Electromagnetic Properties of a Chiral-Plasma Medium

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Abstract- The theoretical properties of a composite chiral-plasma medium are developed. Using the reaction theorem, we obtain the proof of nonreciprocity based upon the constitutive relationships between the electromagnetic vectors $\vec{E}$, $\vec{B}$, $\vec{H}$, $\vec{D}$. Using the Maxwell's equations and the proposed constitutive relations for a chiral-plasma medium, we derive the $\vec{E}$ and $\vec{H}$ vector equations and from these equations, dispersion relations and $\vec{E}$-field polarizations are found.

Keywords: wave polarization, Faraday rotation, chiral-plasma medium

INTRODUCTION.

Chiral medium [1,2] and ferrite medium [3] have been studied over the last decade for many applications. Chiral-medium have been examined as coating for reducing radar cross section, for antennas and arrays, for antenna radomes in waveguides and for microstrip substrate. Here, we examine a chiral-plasma medium, where the plasma part of the composite medium is non-reciprocal due to the external magnetic field. To find the general dispersion relation giving ω against $\vec{k}$ behavior, the vector phasor Helmholtz based equations are derived. We determine the modal eigenvalue properties in the chiral-plasma medium, which is doubly anisotropic. For the case of waves which propagate parallel to the magnetic field it is a cold magnetized chiro-plasma. We compare our results with the typical results obtained for a cold plasma [4]. Also we obtain the chiral-Faraday rotation which can be compared with the typical Faraday rotation for a pair of right- and left-handed circularly polarized waves.

THEORETICAL FOUNDATIONS.

We propose the following constitutive relations for chiral-plasma medium

$$\vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E} + t_1 \vec{H} \quad (1)$$
$$\vec{B} = \mu \vec{H} + t_2 \vec{E} \quad (2)$$

Plasma medium constitutive relations are [4]

$$\vec{D}_p = \overleftrightarrow{\epsilon}_p \cdot \vec{E}_p \quad (3)$$
$$\vec{B}_p = \mu_0 \vec{H}_p \quad (4)$$

where

$$\overleftrightarrow{\epsilon}_p = \begin{vmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{vmatrix} \quad (5)$$

where $\overleftrightarrow{\epsilon}$ and $t_{1,2}$ represent the permittivity tensor and chirality parameters of the composite medium, respectively. The lossless character of the magnetized cold plasma medium is implied by the Hermitian nature of the tensor $(\overleftrightarrow{\epsilon}_p^*)^T = \overleftrightarrow{\epsilon}_p$. The superscripts * and ^{T} denote complex conjugate and transpose, respectively.

In the search for new medium, which displays nonreciprocal properties, it is essential to establish the nature of the chirality parameters t_1 and t_2. The anisotropic reaction theorem [5] is

$$\int \vec{E}_b \cdot \vec{J}_a dv = \int \tilde{E}_a \cdot \tilde{J}_b dv \quad (6)$$

Here, we see that source currents $\vec{J}_a$ and $\vec{J}_b$ produce fields $\vec{E}_a$, and $\vec{E}_b$, respectively, and the *tilde* over the fields indicates a new medium altered from the original medium, thus, we obtain 6×6 constitutive tensors

$$\tilde{A} = \begin{vmatrix} \hat{\epsilon}^T & -\hat{t}'^T \\ -\hat{t}^T & \hat{\mu}^T \end{vmatrix} \quad (7)$$

and

$$\hat{A} = \begin{vmatrix} \hat{\epsilon} & -\hat{t} \\ -\hat{t}' & \hat{\mu} \end{vmatrix} \quad (8)$$

with $\hat{t}$ and $\hat{t}'$ being the optical activity 3×3 tensors.
Reciprocity occurs only if

$$\int \vec{E}_b \cdot \vec{J}_a dv = \int \vec{E}_a \cdot \vec{J}_b dv$$

that is, by (6) it requires that

$$\tilde{A} = \hat{A} \quad (9)$$

For chiral medium we must obtain

$$\hat{\epsilon} = 0, \quad \hat{t} = t_1 I, \quad \hat{t}' - t_2 I, \quad \hat{\mu} = 0 \quad (10)$$

