Electromagnetic waves in a nonlinear dispersive slab

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Abstract **– The electromagnetic scattering of a normal incident plane wave from a strongly nonlinear and dispersive dielectric slab is considered. The dynamics of the polarization density vector are described through a forced nonlinear ordinary differential equation of Duffing type, which takes both dispersive and nonlinear effects into account. The aim of the paper is to study the behaviour of the electromagnetic fields by using the Galerkin method. In particular it is shown that shock waves with infinite slope cannot be developed in real media because of the dispersion.**

I. INTRODUCTION

In recent years the growing interest in nonlinear electromagnetic problems, related to electronic and optical devices, has resulted in several studies on nonlinear propagation phenomena in nonlinear dielectrics.

Often, these problems are solved in the weakly nonlinear limit; in this way the difficulties related to the resolution of nonlinear hyperbolic equations are overcome [1]. In this paper the electromagnetic wave propagation in a *strongly* nonlinear dispersive dielectric is studied in the time domain, solving the set of Maxwell equations by using the Galerkin method directly.

The mathematical model consists of a set of nonlinear fully hyperbolic partial differential equations, if dispersive effects are neglected. When the dispersion is disregarded, field equations lead to a decrease of the smoothness of the solution for increasing time due to nonlinear effects: after a certain time a discontinuity is developed in the electromagnetic wave (shock wave) [2-3].

The discontinuity of the solution is related to the loss of its uniqueness: when the discontinuity is developed the time derivatives of electromagnetic fields are not bounded, and then the uniqueness results known in literature do not hold any more [4]. In fact, it is easy to show that the actual solution of the nonlinear wave equation is not unique. This equation admits several possible families of discontinuous solutions [3]. The non-uniqueness can be resolved by choosing the solutions that are physically meaningful. However, when the time derivatives become steep, just before breaking, the dispersive effects are no longer negligible. These effects must be included to give an improved model and a well posed problem.

We note that in the non dispersive model we cannot use the Galerkin method because it is not guaranteed the continuity and uniqueness of the solution.

The aim of this work is to study the behaviour of the electromagnetic fields by solving Maxwell equations through the Galerkin method when both dispersive and nonlinear effects are taken into account.

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II. FORMULATION OF THE PROBLEM

Let us consider an electromagnetic plane wave and suppose that it is normally incident from the left on a nonlinear, dispersive, isotropic dielectric slab, as shown in Fig. 1.

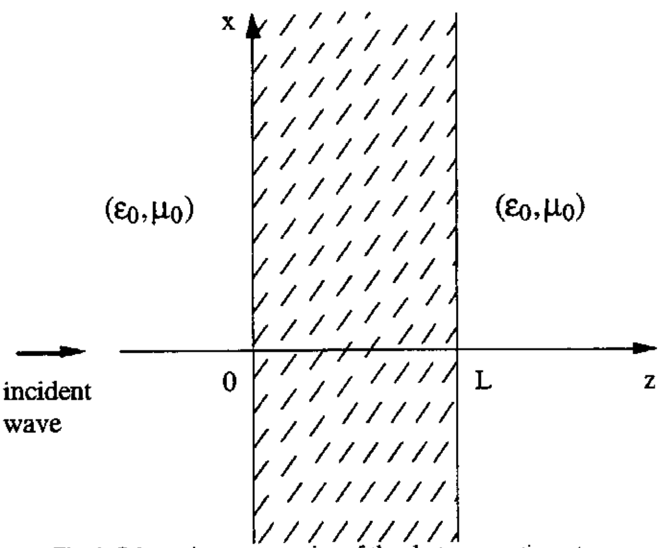

Fig. 1. Schematic representation of the electromagnetic system.

We suppose that the fields $\mathbf{E}$ and $\mathbf{D}$ are directed in the x-direction, while the fields $\mathbf{B}$ and $\mathbf{H}$ are directed in the y-direction, namely

$$\mathbf{E} = \hat{x}\,E, \quad \mathbf{D} = \hat{x}\,D, \quad \mathbf{B} = \hat{y}\,B, \quad \mathbf{H} = \hat{y}\,H\,.$$

The plane wave propagates in the z-direction, and Maxwell equations give

$$\frac{\partial E}{\partial z} + \frac{\partial B}{\partial t} = 0\,, \quad \frac{\partial H}{\partial z} + \frac{\partial D}{\partial t} = 0\,. \tag{1}$$

