Calculation of Associated Legendre Polynomials with Non-Integer Degree

KEITH D. TROTT
Wright Laboratory Armament Directorate
Advanced Guidance Division, Sensor Technology Branch
WL/MNGS Eglin AFB, FL 32542
trott@eglin.af.mil

Abstract

The exact eigenfunction solution for the electromagnetic scattering from a perfectly conducting cone (or any other sectoral body of revolution with a tip) requires the solution in the form of spherical harmonics. The solution for the θ variation of these harmonics is the associated Legendre polynomial. The boundary conditions for the cone generate associated Legendre polynomials with non-integer degree found for a specific cone angle. This paper will discuss the derivation used to calculate the associated Legendre polynomial, the determination of the eigenvalues, the incomplete normalization integral, and a validation technique.

1 Introduction

To compute the exact eigenfunction solution to the cone problem using spherical harmonics, we need to calculate the associated Legendre polynomials $P_\nu^m(x)$ for non-integer degree ν. A more complete discussion of the solution for the electromagnetic scattering from a cone can be found in [1]–[18]. The boundary conditions on the cone surface require solving

$$P_\gamma^m(\cos\theta_0) = 0 \tag{1}$$

and

$$\frac{d}{d\theta} P_\nu^m(\cos\theta_0) = 0 \tag{2}$$

for $\theta_0 = \pi - \alpha$, where α is the cone half-angle. These are solved as functions of the degrees γ and ν respectively. For the case of the cone, these are generally non-integer; therefore, special effort is required to determine a method to evaluate the associated Legendre polynomials. We begin by determining the integral representation used to calculate the associated Legendre polynomial for particular values of m, ν, and θ. This is integrated numerically using Gaussian quadrature.

2 Integral Representation

Solving boundary value problems in a spherical coordinate system generates the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\nu(\nu+1) - \frac{m^2}{1-x^2}\right) y = 0 \tag{3}$$

for which one solution is $P_\nu^m(x)$, the associated Legendre polynomial. Many papers have been published that tabulate these. References include Siegel [2], Lebedev [20], the Bateman Manuscript Project [21], Hobson [22], Snow [23], Abramowitz and Stegun [24], McDonald [25], Carrus and Treuenfels [26], and Siegel [27].

The basic definition of the associated Legendre polynomial is given by

$$P_\nu^m(\cos\theta) = (\sin\theta)^m \frac{d^m P_\nu(\cos\theta)}{d(\cos\theta)^m}. \tag{4}$$

There are different representations of the associated Legendre polynomials. The difference between the various texts entails a $(-1)^m$ factor. This factor also modifies the recursion formulas. The NBS Tables [19] do not use the $(-1)^m$ and neither does Siegel [2]. We have also chosen not to use this factor.

From [21], the following integral is the starting point

$$P_\nu^m(\cos\theta) = \frac{2^m}{\sqrt{\pi}} \frac{1}{(\sin\theta)^m\,\Gamma(1/2-m)} \times \int_0^\pi \frac{(\cos\theta + j\sin\theta\cos t)^{\nu+m}}{(\sin t)^{2m}}\,dt. \tag{5}$$

A change of variables and some manipulation yields

$$P_\nu^m(\cos\theta) = \sqrt{\frac{2}{\pi}} \frac{(\sin\theta)^m}{\Gamma(1/2-m)} \times \int_0^\theta \frac{\cos(\nu+1/2)x}{(\cos x - \cos\theta)^{m+1/2}}\,dx \tag{6}$$

for $0 < \theta < \pi$, integer $m < 1/2$, ν real.

This expression is Equation (27) in [21]. The requirement for $m < 1/2$ keeps the argument of the gamma function positive and is also required for convergence of the integral. An improper integral of the form $\int_a^b x^{-p}\,dx$ will converge for $p < 1$ and diverge for $p \geq 1$, thus to avoid this problem the expression is evaluated by changing m into $-m$, yielding

$$\begin{aligned} P_\nu^{-m}(\cos\theta) &= \sqrt{\frac{2}{\pi}}\frac{1}{(\sin\theta)^m\,\Gamma(m+1/2)}\times \\ &\times\int_0^\theta \frac{\cos(\nu+1/2)x}{(\cos x-\cos\theta)^{1/2-m}}\,dx. \end{aligned} \tag{7}$$