To obtain reciprocity, (9) imposes

$$-t_2 I^T = t_1 I, \quad -t_1 I^T = t_2 I \quad (11)$$

that is,

$$t_1 = -t_2 \quad (12)$$

For plasma medium (3) and (4) hold leading to

$$\hat{\epsilon} = \overleftrightarrow{\epsilon}, \quad \hat{t} = \hat{t}' = 0, \quad \hat{\mu} = \mu_0 = 1. \quad (13)$$

Then for the proposed constitutive relations ((1) and (2)) we have

$$\begin{aligned} \vec{D} &= \overleftrightarrow{\epsilon} \cdot \vec{E} + t_1 \vec{H} && (14) \\ \vec{B} &= \mu_0 \vec{H} + t_2 \vec{E} && (15) \end{aligned}$$

Vector Helmholtz Equations

The E-field vector Helmholtz equation is derived by inserting the constitutive relation (14) and (15) into Maxwell's equations

$$\begin{aligned} \nabla \times \vec{E} &= -i\omega \vec{B} && (16) \\ \nabla \times \vec{H} &= i\omega \vec{D} + \vec{J} && (17) \end{aligned}$$

so

$$\nabla \times \vec{E} = -i\omega\mu_0 \vec{H} - i\omega t_2 \vec{E} \quad (18)$$

$$\nabla \times \vec{H} = i\omega \overleftrightarrow{\epsilon} \cdot \vec{E} + i\omega t_1 \vec{H} \quad (19)$$

Solving for $\vec{H}$, (18) gives

$$\vec{H} = \frac{1}{\mu_0}\left(\frac{i}{\omega}\nabla \times \vec{E} - t_2 \vec{E}\right) \quad (20)$$

and putting this into (19) we obtain

$$\nabla \times \vec{H} = \frac{1}{\mu_0}\frac{i}{\omega}(\nabla \times \nabla \times \vec{E}) - \frac{t_2}{\mu_0}\nabla \times \vec{E} \quad (21)$$

Then the $\vec{E}$-field vector equation is given by

$$\begin{aligned} &\nabla \times \nabla \times \vec{E} + i\omega(t_2 - t_1)\nabla \times \vec{E} \\ &-\omega^2 \mu_0 \epsilon_0 \left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \frac{t_1 t_2}{\mu_0 \epsilon_0}\right) \cdot \vec{E} = 0 \end{aligned} \quad (22)$$

Here, the plasma current is included in the permittivity tensor $\overleftrightarrow{\epsilon}$.

Similarly the $\vec{H}$-field equation is given by

$$\begin{aligned} &\nabla \times \overleftrightarrow{\epsilon}^{-1} \nabla \times \vec{H} + i\omega(t_2 \overleftrightarrow{\epsilon}^{-1} \nabla \times \vec{H} - t_1 \\ &\nabla \times \overleftrightarrow{\epsilon}^{-1} \times \vec{H}) - \omega^2 \mu_0 \left(\boldsymbol{I} - \frac{t_1 t_2}{\mu_0} \overleftrightarrow{\epsilon}^{-1}\right) \vec{H} = 0 \end{aligned} \quad (23)$$

The inverse permittivity tensor is given by

$$\overleftrightarrow{\epsilon}^{-1} = \begin{vmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_3} \end{vmatrix} \quad (24)$$

Dispersion Relation.

Dispersion relation for the propagation vector $\vec{k}$ against ω can be obtained from $\vec{E}$- or $\vec{H}$-vector equation. We start with the $\vec{E}$-field relation which is simpler than the $\vec{H}$-equation.

