THE FINITE DIFFERENCE METHOD IN MAGNETIC FIELD PROBLEMS

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ABSTRACT

Finite difference techniques are widely used in the solution of electromagnetic boundary value problems, but seldom employed with static or quasi-static field problems. Historically this departure was warranted by (1) the relative ease by which problem geometries can be modeled using the finite element counterpart, and (2) the lack of symmetrical properties and large banding in the governing matrices. Presented here are some methods for generalizing the finite difference approach so that problem definition is easily modeled and Hermitian matrices result. The technique uses a conventional finite difference grid placed in the work area irrespective of the problem geometry. Finite difference equations are written in their simplest form across the problem work space. Boundary conditions are then introduced after the bulk equations are in place. The problem is solved using a non square governing matrix in a least square sense. This is accomplished most easily by pre-multiplying the matrix equation by its transpose. An alternative to the preconditioned conjugate gradient technique for solving the resultant matrix equation is to seek the eigenvalues for the system and express the answer as a sum of the eigenvectors. Results are shown for a salient pole motor. The technique is very useful in handling rotating or translating problems where considerable attention must be given to the proper connection and re-connection of the grid points.

BACKGROUND

Finite difference methods are often used to solve complex electromagnetic interaction problems. Because of the way in which the defining equations are represented, the banding of the resulting matrix equations can be quite large. It is also difficult to maintain symmetry in the matrix in order to speed up the solution. Many techniques have been proposed using equivalent circuit models and successive over relaxation techniques to address some of these problems[1], but odd geometries continue to present difficulties in representation of the constitutive equations. On the positive side, finite difference methods are intuitively appealing; governing equations are easily expanded and represented on a point by point basis in space. With homogeneous grids containing equally spaced points, the problem setup is quite straightforward [2].

At issue in this paper is how to apply the attractive features of the finite difference technique in problems with generalized geometries. The following two additional problems are invoked with the generalized geometry :
a. Representing the first and second-order derivatives in a nonuniform grid, one which 'fits' the problem boundaries.
b. Addressing the issue of iteratively refining the grid to enhance solution accuracy without reformulating the problem.

The technique offered in this paper has the flexibility of addressing the above problems, and springboards from the excellent work of Girdinio et al[3],[4]. Among other things, they point out the benefits of using not 4 but 5 points to represent the two-dimensional Laplacian operator as far as the freedom it affords in placement of points. The present work begins at that point and applies the idea not only to the internal volume to be modeled, but also to the enforcement of boundary conditions. The emphasis in this paper is on an easy implementation of the finite difference equations and iterative grid refinement to enhance solution accuracy.

THEORY

Consider a problem where the H field is represented as either a total or reduced scalar potential. For the latter, we use
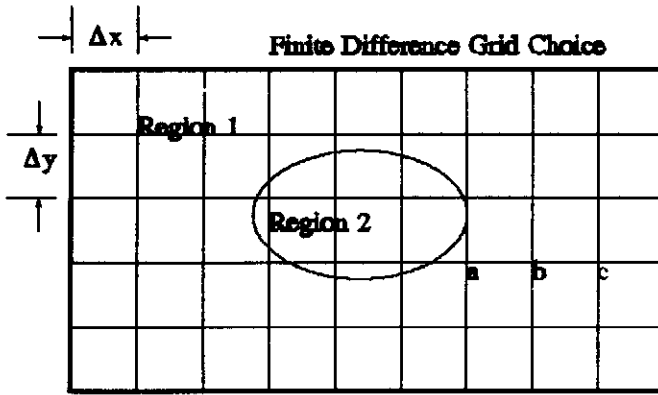
$$\vec{H}=\vec{T}-\nabla\Phi \tag{1}$$

where T is the magnetic intensity source term and is found in the current carrying region using the Biot-Savart law. The problem reduces to solving Laplace's equation for the unknown Φ,

$$\nabla\cdot(\mu\nabla\Phi)=\nabla\mu\cdot\nabla\Phi+\mu\nabla^2\Phi=0. \tag{2}$$

The simplest discretization for a multi-region problem is to continue a homogeneous grid through the problem space as indicated in Figure 1. The Laplacian operator is easily represented in either 2 or 3 dimensions for such a grid.

Referring to Figure 2, the 2-D expansion about point 0 is

Figure at top left with label "Finite Difference Grid Choice", showing Δx, Δy, Region 1, Region 2, and points a, b, c.

