Magnetic Diffusion in One Dimension

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Electromagnetic systems for which conduction currents are the only source of magnetic field can be characterized by Maxwell's equations using the magneto-quasistatic form of Ampere's law, where time varying **D** is neglected over a surface of fixed identity [1]. The differential equation governing the magnetic field in these applications takes on the familiar parabolic form that often denotes a diffusion process.

A canonical problem that readily allows for demonstration of the diffusion phenomena is shown in Figures 1 and 2. A conducting, non-magnetic bar is placed between the poles of an electromagnet excited by a time-varying current source. We wish to determine the magnetic flux density **B** within the conducting bar, which is assumed to be very long in the y-direction. Thus, the imposed x-directed **B** field will only be spatially dependent upon "z".

The analytical description presumes that the time varying magnetic field serves as a source of electric field (as governed by Faraday's law), but the coupling between **E** and **B** through Ampere's law is incomplete in the sense that displacement currents do not play a role here. The magneto-quasistatic approximation to Ampere's law is :

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1}$$

where we take **J** here to be solenoidal. As already stated, the time varying magnetic field will serve as a source of electric field within the conducting slab as described by Faraday's law :

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \tag{2}$$

It may be straightforwardly shown, upon application of a vector identity and inclusion of the constitutive relationship between current density and the electric field through the conductivity σ, that the differential equation governing the magnetic field in the slab may be written:

$$\frac{1}{\sigma\mu}\,\frac{d^2\mathbf{B}}{dz^2} = \frac{d\mathbf{B}}{dt} \tag{3}$$

Given a sinusoidally varying drive current, the magnetic field on either side of the conducting slab can be specified as $\mathrm{Re}\{B_o \exp(j\omega t)\}$, and the analytic solution to this problem may be found by direct application of phasor analysis. The result is:

$$B_x(z,t) = B(z)\cos\left(\omega t + \phi(z)\right) \tag{4}$$

where

$$B(z) = \frac{B_o\left[\cosh\left(\frac{z}{\zeta}\right)\cos\left(\frac{z}{\zeta}\right) + \sinh\left(\frac{z}{\zeta}\right)\sin\left(\frac{z}{\zeta}\right)\right]^{\frac{1}{2}}}{\left[\cosh\left(\frac{d}{2\zeta}\right)\cos\left(\frac{d}{2\zeta}\right) + \sinh\left(\frac{d}{2\zeta}\right)\sin\left(\frac{d}{2\zeta}\right)\right]^{\frac{1}{2}}} \tag{5}$$

$$\phi(z) = \tan^{-1}\frac{\left[\sinh\left(\frac{z}{\zeta}\right)\sin\left(\frac{z}{\zeta}\right)\right]}{\left[\cosh\left(\frac{z}{\zeta}\right)\cos\left(\frac{z}{\zeta}\right)\right]} - \tan^{-1}\frac{\left[\sinh\left(\frac{d}{2\zeta}\right)\sin\left(\frac{d}{2\zeta}\right)\right]}{\left[\cosh\left(\frac{d}{2\zeta}\right)\cos\left(\frac{d}{2\zeta}\right)\right]} \tag{6}$$

and the skin depth ζ is defined to be:

$$\zeta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{7}$$

We have developed a numerical solution to this equation based on the partial discretization method [2] (finite elements in space, analytical solutions in time). The finite element method implemented for this problem uses linear basis functions, and the inversion of the complex stiffness matrix is accomplished by the bi-conjugate gradient iterative scheme [3]. The numerical results are in agreement with the analytical solution of equation (4), as seen in figure 3.

References

[1] Herman A. Haus and James R. Melcher, *Electromagnetic Fields and Energy*, Prentice Hall, 1989

[2] O. C. Zienkiewicz and K. Morgan, *Finite Elements and Approximation*, John Wiley & Sons, 1983

[3] D.A.H. Jacobs, Generalizations of the Conjugate Gradient Method for Solving Non-Symmetric and Complex Systems of Algebraic Equations, Central Electric Research Laboratories, Lab Note #RD/L/N 70/80, Leatherhead, Surrey

