ON THE APPLICABILITY OF THE BICONJUGATE GRADIENT FFT METHOD FOR THE THIN CONDUCTING PLATE PROBLEM

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ABSTRACT: The application of the Biconjugate Gradient FFT method to the thin conducting plate problem is investigated. Upon comparing with a Conjugate Gradient FFT method, it is found that the Biconjugate Gradient solution requires a relatively larger error tolerance to achieve a comparatively well-behaved current distribution. Coupled with the requirement of only one matrix-vector product per iteration, the computational cost of the Biconjugate Gradient method is, thus, much smaller than those previously reported in the literature. Of particular importance is the use of the incident electric field as a starting estimate to alleviate the non-convergence behaviour which is usually associated with the application of a Biconjugate Gradient approach to conducting plates at grazing angle. For other angles of incidence, it is shown that this procedure also accelerates the resulting convergence rates as compared to those obtained by simply using zero as an initial estimate.

1. INTRODUCTION

The application of FFT-based methods for the flat, conducting plate problem has been receiving a lot of attention in the computational electromagnetics community since the late 1980s. Most of these use the Conjugate Gradient method (CGM) [1] as the iterative algorithm in conjunction with the fast Fourier transform (FFT) to solve for the discretized matrix equation [2]–[6]. This combination, commonly known as the CGFFT method, is attractive both in the characteristics of convergence associated with the CGM and the reduced computational costs associated with using the FFTs to approximate all occurring convolutions.

The finite-termination property of the CGM is, however, only valid when the defining matrix (or operator) of the linear system to be solved is Hermitian and positive-definite (HPD). For other types of systems, it is necessary to premultiply (implicitly) both sides of the pertinent system by the Hermitian version A^* of the matrix A so that the resulting matrix is HPD. The latter system, often referred to as the *normal equation*, is, however, more ill-conditioned than the original system, which, in turn, makes the convergence rate of the resulting algorithm slower than that associated with the original system. For this reason, it is worthwhile exploring iterative algorithms that can be applied directly to the system of equations in mind.

One such method is the Biconjugate Gradient (Bi-CGM) method, first developed by Lanczos for finding the eigenvalues of an unsymmetric system [7] and later extended by Fletcher [8] and Jacobs [9] to treat complex indefinite systems. When applied to a symmetric system, the Bi-CGM has one more advantage over the CGM in that only one matrix-vector operation is needed within each iteration step [10]. The combination of the Bi-CGM with the FFT to solve a convolutional matrix equation is also possible in exactly the same way as a conventional CGFFT formulation. Despite this, there are only a few applications of the Bi-CGFFT method to electromagnetic scattering problems in the literature compared with the abundance of solutions obtained by its CGFFT counterpart. Recent works on the performance of the Bi-CGFFT for conducting problems have, however, put the Bi-CGFFT in a very favourable position compared with a conventional CGFFT approach [11]–[12].

In this paper, the performance of the Biconjugate Gradient FFT method when applied to a thin conducting plate under plane-wave incidence is investigated. The non-convergence behaviour which usually happens when the Bi-CGFFT is applied to a conducting plate at grazing incidence is overcome by choosing the initial guess to be the incident electric field $\mathbf{E}^i$. A systematic study is also conducted of the dependence of the Bi-CGFFT method on the use of this initial estimate procedure for other angles of incidence. Greatly accelerated rates of convergence is shown to result when $\mathbf{E}^i$ is used as an estimate for the initial unknown current distribution instead of simply using zero as commonly adopted in most FFT-based implementations. The layout of the paper is as follows. Section 2 shows the functional form of the Bi-CGM when applied to a symmetric, indefinite system. This is followed by Section 3 which deals with the numerical solutions obtained by the Bi-CGFFT and the CGFFT methods for a variety of incident configurations. Finally, Section 4 draws the conclusion on the applicability of the Bi-CGFFT based on the results obtained in Section 3.

2. THE BICONJUGATE GRADIENT METHOD FOR SYMMETRIC COMPLEX SYSTEMS

Table 1 shows the computationally compact form of the Biconjugate Gradient method [10] when applied to a symmetric system of the form

$$A\mathbf{J} = \mathbf{E}^i, \qquad (1)$$

where $\mathbf{J}$ denotes the unknown current distribution to be solved for. The impedance matrix A on the left–hand side of (1) represents the couplings between the different cells on the plate and is often formulated in discrete–convolutional form to enable the FFT to be efficiently applied whenever there is a matrix–vector to be computed [6].