We use the identity

$$P_\nu^m(\cos\theta) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\,P_\nu^{-m}(\cos\theta). \tag{8}$$

Applying the above identity we write

$$\begin{aligned} P_\nu^m(\cos\theta) &= \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(m+1/2)(\sin\theta)^m}\times \\ &\times\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\int_0^\theta \frac{\cos(\nu+1/2)x}{(\cos x-\cos\theta)^{1/2-m}}\,dx \end{aligned} \tag{9}$$

for $0 < \theta < \pi$, m integer, ν real.

This expression, which is identical to (7.12.32) in [20] except for the factor of $(-1)^m$, is used to calculate the associated Legendre polynomials for real ν; therefore, we can evaluate both integer and non-integer degrees. The integration is performed numerically using Gaussian quadrature. This integral has no convergence problems. When $m = 0$, the exponent is $1/2$, which meets the criteria for convergence. For $m > 0$, the denominator moves into the numerator and it is no longer an improper integral. The $m = 0$ case converges slowly; therefore, a recursion formula found in [20] is used to calculate this case. The recursion for $P_\nu(\cos\theta) = P_\nu^0(\cos\theta)$ yields

$$P_\nu(\cos\theta) = \frac{1}{\nu(\nu+1)}\left(2\frac{\cos\theta}{\sin\theta}P_\nu^1(\cos\theta) - P_\nu^2(\cos\theta)\right). \tag{10}$$

Another important recursion relation is

$$\begin{aligned} \sin\theta\frac{d}{d\theta}P_\nu^m(\cos\theta) &= (\nu-m+1)P_{\nu+1}^m(\cos\theta) \\ &\quad -(\nu+1)\cos\theta\,P_\nu^m(\cos\theta). \end{aligned} \tag{11}$$

This formula enables the calculation of the derivative of the associated Legendre polynomial using the polynomials themselves. This expression is also used in the determination of the eigenvalues satisfying Equation (2). Also, from Equation (4), for $m = 0$, we have

$$\frac{d}{d\theta}P_\nu(\cos\theta) = -P_\nu^1(\cos\theta). \tag{12}$$

ν	m	θ	Calculated	Published [3, App A], [19]
1.0	0	165°	-0.96593	-0.96593
1.5	0	165°	0.40531	0.40531
2.5	0	165°	-0.09819	-0.09820
16.5	0	165°	-0.18075	-0.18075
3.0	0	60°	-0.43750	-0.43750
3.0	1	60°	0.32476	0.32476
5.0	0	30°	-0.22327	-0.22327
5.0	1	30°	2.16797	2.16797
10.0	0	45°	0.11511	0.11511
10.0	1	45°	2.88696	2.88696

Table 1: Comparison for $P_\nu^m(\cos\theta)$

There are no restrictions on the values of m, θ, and ν except those noted in Equation (9). This approach calculates the associated Legendre polynomial for particular values of m, θ, and ν. Another software package, written by Olver and Smith [28], uses power series expansions and recursion to generate associated Legendre polynomials. The main goal of the Olver and Smith package was to use extended-range arithmetic to enable calculation of polynomial values for extensive ranges of m and ν without causing overflow or underflow. We are primarily concerned with finding the θ variation for the cone's eigenfunction solution, finding the eigenvalues generated by satisfying the boundary conditions on the cone, and the calculation of the polynomial for these specific non-integer eigenvalues. Moreover, their package calculates the polynomial for $0 < \theta \leq \pi/2$, integer m and real ν. For the cone problem, we need values for $0 < \theta < \pi$, especially $\pi/2 < \theta < \pi$, with integer m and real ν.

Table 1 and Table 2 show comparisons from [3] and [19] for selected values. It is important to note that all computations (including the Gaussian quadrature) were performed on a 486-based PC. In the next section, orthogonality is proven, and the incomplete Legendre integral is determined.