Continue grid into the object; match BC after setup

Figure 1. Grid continuation through the boundary of a dissimilar medium.

$$\frac{\Phi_4+\Phi_3-2\Phi_0}{\Delta x^2}+\frac{\Phi_1+\Phi_2-2\Phi_0}{\Delta y^2}=0, \qquad (3)$$

where Δx, Δy refer to the grid distance between points in the x and y directions. The problem setup proceeds by writing the Laplacian operator for Φ in all regions. It is assumed the material inhomogeneities are at least piecewise homogeneous, i.e., that the $\nabla\mu$ terms are negligible within any such piecewise homogeneous region. The only precaution to be taken is to use points from a common region when using a finite difference representation of the problem operator such as (3). In close proximity to the boundary interface, the one-sided second-order derivative should be used to represent the Laplacian. The second-order horizontal derivative represented in terms of the potential at adjacent horizontal positions b and c in Figure 1 is

$$\frac{\partial^2\Phi}{\partial x^2}=\frac{\Phi_a-2\Phi_b+\Phi_c}{\Delta x^2}+\mathscr{E}(\Delta x^3) \qquad (4)$$

The final term in (4) indicates the error terms are correct to order Δx^3.

The one sided expansion is the same regardless of which side of the boundary the derivative is taken. After writing (2) for all the grid points in the problem, the result is a N by N matrix for N unknowns. Over the bulk region of the problem, the standard finite difference representation of the Laplacian is employed because of its ease of implementation. This is, in fact, the motivation for using the finite difference method as opposed to the finite element method. The contribution of this paper is to show that it is

possible to set up an evenly spaced finite difference grid irrespective of boundaries, and match boundary conditions after the bulk system equations are modeled. Furthermore, in a standard grid, 4 surrounding points (or 6 in 3D) are used to represent the Laplacian. Here it is shown that using five neighboring points (or 9 points in 3D) provides flexibility in modeling a variety of shapes while preserving the numerical accuracy to order Δ^3 where Δ represents the largest distance to any one of the neighboring points. Indeed, the technique allows one to place points on the interface randomly. The only penalty is the requirement of solving a 6 by 6 matrix for each interfacial point. The technique is also useful for increasing accuracy where the field has a high gradient, i.e., adding points randomly within a certain region of the problem.

This is made clear by the alternative representation of the second derivatives on x and y at point 0 in Figure 2, obtained by the set of equations

$$\Phi_j=\Phi_0+\frac{\partial\Phi_0}{\partial x}\Delta x_j+\frac{\partial\Phi_0}{\partial y}\Delta y_j+$$
$$\frac{1}{2}\frac{\partial^2\Phi_0}{\partial x^2}(\Delta x_j)^2+\frac{1}{2}\frac{\partial^2\Phi_0}{\partial y^2}(\Delta y_j)^2+\frac{1}{2}\frac{\partial^2\Phi_0}{\partial x\partial y}\Delta x_j\Delta y_j+\mathscr{E}(\Delta^3)$$

$$(5)$$

where the index j refers to one of the nearest neighbors, and Δx_j, Δy_j refer respectively to the x and y differences from the point j to the field point at Φ_0. When (5) is repeated for each of the 5 nearest neighbors, there results a matrix equation for the unknown partial derivatives of Φ at point 0 in terms of the potential values at the nearest neighboring points and the self point Φ_0,

$$\begin{bmatrix} \Delta x_1 & \Delta y_1 & \frac{\Delta x_1^2}{2} & \frac{\Delta y_1^2}{2} & \frac{\Delta x_1\Delta y_1}{2} \\ \Delta x_2 & \Delta y_2 & \frac{\Delta x_2^2}{2} & \frac{\Delta y_2^2}{2} & \frac{\Delta x_2\Delta y_2}{2} \\ \Delta x_3 & \Delta y_3 & \frac{\Delta x_3^2}{2} & \frac{\Delta y_3^2}{2} & \frac{\Delta x_3\Delta y_3}{2} \\ \Delta x_4 & \Delta y_4 & \frac{\Delta x_4^2}{2} & \frac{\Delta y_4^2}{2} & \frac{\Delta x_4\Delta y_4}{2} \\ \Delta x_5 & \Delta y_5 & \frac{\Delta x_5^2}{2} & \frac{\Delta y_5^2}{2} & \frac{\Delta x_5\Delta y_5}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial\Phi_0}{\partial x} \\ \frac{\partial\Phi_0}{\partial y} \\ \frac{\partial^2\Phi_0}{\partial x^2} \\ \frac{\partial^2\Phi_0}{\partial y^2} \\ \frac{\partial^2\Phi_0}{\partial x\partial y} \end{bmatrix} = \begin{bmatrix} \Phi_1-\Phi_0 \\ \Phi_2-\Phi_0 \\ \Phi_3-\Phi_0 \\ \Phi_4-\Phi_0 \\ \Phi_5-\Phi_0 \end{bmatrix} \qquad (6)$$