Defining $\vec{E}$ as

$$\vec{E} = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}} \quad (25)$$

We obtain

$$\begin{aligned} &-\vec{k} \times \vec{k} \times \vec{E}_0 + \omega(t_2 - t_1)\vec{k} \times \vec{E}_0 \\ &-\omega^2 \mu_0 \epsilon_0 \left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \frac{t_1 t_2}{\mu_0 \epsilon_0}\right) \vec{E}_0 = 0 \end{aligned} \quad (26)$$

Putting $\vec{E}_0$ into rectangular coordinates

$$\vec{E}_0 = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \quad (27)$$

we obtain a three component system of equations which determine the eigenvector, and the determinant of the coefficient component matrix M_k will determine the eigenvalues, thereby yielding the ω against $\vec{k}$ dispersion diagram in phase-space. Writing $Det(M_k) = 0$, with $k_x = 0$ and with symmetry about the z-axis we obtain

$$\begin{vmatrix} 1-\frac{\epsilon_1}{n^2\epsilon_0}\left(1-\frac{t_1t_2}{\mu_0\epsilon_1}\right) & -\frac{i\epsilon_2}{n^2\epsilon_0}-\frac{\cos\theta(t_2-t_1)}{n\sqrt{\mu_0\epsilon_0}} & \frac{\sin\theta(t_2-t_1)}{n\sqrt{\mu_0\epsilon_0}} \\ \frac{i\epsilon_2}{n^2\epsilon_0}+\frac{\cos\theta(t_2-t_1)}{n\sqrt{\mu_0\epsilon_0}} & cos^2\theta-\frac{\epsilon_1}{n^2\epsilon_0}\left(1-\frac{t_1t_2}{\mu_0\epsilon_1}\right) & -\sin\theta\cos\theta \\ -\frac{\sin\theta(t_2-t_1)}{n\sqrt{\mu_0\epsilon_0}} & -\sin\theta\cos\theta & sin^2\theta-\frac{\epsilon_3}{n^2\epsilon_0}\left(1-\frac{t_1t_2}{\epsilon_3\mu_0}\right) \end{vmatrix} = 0 \quad (28)$$

Here, the refractive index n is defined as

$$n=\frac{ck}{\omega}, \quad \text{where} \quad c=\frac{1}{\sqrt{\mu_0\epsilon_0}}.$$

If $\mu_0 = 1$, $\epsilon_0 = 1$, $t_1 = t_2 = 0$ we obtain the same results given by Krall and Trivelpiece for a magneto-plasma [4].

For a lossless chiroplasma, i.e., $t_1 = it\sqrt{\mu_0\epsilon_0}$ and $t_2 = -it\sqrt{\mu_0\epsilon_0}$, the non-trivial solution of this system comes from setting the determinant of the coefficients equal to zero, giving

$$f(0) = F(n^2, \omega, \epsilon_1, \epsilon_2, t, \epsilon_3, k) \quad (29)$$

Equation (29) is then the general dispersion relation for waves propagating in a cold collisionless homogeneous chiroplasma in a uniform magnetic field. For given plasma frequency ω_p, cyclotron frequency ω_c, wave frequency ω and direction of propagation θ, (29) can be solved for the index of refraction n, having as parameter the chirality t.

In terms of k, the dispersion relation is given by

$$a_1k^4 + a_2k^3 + a_3k^2 + a_4k + a_5 = 0 \quad (30)$$

where

$$\begin{aligned} a_1 &= -\left[\omega^2\epsilon_1\mu_0\left(1-\frac{t^2\epsilon_0}{\epsilon_1}\right) sin^2\theta \right. & (31)\\ &\left. -\omega^2\epsilon_3\mu_0\left(1-\frac{t^2\epsilon_0}{\epsilon_3}\right) cos^2\theta\right] \\ a_2 &= 0 & (32)\\ a_3 &= w^4\mu_0^2\left[\epsilon_1\epsilon_3\left(1-\frac{t^2\epsilon_0}{\epsilon_1}\right)\left(1-\frac{t^2\epsilon_0}{\epsilon_3}\right)\right] & (33)\\ & w^4\mu_0^2\left[\epsilon_1\epsilon_3\left(1+\frac{3t^2\epsilon_0}{\epsilon_1}\right)\left(1-\frac{t^2\epsilon_0}{\epsilon_3}\right)\right] cos^2\theta \\ & w^4\mu_0^2\left[\left(\epsilon_1\left(1+\frac{t^2\epsilon_0}{\epsilon_1}\right)\right)^2-\epsilon_2^2-4\epsilon_0^2t^4\right] sin^2\theta \\ a_4 &= -4\mu_0^2\sqrt{\mu_0\epsilon_0}w^5t\epsilon_2\epsilon_3\left(1-\frac{t^2\epsilon_0}{\epsilon_3}\right) cos\theta & (34)\end{aligned}$$