Choose an initial guess $\mathbf{J}^{(0)}$; $n = 0$

$\mathbf{F}^{(0)} = \mathbf{E}^i - A\mathbf{J}^{(0)}$; $\mathbf{p}^{(0)} = \mathbf{F}^{(0)}$

while $\| \mathbf{F}^{(n)} \| / \| \mathbf{F}^{(0)} \| > \epsilon$ **do**

$$\alpha^{(n)} = \frac{< \mathbf{F}^{(n)*}, \mathbf{F}^{(n)} >}{< \mathbf{p}^{(n)*}, A\mathbf{p}^{(n)} >}$$

$$\mathbf{J}^{(n+1)} = \mathbf{J}^{(n)} + \alpha^{(n)}\mathbf{p}^{(n)}$$

$$\mathbf{F}^{(n+1)} = \mathbf{F}^{(n)} - \alpha^{(n)}A\mathbf{p}^{(n)}$$

$$\beta^{(n)} = \frac{< \mathbf{F}^{(n+1)*}, \mathbf{F}^{(n+1)} >}{< \mathbf{F}^{(n)*}, \mathbf{F}^{(n)} >}$$

$$\mathbf{p}^{(n+1)} = \mathbf{F}^{(n+1)} + \beta^{(n)}\mathbf{p}^{(n)}$$

$n = n + 1$

end while

Table 1: The generalized Biconjugate Gradient algorithm for a complex, symmetric indefinite system.

Apart from the definition of the inner product (to calculate $\alpha^{(n)}$ and $\beta^{(n)}$), the form of the Biconjugate Gradient method presented in Table 1 is almost identical to a classical Conjugate Gradient formulation for a real and symmetric positive–definite (SPD) system [1]. In both methods, only one matrix–vector product in the form of $A\mathbf{p}^{(n)}$ is required at each iteration step. For complex indefinite systems, however, the CGM has to be modified to avoid the possible division by zero and substantial error growth when this situation is nearly reached [8]. The main resulting cost is an extra matrix–vector product in the form of $A^*\mathbf{F}^{(n)}$, where A^* denotes the complex conjugate of the matrix A [1], [10]. Thus, the computational cost associated with a symmetric indefinite Bi–CGM approach is expected to be half of those required in a CGM formulation when applied to the same problem[1].

The convergence criteria for a Biconjugate Gradient FFT method is specified in the same way as for its Conjugate Gradient counterpart. Let $\mathbf{F}^{(n)}$ denotes the residual field at the n^{th} step, then the stopping criteria is defined in terms of the *normalized residual norm*,

$$\frac{\| \mathbf{F}^{(n)} \|}{\| \mathbf{F}^{(0)} \|}, \qquad (2)$$

where $\mathbf{F}^{(0)}$ denotes the initial field residual and $\| . \|$ denotes the norm of a complex vector. At each step, (2) is compared to an *error tolerance* ϵ which is specified by the user. The usual stopping criteria for a well–behaved CGFFT solution is 0.01 [5]–[6], but as will be demonstrated in the next section, the corresponding value for a Bi–CGFFT solution can be much larger, thus reducing considerably the required number of iterations, and thus computational costs.

An important parameter which is often neglected in other Biconjugate Gradient formulations is the form of the initial guess $\mathbf{J}^{(0)}$. It is often assumed that this is taken to be zero. In Section 3, we demonstrate that choosing the initial guess to be the incident electric field $\mathbf{E}^i$, not only alleviates the non–convergence problem that is usually associated with the application of the Bi–CGFFT at grazing incidence, but also helps to accelerate the convergence rates associated with other angles of incidence.

3. A COMPARATIVE STUDY OF COMPUTATIONAL EFFICIENCIES OF THE BI–CGFFT AND CGFFT METHODS FOR THE THIN CONDUCTING PLATE

In this section, the performance of the Bi–CGFFT method when applied to the pulse–basis formulation recently proposed by Tran and McCowen [6] is investigated. For each incident configuration, a comparison is made with a corresponding CGFFT formulation. The fact that the Bi–CGFFT generally requires half the computational workload compared to its CGFFT counterpart when applied to a symmetric system is well–known and has been investigated elsewhere [11], [12]. The aim of this section is to show that the efficiency of the Bi–CGFFT can be enhanced further by virtue of the fact that it needs a relatively larger tolerance to achieve a well–behaved current distribution as that generated by the CGFFT. The importance of the use of the incident electric field as a starting estimate in a Bi–CGFFT formulation will also be investigated. All numerical results are performed on a VAX 8820 computer.

A. Broadside incidence

Fig.1 shows the broadside current distributions obtained by the two FFT–based methods on a $1\lambda_0 * 1\lambda_0$ square plate using mesh sizes $\Delta x = \Delta y = 0.0303\lambda_0$ with different error tolerances. It can be seen from this figure that the Bi–CGFFT requires a very coarse tolerance to yield a well–behaved solution, as compared with a much finer tolerance as required by the CGFFT to achieve essentially the same results. When applied with the same tolerance as the Bi–CGFFT, i.e. $\epsilon = 0.1$, the CGFFT results are hardly recognizable. For the x – component, only a vague resemblance of the expected lobes along the front and back edges can be assumed (see [5]–[6] for a discussion of these

1. For non–symmetric systems, the application of a Biconjugate Gradient method still requires two matrix–vector products per iteration step. However, as most EM scattering formulations are symmetric in nature, the algorithm with reduced costs is applicable.

results with respect to different discretization procedures); the ripples in the centre of the scattering plate are too high in comparison with the edge lobes. The behaviour of the $y-$component at the same error tolerance is even more out of shape: the lobes along the four edges are nearly of the same amplitudes, as are the ripples in the centre of the scattering plate. Although these anomalies are self–corrected at the finer tolerance of $\epsilon = 0.1 * 10^{-1}$, the resulting computational cost is much higher compared with the corresponding Bi–CGFFT: the CPU times required by the CGFFT on a VAX 8820 is 08:40 minutes (172 iterations) as compared with only 01:17 minutes (27 iterations) for the Bi–CGFFT—an increase of 676%. Yet there is little difference between the two numerical solutions; the only noticeable discrepancy is in the co–polarized $y-$component, where the CGFFT current density is slightly smoother along the four edges. The predicted current distribution from the Bi–CGFFT method for the dominant $x-$component is virtually indistinguishable from its CGFFT counterpart.