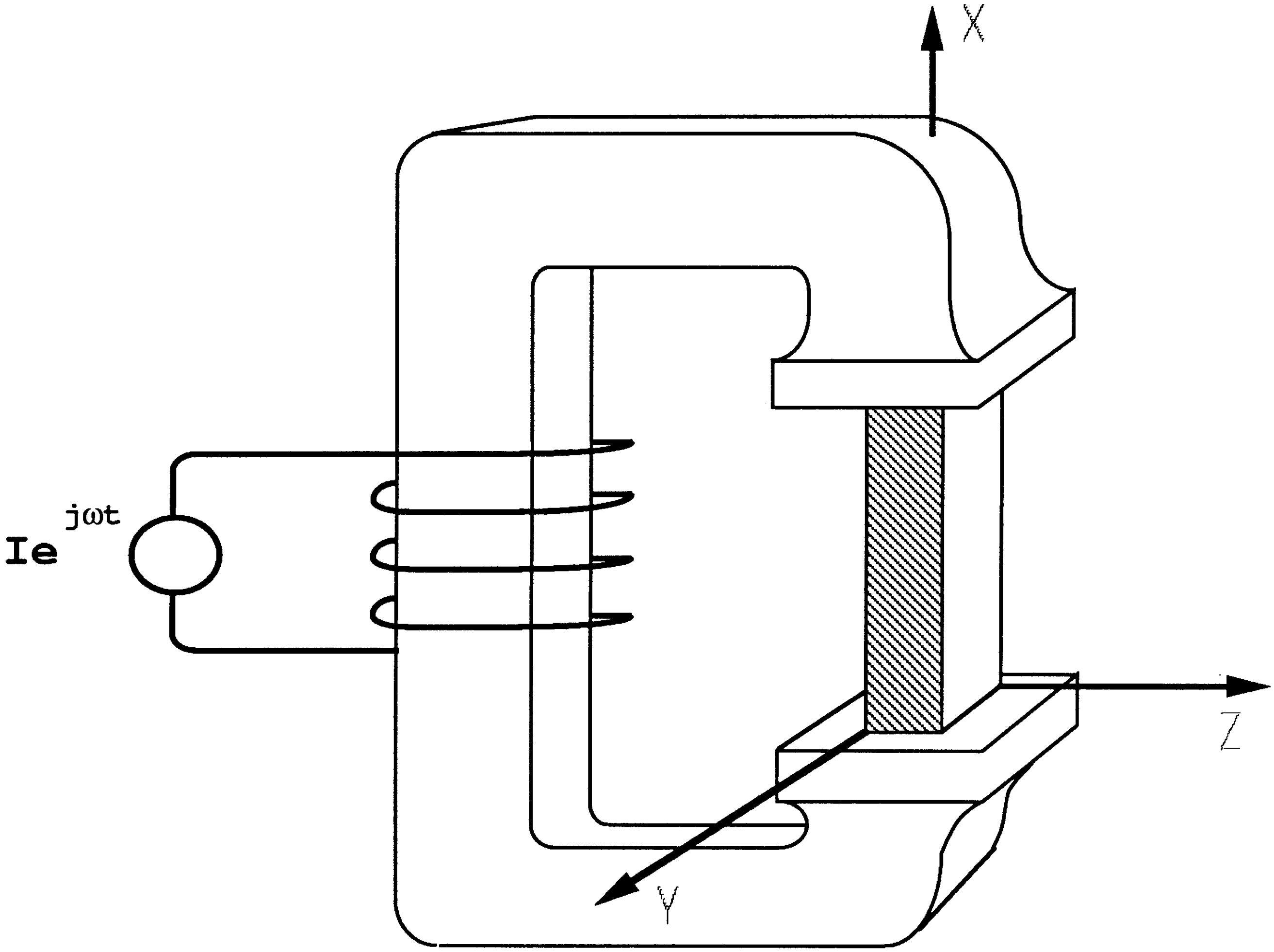

Figure 1. A conducting slab is placed between the pole faces of a sinoidally driven electromagnet. The x-directed **B** field will induce eddy currents inside the structure that tend to "buck-out" the applied field. The extent to which the magnetic field penetrates the slab is described by the a diffusion equation. The slab is long enough in "y" that solenoidal eddy currents are bidirectional.