$$\begin{aligned} a_5 &= \mu_0^3w^6\left[\epsilon_3\left(\epsilon_2^2-\epsilon_1^2\right)\left(1-\frac{t^2\epsilon_0}{\epsilon_3}\right)\right. & (35)\\ &\left. +2\epsilon_0t^2\epsilon_1\left(\epsilon_3-\epsilon_0t^2\right)-\epsilon_0^2t^4\left(\epsilon_3-\epsilon_0t^2\right)\right] \end{aligned}$$

Here, there are four different eigenmodes for $\vec{k}$ as implied by (30). The components of the permittivity tensor are obtained using the constitutive equations (14) and (15), and are given by:

$$\begin{aligned} \epsilon_1 &= 1-t^2-\frac{w_p^2}{w^2-w_{ce}^2} & (36)\\ \epsilon_2 &= \frac{w_{ce}}{w}\frac{w_p^2}{w^2-w_{ce}^2} & (37)\\ \epsilon_3 &= 1-t^2-\frac{w_p^2}{w^2}, & (38)\end{aligned}$$

where w_p is the plasma frequency and w_{ce} is the electron gyrofrequency given by:

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} \quad (39)$$

$$\omega_{ce} = \frac{eB_0}{m_e c}. \quad (40)$$

We can observe that for $t = 0$ we obtain the same expressions given by Krall and Trivelpiece [4] for a plasma medium.

High-Frequency Waves with $\vec{k} \parallel \vec{B}_0$ and $\vec{k} \perp \vec{B}_0$.

Setting $\theta = 0$, it is possible to find circularly polarized waves from (28) by writing the $\vec{E}$-field vector equation in the form

$$\begin{aligned} (n^2-\epsilon_R)E_R &= 0, \\ (n^2-\epsilon_L)E_L &= 0, \\ \epsilon_3\left(1-\frac{t^2}{\epsilon_3}\right)E_z &= 0 & (41)\end{aligned}$$

where

$$\epsilon_{R,L} = \epsilon_1\left(1-\frac{t^2}{\epsilon_1}\right) \pm \epsilon_2\left(1-\frac{2tn}{\epsilon_2}\right) \quad (42)$$

and

$$E_{R,L} = E_x \pm iE_y. \quad (43)$$

It is useful to explore these solutions in terms of the wavenumbers k_R and k_L given by

$$k_R = \frac{t\omega}{c} \pm \frac{\omega}{c}\sqrt{\epsilon_1 - \epsilon_2} \quad (44)$$

and

$$k_L = -\frac{t\omega}{c} \pm \frac{\omega}{c}\sqrt{\epsilon_1 + \epsilon_2} \quad (45)$$

where k_R is the wave number for a circularly polarized wave which drives electrons in the direction of their cyclotron motion, i.e., right circularly polarized waves and k_L is the wave number for a circularly polarized wave which drives electrons in the direction opposite to their cyclotron motion, i.e., left circularly polarized waves. The t parameter modifies the typical plot of $\omega(k)$ shown by Kral and Trivelpiece, where the cutoff frequencies are shifted. In Figure 1 we present the modifications introduced by the parameter t in the dispersion relations of the right and left polarized waves. In this Figure the dispersion relations of the right and left circularly polarized waves are indicated by circles and stars, respectively. When $t \neq 0$, ϵ_1 and ϵ_3 depend on t and k_R and k_L have a linear term, $t\omega/c$, as can be seen in (44) and (45). In this way, rather than to modify the curves that exist for $t = 0$, the parameter t permits that the wave propagates in a region of frequencies that is forbidden in the case $t = 0$. Another effect caused by the presence of the parameter t is a conversion of modes. We can observe in Figure 1 that for $t = 0$, there is no intersection of the dispersion relations of the right and left circularly polarized waves. When $t \neq 0$ we can observe that there is an intersection of these curves, indicating that the presence of the t parameter permits that a wave changes its polarization.

