SOLUTION OF LAPLACE'S EQUATION USING MULTIPLE PATHS METHOD (MPM)

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ABSTRACT

A probabilistic method called the multiple paths method (MPM) is presented to solve the potential equation. Unlike other probabilistic techniques there is no need for the generation of a grid resulting in less computation time and simplicity. The method is based on the calculation of the probability of absorption of a particle at a point on the boundary. The random walk is along a "path" which is a line passing through the starting point, where the potential is to be determined. The results show good agreement with other probabilistic methods and numerical techniques.

Keywords: Probabilistic potential theory, probabilistic methods, Multiple paths method, Laplace's equation, Electrostatics

1 INTRODUCTION

Numerical methods are frequently employed in solving electromagnetic problems [1-4]. Other techniques include probabilistic methods such as Monte Carlo (MCM) [5,6,7] and the exodus method [8]. In this work a new method called the multiple paths method (MPM) based on stochastic processes is presented to solve the potential equation in a two dimensional region. Unlike other probabilistic techniques the generation of a grid is not needed resulting in a less computation time and greater simplicity, particularly in the case of irregular regions. Consider Laplace's equation:

$$\nabla^2 V = 0 \text{ with } V = V_b \text{ on the boundary} \qquad (1)$$

In probabilistic methods the evaluation of the potential, $V(x_0, y_0)$ at a point (x_0, y_0) in the region is based on the following equation:

$$V(x_0, y_0) = \sum_{k=1}^{m} p_k V_p(k) \qquad (2)$$

where p_k is the probability of absorption of a random walking particle at point K on the boundary, $V_p(k)$ is the prescribed potential at that point and m is the number of trial points. The accuracy of the potential increases with an increasing m. In this equation $V_p(k)$ is known and only p_k is to be determined. In the present method different lines called "paths" are drawn passing through the point (x_0, y_0) whose potential $V(x_0, y_0)$ is to be determined and the random walk is considered along these paths. The complexities arising from the irregularities of the boundary are not present in this method. The method is applied to regions of uniform permittivity as well as different dielectric constants. Since there is no need to generate a grid for a random walk, the evaluation of the potential requires much less computation time.

2 MULTIPLE PATHS METHOD (MPM)

Consider Laplace's equation, (1) with Dirichlet boundary conditions in which the potential, $V(x_0, y_0)$ at (x_0, y_0) is to be determined. In order to determine p_k in (2) an arbitrary point in the region is chosen randomly. The line passing through (x_0, y_0) and this point intersects the boundary at points K and K' with the prescribed potential $V_p(k)$ and $V_p(k')$ respectively and KK' which includes the point (x_0, y_0) is called a path. We divide the path into n segments with equally spaced stations one of which is

located at (x_0, y_0). Figure 1 shows such a segmentation. The length of KK' is denoted by A while B is the distance of K' to the point (x_0, y_0). If B is divided into m stations, we have $m = B/d$ and therefore;

$$n = \left(\frac{A}{B}\right) \times m \tag{3}$$

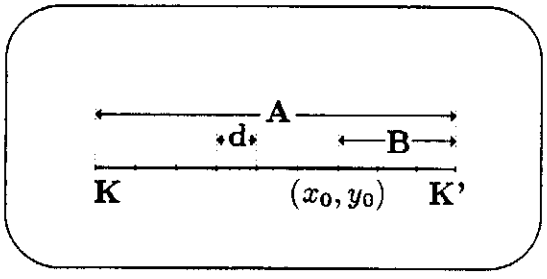

Fig. 1: A Segmented path with a separation 'd' between two neighboring stations.

We consider two cases; (I): n is an integer which is the ideal case and (II): n is not an integer, in which case we can either change m and continue the process until case (I) is reached or we can round off n to an integer. In the latter case we will introduce an error which is not significant for small segments corresponding to a large n. We will show, however, that for homogeneous regions the process is independent of choosing m and such a consideration for m and n is irrelevant. Now consider a particle at (x_0, y_0) for this path and let the probabilities of absorption of the particle at K and K' be p_k and $p_{k'}$ respectively. The problem is now one of a random walk on the specified path (Fig. 2). The probability of absorption to state c when we start from state i (transition state) is denoted by $\prod_i(c)$ and the probability of absorption to state c starting from state i, after n walks, is denoted by $\prod_i^n(c)$ Hence,

$$\prod_i(c) = \sum_{n=1}^{\infty} \prod_i^n(c) \tag{4}$$

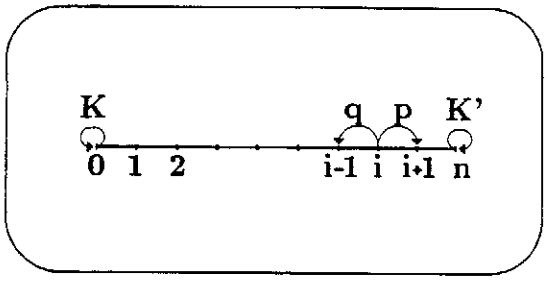

Fig. 2: A path KK' showing the motion of a particle from one state to the next state.