3 Orthogonality & Normalization

The cone presents a special case in terms of orthogonality and normalization. Orthogonality on the cone is used to find the incomplete Legendre integral. As seen in [1], these incomplete Legendre integrals have the form

$$\int_0^{\theta_0}[P_\nu^m(\cos\theta)]^2\sin\theta\,d\theta. \tag{13}$$

First we prove the original statement of orthogonality. We begin with the original differential equation for different ν and μ. We will work with the form in terms of the variable x, and then convert to $\cos\theta$. Multiplying

ν	m	θ	Calculated	Published [3, App A], [19]
1.0	0	165°	-0.2588	-0.2588
1.5	0	165°	2.8331	2.8331
2.5	0	165°	-2.9988	-2.9988
16.5	0	165°	5.3684	5.3684
3.0	1	60°	-5.4375	-5.4375
3.0	2	60°	-3.2476	-3.2476
5.0	1	30°	-10.4532	-10.4532
5.0	2	30°	11.4844	11.4844
10.0	1	45°	9.7753	9.7754
10.0	2	45°	325.569	325.569

Table 2: Comparison for $\frac{d}{d\theta}P_\nu^m(\cos\theta)$

the equations in ν and μ, then subtracting, we have

$$P_\nu^m(x)P_\mu^m(x) = \frac{d}{dx}\left[\frac{(1-x^2)}{(\nu-\mu)(\nu+\mu+1)} \times \left(P_\nu^m(x)\frac{d}{dx}P_\mu^m(x) - P_\mu^m(x)\frac{d}{dx}P_\nu^m(x)\right)\right]. \quad (14)$$

Integrating this from 1 to x_0 in x (equivalent to 0 to θ_0 in theta) yields

$$\int_1^{x_0} P_\nu^m(x)P_\mu^m(x)\,dx = \frac{(1-x^2)}{(\nu-\mu)(\nu+\mu+1)} \times \left(P_\nu^m(x)\frac{d}{dx}P_\mu^m(x) - P_\mu^m(x)\frac{d}{dx}P_\nu^m(x)\right)\Bigg]_1^{x_0}. \quad (15)$$

When $\nu \neq \mu$, the integral goes to zero at the lower limit because of the $(1-x^2)$ term, and it goes to zero at the upper limit due the boundary conditions. Therefore, orthogonality has been shown. When $\nu = \mu$, we have a $\frac{0}{0}$ condition. Applying L'Hospital's rule in ν yields

$$\int_1^{x_0} [P_\nu^m(x)]^2\,dx = \frac{(1-x_0^2)}{2\nu+1} \times \left[\frac{\partial}{\partial\nu}P_\nu^m(x_0)\frac{\partial}{\partial x}P_\nu^m(x_0) - P_\nu^m(x_0)\frac{\partial^2}{\partial\nu\partial x}P_\nu^m(x_0)\right]. \quad (16)$$

Performing a change of variables such that $x = \cos\theta$, the expression has two forms depending upon which boundary condition and eigenvalue is used. When the boundary condition is $P_\gamma^m(\cos\theta_0) = 0$, the incomplete Legendre integral is

$$\int_0^{\theta_0} [P_\gamma^m(\cos\theta)]^2 \sin\theta\,d\theta = \frac{\sin\theta_0}{2\gamma+1} \times \frac{\partial}{\partial\gamma}P_\gamma^m(\cos\theta_0)\frac{\partial}{\partial\theta}P_\gamma^m(\cos\theta_0). \quad (17)$$

For the boundary condition $\frac{d}{d\theta}P_\nu^m(\cos\theta_0) = 0$, the incomplete Legendre integral is given by

$$\int_0^{\theta_0} [P_\nu^m(\cos\theta)]^2 \sin\theta\,d\theta = -\frac{\sin\theta_0}{2\nu+1} \times P_\nu^m(\cos\theta_0)\frac{\partial^2}{\partial\nu\partial\theta}P_\nu^m(\cos\theta_0). \quad (18)$$

The derivative with respect to theta was done using recursion formulas; whereas, the derivative with respect to the degree was done using a double precision library subroutine. Equation (17) and Equation (18) match Equation (18.266) and Equation (18.267) given in [18]. Moreover, the results check with the tables given in [3]. In the next section, the determination of the particular eigenvalues ν and γ will be discussed.