In Figure 1 for $t = 0$ we can also observe that there is a region where only right circularly polarized waves propagate, a region where only left circularly polarized waves propagate and a region where both propagate. If their amplitudes are equal, the effect of the superposition of a left and right circularly polarized wave is to produce a plane wave with a particular plane of polarization. Because the two polarizations propagate at different velocities, the plane of polarization rotates as the wave propagates along the magnetic field. This effect is called Faraday rotation.

The global rotation of the plane of polarization as a function of distance in the direction of propagation is given by

$$\frac{E_x}{E_y} = cot\left(\frac{k_L - k_R}{2}\right) z. \quad (46)$$

which means that the presence of the t parameter affects also the Faraday rotation. The chiral-Faraday rotation can be used as a plasma probe. In a laboratory experiment this would be done by launching a planewave along the magnetic field in a chiroplasma. Considering that the plane of polarization of this wave can be determined by an antenna and that we know the magnetic field, the density of the plasma and the frequency of the launched wave, the measurement of the plane of polarization away from the source can determine the value of the parameter t. For instance, considering for the plasma frequency, $w_p = 5s^{-1}$, for the electron gyrofrequency, $w_{ce} = 2s^{-1}$, and for the launched wave, $w = 6.5s^{-1}$, the value of the plane of polarization 10 mm away from the source is $E_x/E_y = 85.76$, $E_x/E_y = 118.17$ and $E_x/E_y = 186.11$ for $t = 0$, $t = 0.05$ and $t = 0.1$, respectively.

Setting $\theta = \pi/2$, we obtain the following dispersion relations:

$$k_X = \pm\frac{\sqrt{A - \sqrt{B}}}{\sqrt{2(\epsilon_1 - t^2)}} \quad (47)$$

and

$$k_O = \pm\frac{\sqrt{A + \sqrt{B}}}{\sqrt{2(\epsilon_1 - t^2)}}, \quad (48)$$

where

$$A = \frac{\omega^2}{c^2}\left[\epsilon_1^2 - \epsilon_2^2 + \epsilon_1\epsilon_3 + t^2(\epsilon_1 - \epsilon_3) - 2t^4\right] \quad (49)$$

and

$$\begin{aligned} B \quad = \quad & \frac{\omega^4}{c^4}\Big[\left((\epsilon_1^2 - \epsilon_2^2) - \epsilon_1\epsilon_3\right)^2 + \\ & t^2(6\epsilon_1^3 - 6\epsilon_1\epsilon_2^2 - 2\epsilon_1\epsilon_3^2 + 12\epsilon_1^2\epsilon_3 - 2\epsilon_2^2\epsilon_3) + \\ & t^4(-15\epsilon_1^2 + 8\epsilon_2^2 - 18\epsilon_1\epsilon_3 + \epsilon_3^2) + \\ & 8t^6(\epsilon_1 + \epsilon_3)\Big] \end{aligned} \quad (50)$$

It should be pointed out that the eletric field of the extraordinary wave, k_X is perpendicular to the magnetic field and the eletric field of the ordinary wave, k_O, is parallel to the magnetic field.

In Figure 2 we present the effect of the parameter t on the dispersion relations for the case $\theta = \pi/2$. In this Figure the ordinary and extraordinary waves are indicated by circles and stars, respectively. When $t = 0.05$, for $\theta = \pi/2$, the effect of the parameter is very small. We can observe that the dispersion relations are a little modified, but the parameter is not able to break up the forbidden regions that exist when $t = 0$. When $t = 0.5$, the dispersion relations show very different curves with respect to the curves for $t = 0$, and there is no more bands of forbidden frequencies.

The difference in the way the t parameter acts in the parallel and perpendicular directions is due to the kind of equations we have. In (44) and (45) the t parameter appears as a linear term and in (47) and (48) the t parameter appears just inside a square root. We observe also that for $\theta = \pi/2$ the parameter t does not lead to the conversion of modes, as it happens for $\theta = 0$.