It is straightforward to invert this 5 by 5 matrix to arrive at the result

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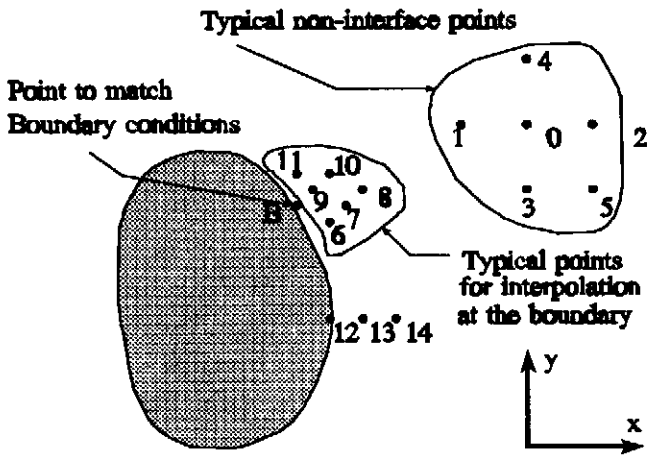**Figure 2.** *Typical points used for evaluating the Laplacian operator in the bulk and for interpolation near interfaces.*

$$\begin{bmatrix} \dfrac{\partial \Phi_0}{\partial x} \\[2mm] \dfrac{\partial \Phi_0}{\partial y} \\[2mm] \dfrac{\partial^2 \Phi_0}{\partial x^2} \\[2mm] \dfrac{\partial^2 \Phi_0}{\partial y^2} \\[2mm] \dfrac{\partial^2 \Phi_0}{\partial x \partial y} \end{bmatrix} = C \begin{bmatrix} \Phi_1 - \Phi_0 \\ \Phi_2 - \Phi_0 \\ \Phi_3 - \Phi_0 \\ \Phi_4 - \Phi_0 \\ \Phi_5 - \Phi_0 \end{bmatrix} \cdot \qquad (7)$$

C is the 5 by 5 inverse of the left hand side of (6). Thus, the Laplacian equation could be written at the point Φ_0 as

$$\sum_{j=1}^{5} C_{3j} \Phi_j + \sum_{j=1}^{5} C_{4j} \Phi_j - \sum_{j=1}^{5} (C_{3j} + C_{4j}) \Phi_0 = 0. \qquad (8)$$

The entire matrix is then built up from this principle, implementing (2) for every point in a piecewise homogeneous region. Because the $\nabla \mu$ term of (2) is zero in such regions, the permeability of the material will not appear in any of these equations.

The remaining step necessary to properly model the problem is to impose the boundary conditions. Referring back to Figure 2, it is necessary to interpolate Φ at the boundary points B in terms of the potential values on one side of the boundary only. As the boundary is approached

from a given region, this interpolation must be realized using potentials located in the same region. If the points lie on a line in 2-D this interpolation requires 6 points, 10 in 3-D. The 6 conditions come from the Taylor expansion of the sought after boundary potential in terms of 6 nearest neighbors (points 6-11).

$$\Phi_j = \Phi_B + \frac{\partial \Phi_0}{\partial x} \Delta x_j + \frac{\partial \Phi_B}{\partial y} \Delta y_j +$$

$$\frac{1}{2} \frac{\partial^2 \Phi_B}{\partial x^2} (\Delta x_j)^2 + \frac{1}{2} \frac{\partial^2 \Phi_B}{\partial y^2} (\Delta y_j)^2 + \frac{1}{2} \frac{\partial^2 \Phi_B}{\partial x \partial y} \Delta x_j \Delta y_j + \mathcal{O}(\Delta^3)$$

$$(9)$$

Equation (5) is repeated for all 6 nearest neighbors; it is understood that all the derivatives are evaluated at the boundary point B, and that Δx_j, Δy_j refer, respectively, to the x and y differences from the point j to the boundary point B. The solution of this 6 by 5 matrix equation is very fast, and yields not only the desired Φ, $\partial \Phi / \partial x$, and $\partial \Phi / \partial y$, but the higher-order derivatives as well. Any Dirichlet or Neuman condition can be matched with these derivative values.

