used in the symmetrization. In the symmetry decomposition (6) the distinction between the field in the principal domain and its mirror images is more explicit. Because of this one-sidedness, the first way in which the field points are mirrored is better suited for the implementation.

An expansion for the principal domain can be inside or outside the principal domain. Other than in (5), its mirror image will not be used for the principal domain anymore. For evaluation of the field on the ***negative side*** of a symmetry plane, i.e., outside the principal domain, the field is always evaluated in the corresponding point on the positive side, i.e., in the principal domain, and subsequently "mirrored" back to the negative side, according to equations (8) and (7). For the field in a ***symmetry plane***, the definition range of the symmetrized functions (6a) and (6b) on one side of the symmetry plane can be extended to include the symmetry plane itself, e.g., from $x > 0$ to $x \geq 0$.

The elimination of superfluous (due to symmetry considerations) components of the boundary conditions in the matching points is already present in the code [1]. Therefore, simple matching points with the *full* boundary conditions, i.e., for all components, may be used to


Figure 8: Model of an obstacle in a two-dimensional waveguide without and with symmetries. In region a), an additional wall is introduced. Boundary b) is only needed in certain cases.

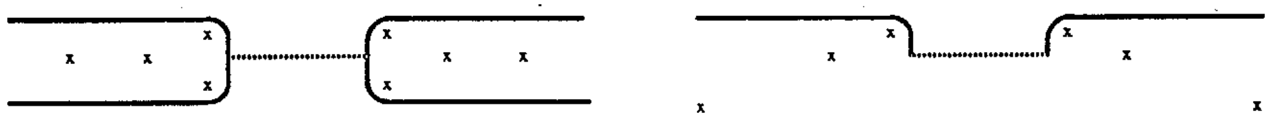

Figure 9: Model of an opening in an infinite plane without and with symmetries. For better visualization, the horizontal symmetry is not exploited.

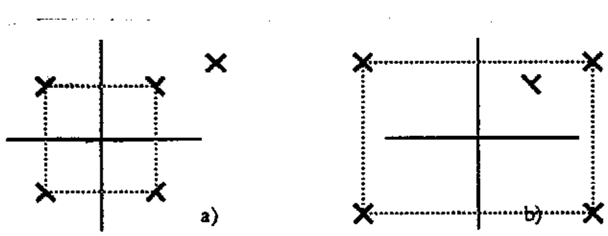


Figure 10: Two alternative but equivalent ways of performing the symmetry decomposition (5). In a), the mirror images of the points are used, in b) the mirror images of the expansions.

enforce the symmetry conditions in the symmetry plane; the necessary components are automatically selected. This eliminates the need for separate models with different boundary conditions for different symmetry components of the problem.

This internal change in the 3D MMP code still gives correct behavior for models with "conventional" symmetries even if expansions or matching points are not in the principal domain; it is therefore completely compatible with older versions.

## 7. CONCLUSION

An additional kind of symmetry decomposition for the field functions in the MMP code has been introduced. This complements the existing symmetries in the MMP codes and allows the use of symmetries for problems which could not be treated with the existing implementation. The functions resulting from the new symmetry decomposition are in principle discontinuous across symmetry planes. Therefore, additional matching points have to be introduced on the symmetry planes in order to satisfy the continuity of the odd components of the field function. This is equivalent to introducing a wall on the symmetry plane which represents an ideal electrical conductor in the odd case and a magnetic one in the even case. Not only is the total amount of computation reduced, but due to the one-sided nature of the model the user is much freer in the placing of expansions.

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