The dynamics of the polarization density vector

$$\mathbf{P} = \hat{x}\, P = \hat{x}\, (D - \varepsilon_0\, E) \tag{2}$$

are described by the following model [5]

$$\frac{\partial^2 P}{\partial t^2} + \omega_0^2\, P + F(P^2)\, P = \chi\, \varepsilon_0\, \omega_0^2\, E\, . \tag{3}$$

Equation (3) is a Duffing equation without losses (the medium behaves as a "lattice" of nonlinear driven oscillators), and represents the simpler way to take into account both dispersive and nonlinear effects (included *anomalous* dispersion) in a dielectric. It may be regarded naively as the motion equation of the "electrical dipoles" in the dielectric medium, driven by the external electric field. The nonlinearity of the medium is expressed by the function $F(\cdot)$ with $F(0) = 0$. In the limit of small values of P, namely the linear case, the dipoles vibrate with a natural frequency ω_0, representing the resonance frequency of the medium. Moreover, the constant χ coincides with the dielectric susceptibility in the linear and stationary case.

Assuming $B = \mu_0 H$, and introducing the polarization current $\mathbf{J}_p = \hat{x}\, J_p$, we can derive the following mathematical model for the electromagnetic propagation in the slab

$$\begin{cases} \dfrac{\partial H}{\partial t} = -\dfrac{1}{\mu_0}\dfrac{\partial E}{\partial z}\,, \quad \dfrac{\partial E}{\partial t} = -\dfrac{\partial H}{\partial z} - \dfrac{J_p}{\varepsilon_0}\,, \\ \dfrac{\partial P}{\partial t} = J_p\,, \quad \dfrac{\partial J_p}{\partial t} = -\,\omega_0^2\, P - F(P^2)\, P + \chi\, \varepsilon_0\, \omega_0^2\, E\, . \end{cases} \tag{4}$$

We will consider a homogeneous initial conditions in the slab, and the following boundary conditions

$$\begin{cases} Z_0\, H(z = 0^+, t) + E(z = 0^+, t) = 2\, E_i(-ct) \\ Z_0\, H(z = L^-, t) - E(z = L^-, t) = 0 \end{cases} \tag{5}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ and c are respectively the characteristic impedance and the speed of the light in the medium surrounding the slab, L is the slab width, and E_i is the amplitude of the incident electric field, which is supposed to be

$$E_i(-ct) = \begin{cases} E_m \sin(2\pi f_c t)\,, & t \in [0, 1/(2f_c)]\,, \\ 0\,, & \text{elsewhere}\,, \end{cases} \tag{6}$$

where E_m, and f_c, are the amplitude and the carrier frequency of the incident wave, respectively.

The aim of the next section will be the study of the behaviour of numerical solutions of the system made by equations (2) and (4), with the boundary conditions (5), and with homogeneous initial conditions, using the Galerkin method.

III. Numerical Solution

In order to perform a numerical study of system (4), it is convenient to put it in a dimensionless form making use of the new variables ($H_m = E_m/Z_0$)

$$e = \frac{E}{E_m}\,,\ h = \frac{H}{H_m}\,,\ p = \frac{P}{P_0}\,,\ j_p = \frac{J_p}{J_0}\,,\ \tau = \frac{t}{T_0}\,,\ \zeta = \frac{z}{L_0}\,;$$

therefore we obtain

$$\begin{cases} \dfrac{\partial h}{\partial \tau} = -\dfrac{\partial e}{\partial \zeta}\,, \quad \dfrac{\partial e}{\partial \tau} = -\dfrac{\partial h}{\partial \zeta} - \Omega\, j_p\,, \\ \dfrac{\partial p}{\partial \tau} = j_p\,, \quad \dfrac{\partial j_p}{\partial t} = -\, p - \alpha\, p^3 + \Omega\, e\,, \end{cases} \tag{7}$$

where the function $F(x) = \eta\, x^2$ has been selected to describe, by means of the quantity η, the nonlinearity, and the following constants have been introduced

$$T_0 = \frac{1}{\omega_0} = \frac{L_0}{c}\,,\ \Omega = \chi,\ P_0 = \Omega \varepsilon_0 E_m,\ \alpha = \eta \left(\frac{E_m \Omega \varepsilon_0}{\omega_0}\right)^2 .$$