4 Eigenvalues

The determination of the zeros shown in Equation (1) and Equation (2) is a very important aspect of the solution to the cone problem. The solutions determine over which values the eigenfunction series is summed. There have been several articles published on this subject. In [26], there are tables for both $P_\gamma^m(\cos\theta) = 0$ and $\frac{d}{d\theta}P_\nu^m(\cos\theta) = 0$. The results obtained in this derivation compare very well with those published. Also, the values of ν and γ used in the solution given in [3] compare very well with ours. To find the zeros, we begin with Equation (9). The leading coefficients will not be zero in the specified range; therefore, the zeros of $P_\gamma^m(\cos\theta)$ occur where the integral is zero. That is,

$$f(\nu,m,\theta) = \int_0^\theta \frac{\cos(\nu+1/2)x}{(\cos x - \cos\theta)^{1/2-m}}\,dx = 0 \quad (19)$$

as a function of ν, for a given m and θ.

The determination of the zeros of $\frac{d}{d\theta}P_\nu^m(\cos\theta)$ employs the recursion formula in Equation (11). The integral from Equation (9) is substituted into the recursion expression. After manipulating the integrals, applying several gamma function identities, and factoring out common terms, we have

$$\frac{d}{d\theta}P_\nu^m(\cos\theta) = \frac{1}{(\sin\theta)^{m+1}}\sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)\Gamma(m+1/2)} \times [(\nu+m+1)I_1 - (\nu+1)I_0\cos\theta] = 0, \quad (20)$$

where

$$I_0 = \int_0^\theta \frac{\cos(\nu+1/2)x}{(\cos x - \cos\theta)^{1/2-m}}\,dx \quad (21)$$

and

$$I_1 = \int_0^\theta \frac{\cos(\nu+3/2)x}{(\cos x - \cos\theta)^{1/2-m}}\,dx. \quad (22)$$

i	m	θ	γ_i Calculated	γ_i Published in [3, App B]
1	1	165°	1.03163	1.03163
2	1	165°	2.08443	2.08443
3	1	165°	3.14992	3.14992
4	1	165°	4.22309	4.22309
5	1	165°	5.30108	5.30108

Table 3: Eigenvalues for $P_\gamma^m(\cos\theta) = 0$

i	m	θ	ν_i Calculated	ν_i Published in [3, App A]
1	1	165°	0.9671	0.9673
2	1	165°	1.9189	1.9189
3	1	165°	2.8871	2.8890
4	1	165°	3.8879	3.8900
5	1	165°	4.9171	4.9180

Table 4: Eigenvalues for $\frac{d}{d\theta}P_\nu^m(\cos\theta) = 0$

As before, the leading coefficients are not zero; therefore, finding the zeros only requires the solution of

$$(\nu + m + 1)I_1 - (\nu + 1)I_0 \cos\theta = 0 \tag{23}$$

as a function of ν, for a given m and θ.

These results also compared very well with the published values. All the integrals were performed using a 32 point Gaussian quadrature routine and the roots were extracted using existing library subroutines. Comparison of calculated values and published values are shown in Table 3 and Table 4. The published values for γ_i and ν_i are from [3, App B] and [3, App A] respectively. Our calculations support Siegel's contention in [27] that the first few eigenvalues given in [26] are incorrect. In the next section, the method employed to validate the associated Legendre function routines is discussed.

5 Validation

In [18], a Wronskian result is given. A more general form of this expression is

$$\begin{aligned} &P_\nu^m(\cos\theta_1)\sin\theta_2\frac{d}{d\theta_2}P_\nu^m(\cos\theta_2) \\ &\quad -P_\nu^m(\cos\theta_2)\sin\theta_1\frac{d}{d\theta_1}P_\nu^m(\cos\theta_1) \\ &\quad = \frac{2}{\pi}\sin[(\nu-m)\pi]\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}. \end{aligned} \tag{24}$$

For $\theta_2 = \pi - \theta_1$, we have

$$\frac{d}{d\theta_2}P_\nu^m(\cos\theta_2) = -\frac{d}{d\theta_1}P_\nu^m(-\cos\theta_1) \tag{25}$$

and

$$P_\nu^m(\cos\theta_2) = P_\nu^m(-\cos\theta_1). \tag{26}$$

m	θ	ν	LHS	RHS
0	165°	1.03163	0.2440	0.2440
0	165°	0.96714	-0.2535	-0.2535
1	165°	1.03163	-0.5114	-0.5114
1	165°	0.96714	0.4882	0.4882
1	165°	1.9189	-3.4723	-3.4723
1	165°	2.08443	4.1456	4.1456
1	140°	0.6	0.9043	0.9043
0	110°	4.5	-0.6775	-0.6775