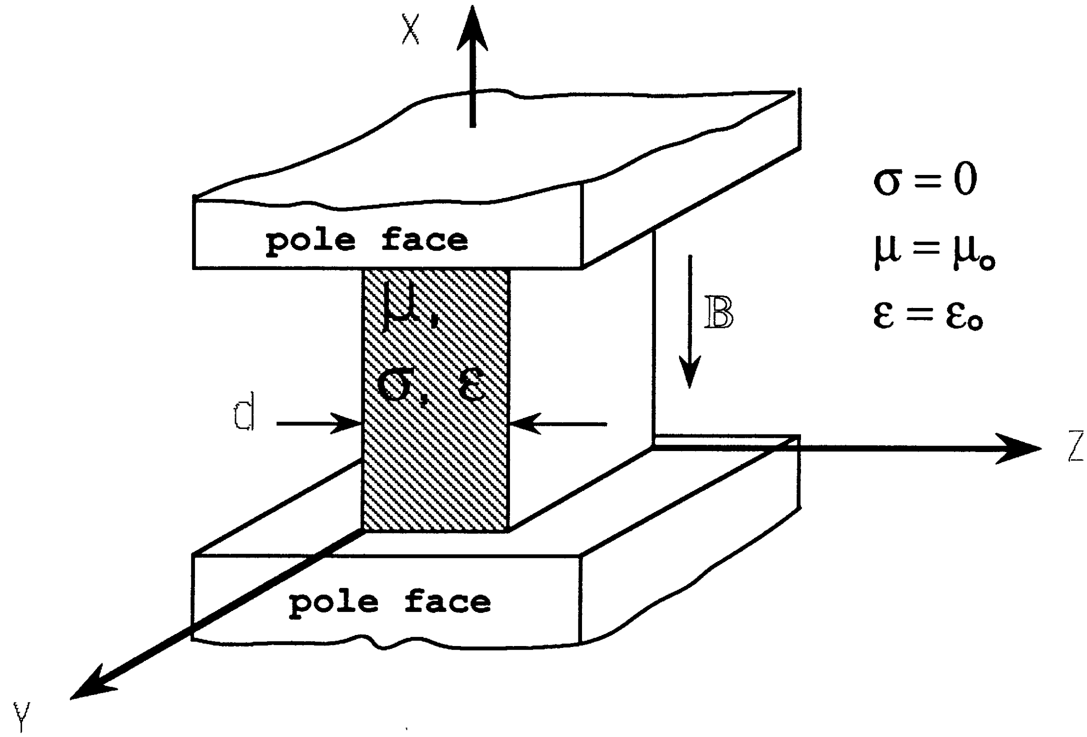

Figure 2. The conducting slab between the pole faces is of "infinite" extent in the "y" direction and the induced eddy currents are y-directed. The x-directed magnetic field is a function of "z" only and is described by a one dimensional parabolic differential equation.

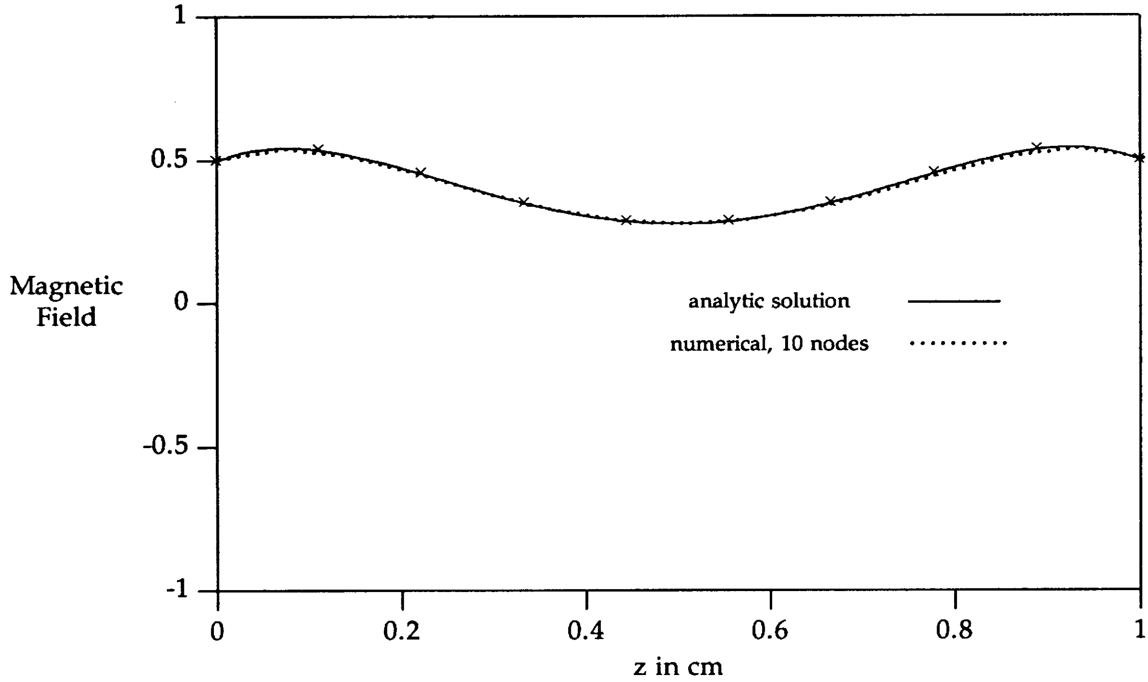

Figure 3. A comparision between the analytical solution and the numerical solution obtained by the partial discretization method. The conducting slab is aluminum with d = 1cm, and material properties $\sigma = 38.2$ MS/m and $\mu = \mu_o$. The frequency of excitation is 800 Hz. The graph shows B as a function of "z" at the instant when the phase angle of the excitation is equal to 60 degrees.