It should be mentioned in passing that when the points lie on a line collinear with the local normal as the points 12-14 do in Figure 2, the first-order derivative at location 12 with respect to the local normal (x direction) is computed from 2 Taylor expansions about the point 12; since

$$\Phi_{13} = \Phi_{12} + \frac{\partial \Phi_{12}}{\partial x} \Delta x + \frac{\partial^2 \Phi_{12}}{\partial x^2} \frac{\Delta x^2}{2!} + \dots \qquad (10)$$

and

$$\Phi_{14} = \Phi_{12} + \frac{\partial \Phi_{12}}{\partial x} 2 \Delta x + \frac{\partial^2 \Phi_{12}}{\partial x^2} \frac{4 \Delta x^2}{2!} + \dots \qquad (11)$$

it follows that

$$\frac{\partial \Phi_{12}}{\partial x} = \frac{4 \Phi_{13} - \Phi_{14} - 3 \Phi_{12}}{2 \Delta x}. \qquad (12)$$

This conventional alternative for enforcing the boundary conditions will be employed by way of making a comparison in the results section.

MATRIX SOLUTION

Suppose there are a total of N unknown potential points Φ. Also consider that (2) is implemented at every one of these N points yielding N equations. Suppose that the boundary conditions, that tangential H and normal B be continuous, are imposed at P boundary points yielding an additional 2P constraints. The total number of equations is now 2P+N with only N unknowns. It is best to solve this non-square matrix in a least square sense using eigenvalues.

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The resulting matrix for the N unknowns could be written as

$$A\vec{\Phi} = \vec{b} \qquad (13)$$

where Φ is the vector of unknown potentials, A contains all the geometry, and the right hand side b contains the source terms. Pre-multiplying by the transpose of A yields the positive definite matrix H in the equation

$$H\vec{\Phi} = A^t\vec{b} = \vec{z}. \qquad (14)$$

Although this new matrix H is more ill conditioned than the original matrix A, it is positive definite and can be treated by some valuable techniques [5]. In particular, one can use the eigenvalues λ and eigenvectors $\vec{q}$ directly to get an accurate solution. Because the eigenvectors of H are orthogonal, both sides of (14) can be multiplied by each of the eigenvectors to obtain the solution vector Φ as a weighted sum of the eigenvectors. That is, assuming

$$\vec{\Phi} = \sum_{i=1}^{N} \alpha_i \vec{q}_i, \qquad (15)$$

where we express Φ as a weighted sum of all eigenvectors $\vec{q}_i$, it then follows that

$$\alpha_i = \frac{\vec{q}_i \cdot \vec{z}}{\lambda_i}. \qquad (16)$$

GRID REFINEMENT

Adding points to refine the grid is usually quite desirable when solution accuracy is suspect in a given region or where the field is varying considerably. Adding extra points would normally mean preprocessing from scratch to set all the bulk constitutive equations since the counting sequence on unknowns would be all wrong. Using the Taylor's expansion in (5) obviates the need to redo the preprocessing of the problem; one simply adds a number of additional equations for each of the additional points added to the grid. With this approach the designer has the liberty of literally placing the grid points anywhere. It is necessary to keep a table of each point's coordinates and material region. The program must in turn identify the 5 (9 for 3D) nearest neighbors still within its own region. Points could be placed on the boundary itself if desired to increase accuracy in these regions.

RESULTS

The motor of Figure 3 was analyzed in attempting to assess the accuracy of this method. The stator is doubled
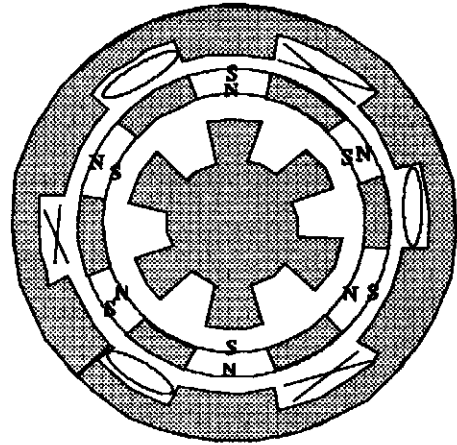

Figure 3. Double sided motor with permanent magnet shell rotor.

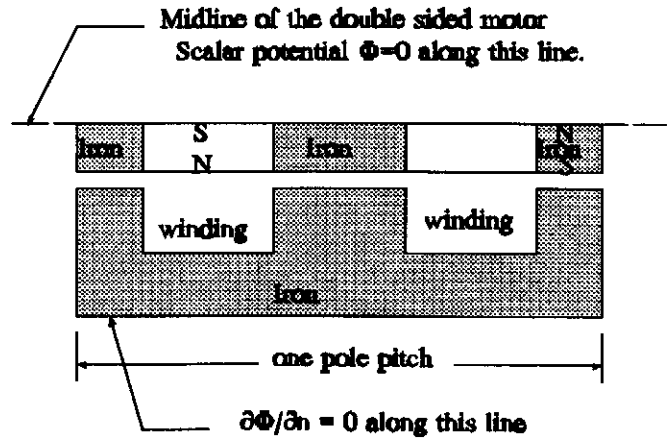sided. Sandwiched in between is a permanent magnet rotor comprising the bulk of the air gap.