Table 5: Calculated (LHS) vs Exact (RHS)

Substituting the preceeding equations into Equation (24) and eliminating the subscripts provides the Wronskian relationship we use for validating our calculated values. The Wronskian relationship is given by

$$\begin{aligned} &P_\nu^m(\cos\theta)\frac{d}{d\theta}P_\nu^m(-\cos\theta) + P_\nu^m(-\cos\theta)\frac{d}{d\theta}P_\nu^m(\cos\theta) \\ &\quad = -\frac{2}{\pi}\frac{\sin[(\nu-m)\pi]}{\sin\theta}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}. \end{aligned} \tag{27}$$

This Wronskian provides an alternative method of checking the accuracy of our integral calculation. Whenever possible, routines were checked against existing tabulated results; however, for non–tabulated parameters, Equation (27) was used for validation. This validation only works for non–integer ν. In effect, the Wronskian checks the $P_\nu^m(\cos\theta)$ and $\frac{d}{d\theta}P_\nu^m(\cos\theta)$ routines simultaneously. Example comparisons are shown in Table 5.

The Wronskian validation and comparison with previously published data are not meant to provide definitive proof this algorithm will work for all cases; however, the satisfactory comparison for these random cases is encouraging. Moreover, the Wronskian provides a spot check (as required) for any non-tabulated parameters.

6 Conclusion

In order to complete the eigenfunction solution for the cone, we derived a method to calculate the associated Legendre functions for non–integer degree. Our integral formulation was also used to determine the eigenvalues required to complete the cone's eigenfunction solution. In addition, we derived expressions for the incomplete Legendre normalization integral. Excellent agreement with previously published results was achieved using a 486–based PC to compute the associated Legendre functions and the eigenvalues with our integral formulation. The Wronskian validation method provided another check of our results.

Acknowledgements

The author would like to thank the reviewers for their constructive comments. The incorporation of these comments have made this a much stronger paper.

References

[1] K.D. Trott, "A High Frequency Analysis of Electromagnetic Plane Wave Scattering by a Fully Illuminated Perfectly Conducting Semi-Infinite Cone", Ph.D. dissertation, Ohio State Univ., Columbus, OH, 1986. (also *IEEE Trans. Antennas and Propagat.*, Vol 38, No. 8, pp. 1150–1160, Aug., 1990.)

[2] K.M. Siegel and H.A. Alperin, "Studies in Radar Cross Sections-III; Scattering by a Cone," UMM 87, Willow Run Research Center, University of Michigan, Jan., 1952.

[3] K.M. Siegel, H.A. Alperin, J.W. Crispin, H.E. Hunter, R.E. Kleinman, W.C. Orthwein, and C.E. Schensted, "Studies in Radar Cross Section-IV; Comparison Between Theory and Experiment of Cross-Section of a Cone," UMM 92, Willow Run Research Center, University of Michigan, Feb., 1953.

[4] L.L. Bailin and Samuel Silver, "Exterior Electromagnetic Boundary Value Problems for Spheres and Cones," *IRE Trans. Antennas and Propagat.*, Vol AP-4, pp. 5–16, Jan., 1956.

[5] L.B. Felsen, "Alternative Representations in Regions Bounded by Spheres, Cones, and Planes," *IRE Trans. Antennas and Propagat.*, Vol AP-5, pp. 109–121, Jan., 1957.

[6] L.B. Felsen, "Plane Wave Scattering by Small Angle Cones," *IRE Trans. Antennas and Propagat.*, Vol AP-5 pp. 121–129, Jan., 1957.

[7] L.B. Felsen, "Asymptotic Expansion of the Diffracted Wave for a Semi-Infinite Cone," *IRE Trans. Antennas and Propagat.*, Vol AP-5, pp. 402–404, Oct., 1957.