For the probability of absorption after n walks, employing mathematical induction, it can be easily shown that

$$\prod_i^n(c) = \sum_{j \in T} p_{ij} \prod_j^{n-1}(c) \;\; ; \;\; \forall n > 1, i \in T \tag{5}$$

where T is the state space. From system of equations (5), we deduce that

$$\prod_i(c) = \prod_i^1(c) + \sum_{j \in T} p_{ij} \prod_j(c) \;\; ; \;\; \forall n > 1, i \in T \tag{6}$$

Since $\prod$ is a probability function; $\prod_j \geq 0$ and $\prod_i(c)$ is the unique, bounded, solution of system of equations (6). P_{ij}'s are the elements of the state transition matrix [9]. In view of the system of equations (6) and Fig. 2, we have, $T = 1, 2, \ldots, n-1$ and the transition matrix, $\mathbf{P}$, is given by:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ q & 0 & p & \cdots & \cdots & 0 \\ 0 & q & 0 & p & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{7}$$

in which, $p + q = 1$.

Consider Fig. 2 and suppose that u_i is the probability of starting from state i and ending at state zero (point K) and similarly v_i is the probability of starting at state i and ending at state n (point K'). Hence $u_i = \prod_i(0)$, and $v_i = \prod_i(n)$. From the system of equations (6) we will have:

$$\begin{aligned} u_1 &= q + pu_2 \\ u_2 &= qu_1 + pu_3 \\ &\vdots \\ u_j &= qu_{j-1} + pu_{j+1} \\ &\vdots \\ u_{n-1} &= qu_{n-2} \end{aligned}$$

and we will finally deduce, for the probability u_r, [9]:

$$u_r = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^r}{\left(\frac{q}{p}\right)^n - 1} \;\; ; \;\; p \neq q \tag{8}$$

and

$$u_r = \frac{n - r}{n} \;\; ; \;\; p = q = 1/2 \tag{9}$$

By a similar procedure v_r can be determined which finally yields:

$$v_r = 1 - u_r \tag{10}$$

Therefore the probability of absorption at each boundary point of the region can be determined from (8), (9) and (10) accordingly. In view of these equations, the computation time reduces by a factor of two since determination of u_k provides v_k simply by (10) and no further computation is needed. The potential $V_1(x_0, y_0)$, based on the first path is therefore given by:

$$V_1(x_0, y_0) = u_i(0)V_p(K) + v_i(n)V_p(K')$$

or equivalently,

$$V_1(x_0, y_0) = [V_p(K) - V_p(K')]\, u_i + V_p(K') \tag{11}$$

where $V_1(x_0, y_0)$ is the potential at (x_0, y_0) obtained from path (1) and $V_p(K)$ and $V_p(K')$ are the potentials at K and K' on the boundary respectively.

Equation (11) can repeatedly be used to find the potential at (x_0, y_0) for different paths(Fig. 3). If m paths are selected and $V_s(x_0, y_0)$ is the potential obtained at (x_0, y_0) for path s, then for the potential, $V(x_0, y_0)$, at (x_0, y_0), [5]:

$$V(x_0, y_0) = \frac{\sum_{s=1}^{m} V_s(x_0, y_0)}{m} \tag{12}$$

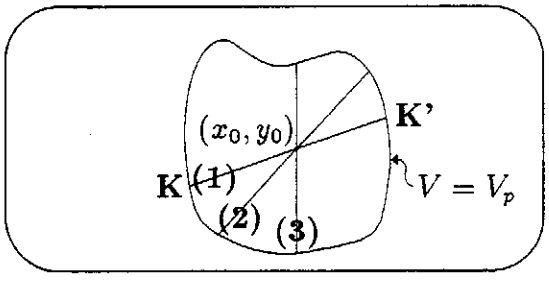

Fig. 3: Some paths used for calculating the potential at the point of interest, (x_0, y_0).

Consider Fig. 1 where the length of KK' is A and $m = B/d$ with each segment having a length 'd'. Then for the total number of segments in the path , n, equation (3) yields: $n = A/d$ and since for homogeneous regions, $p = q = 1/2$, from (9) and (10) we obtain

$$u_r = \frac{B}{A} \quad \text{and} \quad v_r = 1 - \frac{B}{A} \tag{13}$$

In this case a solution will clearly be reached quickly and accurately since u_r and v_r depend on A and B and not the segmentation of the path. As was indicated, the generation of a grid in the region is not needed in this method. This is particularly useful for regions with irregular boundaries.