Boundary conditions (5) in a dimensionless form are

$$\begin{cases} h(\zeta = 0^+, \tau) + e(\zeta = 0^+, \tau) = 2\, e_i(-\tau) \\ h(\zeta = \lambda^-, \tau) - e(\zeta = \lambda^-, \tau) = 0 \end{cases} \tag{8}$$

where $\ell = L/L_0$; the normalized incident wave is

$$e_i(-\tau) = \begin{cases} \sin(\nu_c \tau)\,, & \tau \in [0, \pi/\nu_c]\,, \\ 0\,, & \text{elsewhere}\,, \end{cases} \tag{9}$$

where we have

$$\nu_c = \frac{2\pi f_c}{\omega_0}\,.$$

According to the definition (10), the weak form of the equation (7) becomes

$$(u,v) = \int_0^{\ell} u(\zeta)\, v(\zeta)\, d\zeta\, . \tag{10}$$

Using the definition (10) and multiplying the differential system (7) for a test function $w(\zeta)$, we have

$$\begin{cases} \dfrac{d}{d\tau}(w,h) = -\left(w, \dfrac{de}{d\zeta}\right), \\ \dfrac{d}{d\tau}(w,e) = -\left(w, \dfrac{dh}{d\zeta}\right) - \Omega\, (w, j_p)\,, \\ \dfrac{d}{d\tau}(w,p) = (w, j_p)\,, \\ \dfrac{d}{d\tau}(w, j_p) = -\,(w,p) - \alpha\, (w, p^3) + \Omega\, (w,e)\, . \end{cases} \tag{11}$$

Boundary conditions (8) can be automatically taken into account by means of the integration by parts

$$\left(w,\frac{de}{d\zeta}\right) = -\left(\frac{dw}{d\zeta},e\right) + we|_\ell - we|_0 \ ,$$
$$\left(w,\frac{dh}{d\zeta}\right) = -\left(\frac{dw}{d\zeta},h\right) + wh|_\ell - wh|_0 \ ,$$

(the subscripts 0 and ℓ mean $\zeta = 0$ and $\zeta = \ell$, respectively) producing the following weak formulation of the problem

$$\begin{cases} \dfrac{d(w,h)}{d\tau} = \left(\dfrac{dw}{d\zeta},e\right) - we|_l + we|_0 \\ \dfrac{d(w,e)}{d\tau} = \left(\dfrac{dw}{d\zeta},h\right) - we|_l + we|_0 + 2w|_0\, \varepsilon_i - \Omega(w,j_p) \\ \dfrac{d(w,p)}{d\tau} = (w,j_p) \\ \dfrac{d(w,j_p)}{d\tau} = -(w,p) - \alpha(w,p^3) + \Omega(w,e) \ . \end{cases} \quad (12)$$

Consider, now, a functional space of finite dimension N, whose basis functions $w_i(\zeta)$ are continuous piecewise linear with a compact support: they take the value 1 at $\zeta_i = i\,\Delta$, and the value 0 at other node points.

If we expand each unknown field in this space, namely

$$e(\zeta,\tau) \approx \sum_{i=1}^{N} e_i(\tau)\, w_i(\zeta) \ ,$$

and similarly for the other unknown fields, introducing the vectors $\mathbf{e}(\tau) = [e_1(\tau), e_2(\tau), \cdots, e_N(\tau)]^T$, $\mathbf{h}(\tau)$, $\mathbf{p}(\tau)$, and $\mathbf{j}_p(\tau)$, the system (12) becomes the following system of ordinary differential equations

$$\begin{cases} \mathbf{L}\dfrac{d\mathbf{h}}{d\tau} = \mathbf{D}\ \mathbf{e} + \mathbf{B}\ \mathbf{e} \ , \\ \mathbf{L}\dfrac{d\mathbf{e}}{d\tau} = \mathbf{D}\ \mathbf{h} - \mathbf{R}\ \mathbf{e} + 2\,\mathbf{c}\,\varepsilon_i(\tau) - \Omega\ \mathbf{L}\ \mathbf{j}_p \ , \\ \dfrac{d\mathbf{p}}{d\tau} = \mathbf{j}_p \ , \\ \dfrac{d\mathbf{j}_p}{d\tau} = -\,\mathbf{p} + \alpha\, \mathbf{L}^{-1}\, \mathbf{F}(\mathbf{p}) + \Omega\ \mathbf{e} \ , \end{cases} \quad (13)$$

where $\mathbf{c} = [1, 0, \cdots, 0]^T$, $\mathbf{L} = (w_i,w_j)$ is a tridiagonal and symmetric matrix, given by

$$\mathbf{L}_{i,i} = \frac{\Delta}{3}\left(2 - \delta_{i,1} - \delta_{i,N}\right) \quad i = 1, 2, \cdots, N \ ,$$
$$\mathbf{L}_{i,i+1} = \frac{\Delta}{6} \qquad\qquad i = 1, 2, \cdots, N\text{-}1 \ ,$$