[8] F.V. Schultz, G.M. Ruckgaber, J.K. Schindler, and C.C. Rogers, "The Theoretical and Numerical Determination of Radar Cross Section of a Finite Cone," *Proceedings IEEE*, Vol 53, No. 8, pp.1065–1067, Aug., 1965.

[9] M.E. Bechtel, "Application of Geometric Diffraction Theory to Scattering from Cones and Disks," *Proceedings IEEE*, Vol 53, No. 8, pp. 877–882, Aug., 1965.

[10] T.B.A. Senior and P.L.E. Uslenghi, "High Frequency Backscattering from a Finite Cone," *Radio Science*, Vol 6, No. 3, pp. 393–406, Mar., 1971.

[11] W.D. Burnside and L. Peters, Jr., "Axial-Radar Cross Section of Finite Cones by the Equivalent Current Concept with Higher Order Diffraction," *Radio Science*, Vol 7, No. 10, pp. 943–948, Oct., 1972.

[12] R.G. Kouyoumjian, "High Frequency Target Strength of Cone Shaped Scatterers, Part I—Flat Backed Cone," Tech Report ESL 4720-1, Aug., 1977.

[13] K.K. Chan, L.B. Felsen, A. Hessel, and J. Shmoys, "Creeping Waves on a Perfectly Conducting Cone," *IEEE Trans. on Antennas and Propagat.*, Vol AP-25, No. 5, Sept., 1977.

[14] K.K Chan and L.B. Felsen, "Transient and Time Harmonic Diffraction by a Semi-Infinite Cone," *IEEE Trans. on Antennas and Propagat.*, Vol AP-25, No. 6, pp. 802–804, Nov., 1977.

[15] K.K Chan and L.B. Felsen, "Transient and Time Harmonic Dyadic Green's Functions for a Perfectly Conducting Cone," *IEEE Trans. on Antennas and Propagat.*, Vol AP-27, No. 1, Jan., 1979.

[16] D.S. Wang and L.N. Medgyesi-Mitschang, "Electromagnetic Scattering from Finite Circular and Elliptical Cones," *IEEE Trans. on Antennas and Propagat.*, Vol Ap-33, No. 5, May, 1985.

[17] A.S. Goryainov, "Diffraction of a Plane Electromagnetic Wave Propagating along the Axis of a Cone", Radio Eng Electron No. 6, 1961.

[18] J.J. Bowman, T.B.A. Senior, and P.L.E. Uslenghi (Editors), *Electromagnetic and Acoustic Scattering By Simple Shapes*, Chap. 18, North Holland Pub. Co., Amsterdam, 1969.

[19] Mathematical Tables Project, *Tables of Associated Legendre Functions*, New York, Columbia University Press, 1945.

[20] N.N. Lebedev, *Special Functions and Their Applications*, New York, Dover Pub. Co., 1972.

[21] Bateman Manuscript Project, *Higher Transcendental Functions*, Vol I, McGraw-Hill, 1953.

[22] E.W. Hobson, *Spherical and Ellipsoidal Harmonics*, Cambridge University Press, Cambridge England, 1939.

[23] C. Snow, *Hypergeometric and Legendre Functions with Applications to Potential Theory* (NBS Applied Math Series No. 55) Washington, D.C., NBS, 1964.

[24] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (NBS Applied Math Series No. 55) Washington, D.C., NBS, 1964.

[25] H.M. MacDonald, "Zeros of the Spherical Harmonic $P_n^m(\mu)$ Considered as a Function of n," Proceedings of London Mathematical Society, Vol XXXI, May, 1899.

[26] P. Carrus and C.G. Treuenfels, "Tables of Roots and Incomplete Integrals of Associated Legendre Functions of Fractional Order," Journal of Mathematics and Physics, Vol XXIX, 1950.

[27] K.M. Siegel, D.M. Brown, H.E. Hunter, H.A. Alperin, and C.W. Quillen, "Studies in Radar Cross Section-II: Zeros of associated Legendre Functions $P_n^m(\mu')$ of Non-Integral Degree", UMM 82, Willow Run Research Center, University of Michigan, Apr., 1951.

[28] F.W.J. Olver and J.M. Smith, "Associated Legendre Functions on the Cut", Journal of Computational Physics, Vol 51, No 3, Sept., 1983.