3 RESULTS

In probabilistic methods such as the Monte Carlo method with either fixed or random walk, or the exodus method, the region is subdivided into a grid with square cells and the probability of absorption of a random walking particle to a boundary point, u_r, is considered. For each trial particle, the calculation of this probability needs considerable time since we have a two dimensional grid with the random walking particle along the sides of the square cells in the grid which results in an increasingly higher computation time for the calculation of the probability. In the MPM however, each single path yields two probabilities associated with the end points of the path simply by equation (13) without further computation. This feature of the method along with the elimination of the grid considerably reduces the computation time.

For a better understanding of the method, consider Laplace's equation within a normalized 1 by 1 square with zero potentials on each side except on the upper boundary which it is assumed to be 100 volts. The results are tabulated in Table (I) for three points and different numbers of paths, N. Close agreement is observed with the exact solution with an error less than 5 percent. The potential is also plotted for different numbers of paths, N, when N is varied in steps of 10 (Fig. 4). The smooth variations of potential vs. N shows the convergence of the solution.

Table(I): Comparison of the MPM with the Exact Solution in a 1 by 1 Square

(x,y)	N	MPM	N	MCM	Exact
(0.25,0.75)	50	44.1	500	41.8	43.2
	60	44.25	1000	41.10	
	70	42.75	1500	42.48	
	80	43.73	2000	43.35	
(0.5,0.5)	50	24.00	500	23.6	25.00
	60	25.00	1000	25.8	
	70	25.71	1500	25.27	
	80	25.00	2000	25.1	
(0.75,0.25)	30	5.83	500	6.6	6.80
	40	6.25	1000	7.5	
	50	6.00	1500	7.6	
	60	7.08	2000	7.3	

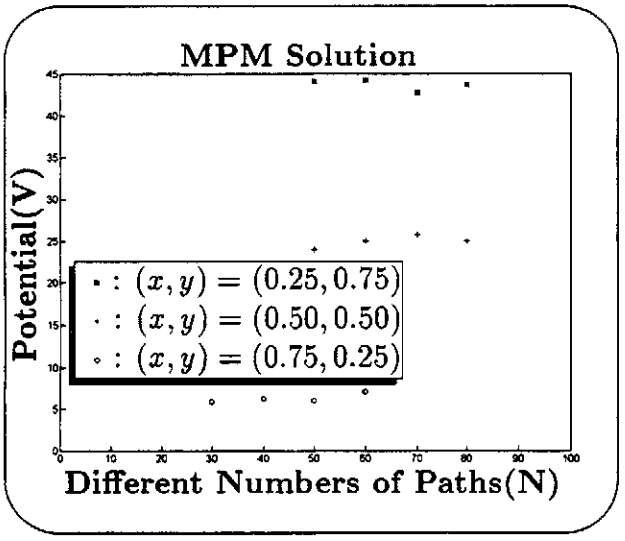

Fig. 4: Potential, V, at three points for different numbers of paths, N.

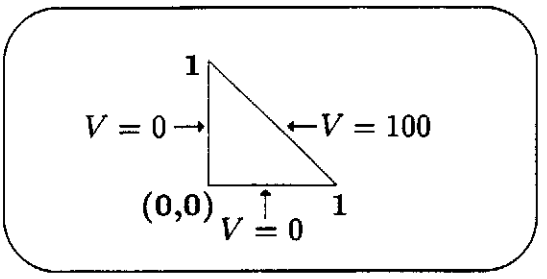

Fig. 5: The triangular region under consideration.

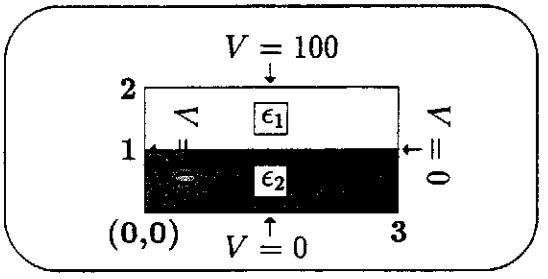

Fig. 6: A 3 by 2 region with two dielectrics.

The Monte Carlo solution with fixed random walk from previous data [7] is also included in Table (I) along with the corresponding N, numbers of walks, for each point. It is observed that for the numbers of paths ranging from 50 to 80, multiple paths method yields comparable results to MCM with 500 to 2000 walks. This justifies the considerable reduction in computation time for MPM.