$\mathbf{D} = (dw_i/d\zeta, w_j)$ is a tridiagonal matrix with $\mathbf{D}_{i,j} = -\,\mathbf{D}_{i,j}$ ($i \neq j$), given by

$$\mathbf{D}_{i,i} = \frac{1}{2}\left(\delta_{i,N} - \delta_{i,1}\right) \quad i = 1, 2, \cdots, N \ ,$$
$$\mathbf{D}_{i,i+1} = -\frac{1}{2} \qquad\qquad i = 1, 2, \cdots, N\text{-}1 \ ,$$

and $\mathbf{B} = \mathrm{diag}(1, 0, 0, \cdots, -1)$, $\mathbf{R} = \mathrm{diag}(1, 0, 0, \cdots, 1)$,

$$\mathbf{F}_i(\mathbf{p}) = \left(w_i, \left[\sum_{j=1}^{N} w_j\, p_j\right]^3\right) .$$

IV. Numerical Results and Final Remarks

All the numerical results discussed in the paper are obtained with the following set of parameters

$$\Omega = 5,\ \nu_c = 0.01,\ \ell = 40 \ .$$

The incident wave is the sinusoidal pulse given by (6), with a duration much longer than the characteristic response time of the material polarization.

In order to prove the convergence of the proposed algorithm, we start by doing numerical experiments for different numbers N of nodes. The r.m.s. error defined by

$$\varepsilon^N_{rms}(\tau) = \sqrt{\frac{1}{\ell}\int_0^\ell \left[e_N(\zeta,\tau) - e_{N-1}(\zeta,\tau)\right]^2 d\zeta}$$

is reported in Table 1 for different values of N, at $\tau = 100$ and for the nonlinear parameter $\alpha = 0.1$: a few hundred elements, even in the nonlinear case, are sufficient to give a satisfactory accuracy ($e_N(\zeta,\tau)$ is the electric field obtained with N elements). Equations (13) have been solved by the fourth order Runge-Kutta method with a time step smaller than the smallest characteristic time of the discrete model, related to the incident pulse and the characteristic frequencies of the dielectric slab.

Table 1

N	300	400	500
$\varepsilon^N_{rms}(\tau{=}100)$	$2.7\ 10^{-3}$	$1.5\ 10^{-3}$	$1.1\ 10^{-3}$

Then we studied the influence of the dispersion phenomena, modelled by equation (3), on the nonlinear propagation.

We considered the liner case ($\alpha = 0$), first. The electric field profiles at different times are shown in Fig. 2. Since the pulse duration is much longer than the characteristic response time of the material polarization, the propagation will be almost dispersionless. In order to show that the effects of the dispersion are negligible, we plotted in Fig. 3 the pairs (p,e) at $\tau = 100$ for all the nodes: the loci are located close to the straight line p = 5e, as in the non-dispersive case. The wiggling tails in Fig. 2 are due to the frequency components of the incident pulse (even if their amplitudes are small) close to the characteristic frequency of the dielectric.

We tested also the algorithm for a linear and really dispersive case. The behaviour of the field is in a good agreement with the analytical results.

Finally we considered the same incident pulse wave, but in a dispersive nonlinear dielectric with $\alpha = 0.1$. The pulse shapes of the electric field along the slab at different times are given in Fig. 4, whereas the pairs (p,e) at $\tau = 25$ and $\tau = 175$ are plotted in Figs. 5.

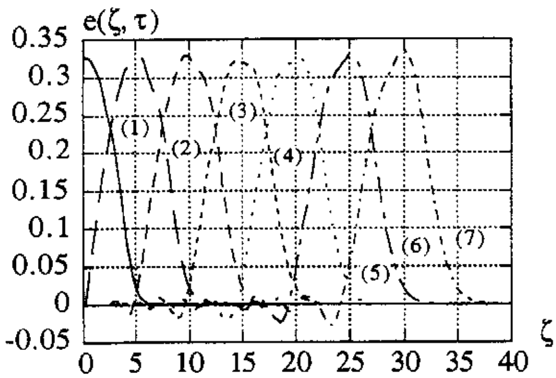

Fig. 2. Electric field profiles at $\tau = 25i$ ($i = 1, 2, ..., 7$) for $\alpha = 0$.