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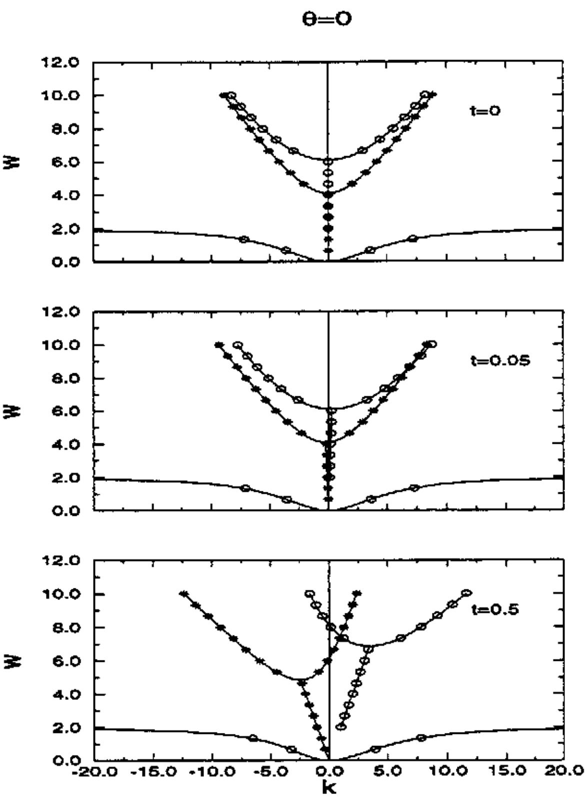

Figure 1: Dispersion relations for various values of the parameter t when the direction of propagation is parallel to the magnetic field ($\theta = 0$). W is the angular frequency normalized to hte plasma frequency.

REFERENCES

[1] D. L. Jaggard, J. C. Liu, and X. Sun, *Electron. Lett.*, **27**, 77, (1991).

[2] C. M. Krowne, *IEE Proceedings-H*, Vol. **140**, 242, (1993).

[3] N. Engheta and P. Pelet, *Optical Letters*, **14**, 593, (1989).

[4] N. A. Krall and A. K. Trivelpiece, *"Principles of Plasma Physics"*, McGraw Hill, (1973).

[5] C. M. Krowne, *IEEE Trans. Antennas Propag.*, **32**, 1273 (1984).

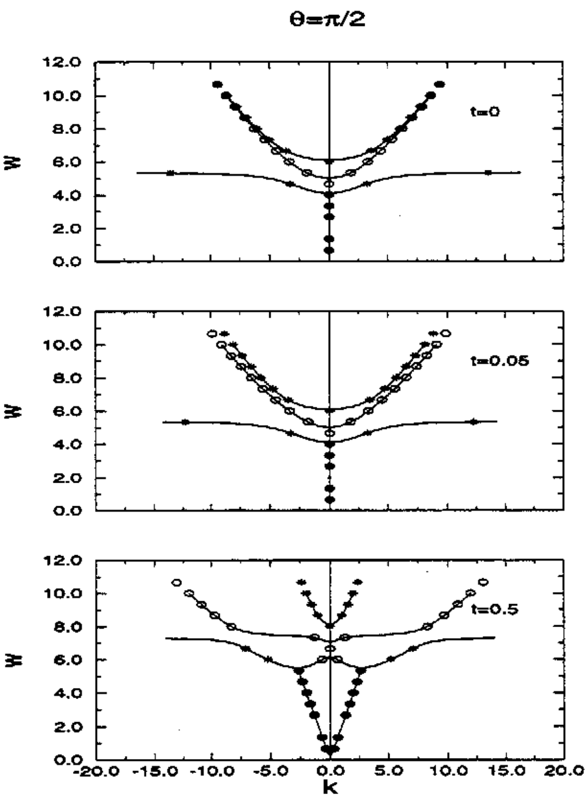

Figure 2: Dispersion relations for various values of the parameter t when the direction of propagation is perpendicular to the magnetic field ($\theta = \pi/2$).