Table (II): Comparison of the (MPM) Solutions to the Results of Finite Element (FE), Finite Difference (FD) and the Monte Carlo Method (MCM)

(X,Y)	MPM	FEM	FDM	MCM
(0.2,0.4)	36.35	36.36	34.88	36.34
(0.2,0.6)	58.31	59.09	58.72	55.46
(0.4,0.2)	36.20	36.36	26.74	36.93
(0.4,0.4)	68.63	65.15	65.41	69.60
(0.6,0.2)	53.07	59.09	56.69	53.76

4 COMPARISON WITH OTHER METHODS

Consider Laplace's equation in a triangular region (Fig. 5). The MPM results are given in Table (II) and compared to the known solutions using finite element (FE), finite difference (FD) and the Monte Carlo methods [2,7] for several points in the region. It is observed that the MPM results agree with other methods. In the case of nonhomogeneous regions a modification of the random walk probabilities at the interface of the dielectrics is needed [9]. In this case (8) is used for u_r at the interface and the random walk probabilities should be evaluated using the appropriate boundary conditions. For instance in Fig. 6 the random walk probabilities in a direction normal to the interface are given by [8,9]:

$$P_{n+} = \frac{\epsilon_1}{2(\epsilon_1+\epsilon_2)} \qquad P_{n-} = \frac{\epsilon_2}{2(\epsilon_1+\epsilon_2)} \tag{14}$$

While for the tangential probabilities:

$$P_{t+} = P_{t-} = 1/4 \tag{15}$$

The multiple paths method is employed to obtain the solution in a 3 by 2 rectangular region of Fig. 6 with $\epsilon_1 = \epsilon_0$ and $\epsilon_2 = 2.25\epsilon_0$. The results are compared to the previously given data [8]. For simplicity, the values of normal random walk probabilities are used as an average which simplifies the calculations. Since the potential at the interface does not change significantly, particularly for smaller steps an average value is justifiable [10] in return for much less computation time. The solutions show good agreement with the data obtained by previous techniques. These results are given in Table (III) and compared with the exodus method, Monte Carlo method and finite differences [8]. The exact value is also given as a reference.

To have an estimate of computation time and convergence, the MPM results of Table (III) are compared with the MCM, FD, and the exact solution in Table (IV). For the multiple paths method (MPM), the corresponding

numbers of paths, N, is also presented ranging from 50 to 100. The MCM results are for $N = 2000$ walks and the FD results are reported for 1000 iterations [8]. The fast convergence makes the method suitable for real time computation.

Table (III): Comparison of the MPM with Other Methods, Exodus, Monte Carlo and Finite Differences. The Exact Value is Also Given as a Reference

(x,y)	Exodus	MCM	FD	MPM	Exact
(0.5,1.0)	13.41	13.40±1.113	13.16	13.18	13.41
(1.0,1.0)	21.13	20.85±1.162	20.74	21.69	21.13
(1.5,1.0)	23.43	23.58±1.213	22.99	23.29	23.43
(1.5,0.5)	10.52	10.13±0.879	10.21	11.62	10.51
(1.5,1.5)	59.36	58.89±2.138	59.06	59.41	59.34

Table (IV): Comparison of Convergence of Multiple Paths Method (MPM) with MCM and FD

(x,y)	N	MPM	N	MCM	N	FD
(0.5,1.0)	60	13.18	2000	13.40±1.113	1000	13.16
(1.0,1.0)	50	21.69	2000	20.85±1.162	1000	20.74
(1.5,1.0)	60	23.29	2000	23.58±1.213	1000	22.99
(1.5,0.5)	200	11.62	2000	10.13±0.879	1000	10.21
(1.5,1.5)	100	59.41	2000	58.89±2.138	1000	59.06

5 CONCLUSIONS AND DISCUSSION

A probabilistic method called the multiple paths method (MPM) was introduced in this work. The method is based on the calculation of the probability of absorption of a particle on the boundary. The random walk is chosen along an arbitrary path passing through the point whose potential is to be determined. A random number generator is used to select a random path. Unlike other probabilistic methods in which a grid is generated a simple path is used for the random walk. The simple result given by (9) and (10) provides less computation time and faster convergence. In rectangular grids, however, each random walk ends up at one point on the boundary and a large array storage is needed. The MPM results are comparable to other probabilistic techniques and numerical methods as well. There are two factors affecting the error and computation time in probabilistic methods: The finite number of trials and the finite number of steps in a grid. In the new method the probabilities p and q are independent of the number of steps for a homogeneous region and therefore the method is fast and accurate for a uniform medium.

Although MPM introduces less computation time and is simple to apply it still needs improvements. For concave regions and near the boundary, there may arise situations where a "shadow region" is not accessible by any path and hence part of the boundary may not incorporate in the calculation which can be a source of error. If part of the boundary is occluded by another and the concave region can be divided in two convex regions the shrinking boundary method can be used. The value of the potential on the border line of the two regions can be incorporated as new boundary points into the calculation. A combination of the MPM and a floating random walk, shrinking boundary method or an exodus method routine can be effective in these special cases.

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