Figure 4. Section of the motor to be analyzed in a linear geometry.

Due to symmetry, only one pole pitch of the motor has been analyzed. The problem is divided down the center of a stator tooth and spans a complete pole pitch. Since no flux exits the lower portion of the stator, a Neuman condition $\frac{\partial\Phi}{\partial n} = 0$ exists there. Also at the midline inside the shell rotor of this double sided motor, no tangential B field exists. Thus the potential Φ can be set to zero along this line as shown in Figure 4. A polar grid would necessarily be used for this inherently cylindrical problem if the radius is small. In practice, very little error was found in examining

one pole pitch in this quasi-linear grid and then multiplying by the number of such poles to get the total torque.

The problem is analyzed by two methods, first using a conventional reluctance grid as suggested in [6]; this approach is delineated "normal" in the figures. Here an equal grid distribution is employed and the spacing chosen to force the unknown potentials to have a spacing wherein some of the potentials naturally fell on the interfaces. Second, the boundary constituted by the iron - air interface is modeled using the method detailed above (except at corners where for convenience the reluctances were kept). The bulk equations were altered to be one sided as in (4) in close proximity to the teeth. Furthermore, the potential derivatives needed to insure continuity of field density B across the interface were modeled using the one sided derivative expressed in (12).

The x directed field was predicted in both cases along the air gap, midway between the stator teeth and rotor
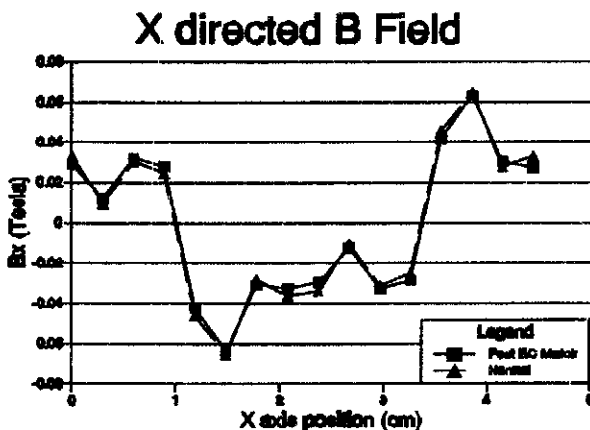
Y directed B Field

Figure 6 Y directed B field along the air gap.

X directed B Field

Figure 5 X directed B along the air gap.

magnets. The results are shown in Figure 5. The number of unknowns was 128. With the post boundary condition technique, an additional 20 equations were added to insure continuity of the normal component of the magnetic field density. The post boundary condition match has good agreement with the normal field approach.

The additional prediction of the y directed B fields is shown in Figure 6.

CONCLUSIONS

A method is presented for predicting magnetic fields using a uniform grid placed incognizant of the interfacial inhomogeneities. Boundary conditions are imposed after the bulk equations are in place and the non-square system is solved for the least square solution. The
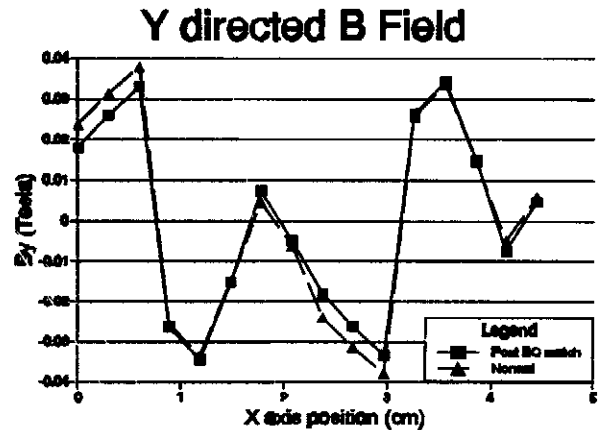
primary advantage of the method is the flexibility afforded in the arbitrary placement of modeling points and in the improvement of accuracy by the ease of adding additional points. The fact that the resulting matrices are always non-square having more rows than columns does not appear to incur numerical inaccuracies for the problems studied. The author has found in parallel that the non-square matrix solution presents no problem for matrices sized at 3000 by 1000, indicating the method would be feasible for large problems.

ACKNOWLEDGMENT

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