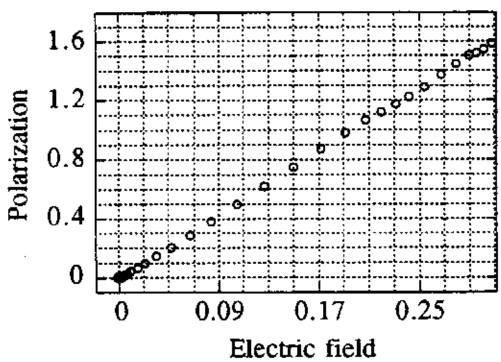

Fig. 3. Plot of $p(\zeta,\tau)$ versus $e(\zeta,\tau)$ for $\tau=100$ and $\alpha = 0$.

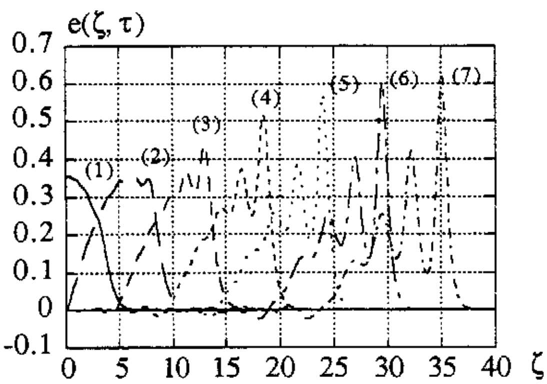

Fig. 4. Electric field profiles at $\tau = 25i$ ($i = 1, 2, ..., 7$) for $\alpha = 0.1$.

Figures 4 and 5 show the interaction between the nonlinear and dispersive behaviour. The plot of $p(\zeta,\tau)$ versus $e(\zeta,\tau)$ is close to the straight line $p = 5e$ in the linear case and in the nonlinear dispersionless case is the punctual function

$$e = 0.2\, p + 0.02\, p^3 . \qquad (14)$$

On the contrary, the interaction between the nonlinear and dispersive phenomena gives a cluster of points in the neighbourhood of the static constitutive relationship (14). Since the smoothness of the solution is decreasing in time, at the beginning the pairs (p,e) are close to the static characteristic, whereas after a certain time, depending on the pulse intensity, the points (p,e) are scattered all around the characteristic. Despite the incident pulse being the same as in Figs. 2 and 3, where the dispersion was negligible, the nonlinear behaviour produces higher harmonics for which the dispersion is relevant. Moreover we saw that the competition between nonlinearity and dispersion does not allow solutions with sharp discontinuities, as noted in [4]. When dispersion is ignored, the equations have discontinuous solutions. As the gradients become steep, just before breaking, the effects of dispersion are no longer negligible.

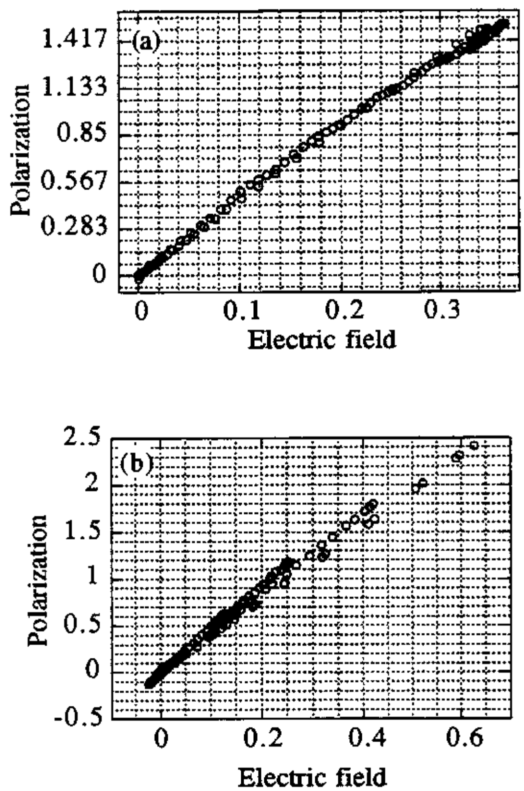

Fig. 5. Plot of $p(\zeta,\tau)$ versus $e(\zeta,\tau)$ for $\alpha = 0.1$: (a) $\tau = 25$, and (b) $\tau = 175$.

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