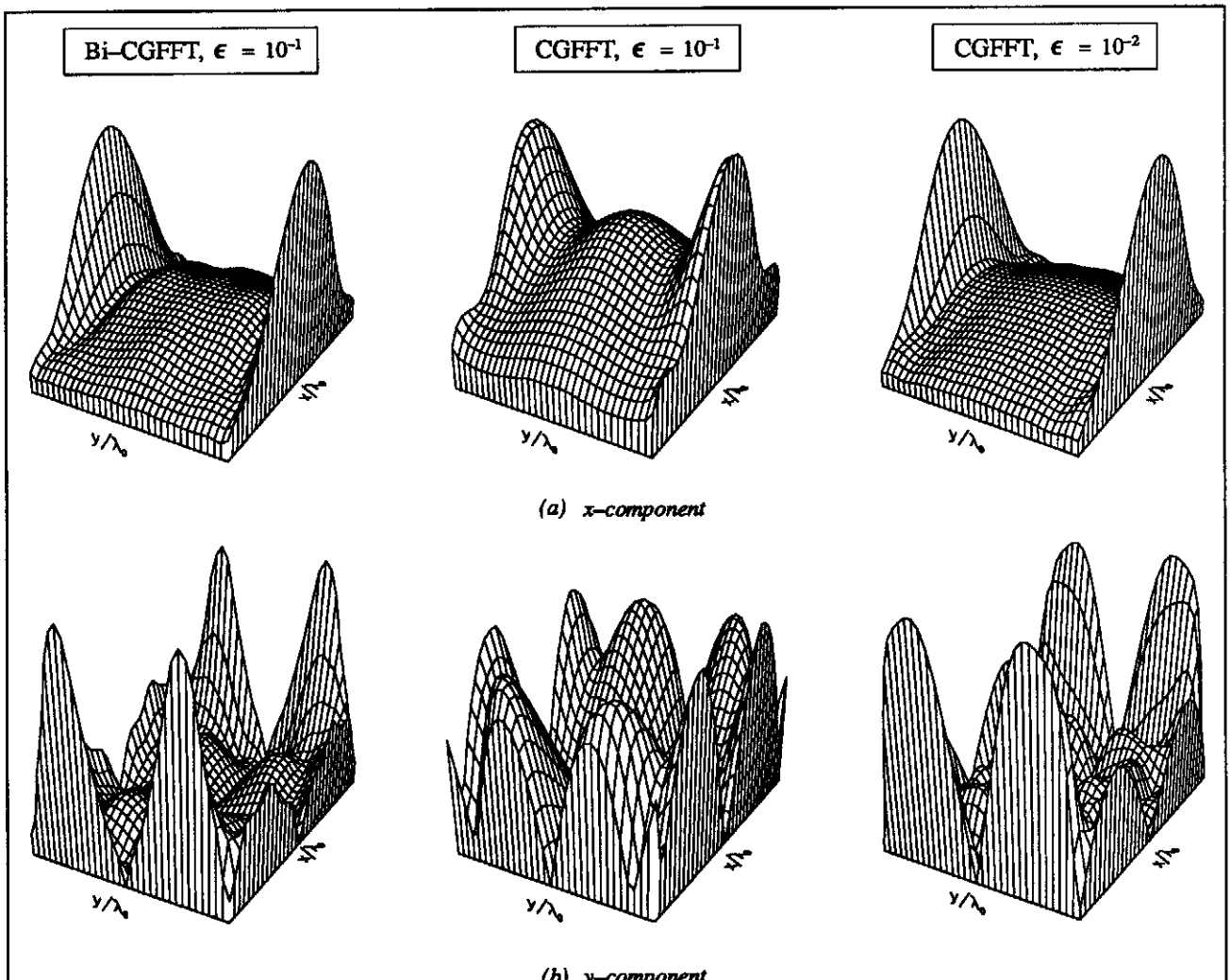


Fig. 1: Comparison of the current distributions by the CGFFT and Bi–CGFFT methods with different error tolerances on a square PEC plate of size $1\lambda_0 * 1\lambda_0$. Normal incidence.

B. Grazing incidence

An important incident configuration of the plate scattering problem is the grazing incidence case. When applied to a geometric shape that possesses edges and corners such as a rectangular plate, the rapidly changing behaviour of the current density near the edge and the corner diffraction effects makes its computation a particularly difficult one [13]. The Bi–CGFFT that gives the solution for the broadside incidence in Fig. 1 fails to give a convergent result when the plate is subject to the grazing angle. This undesirable behaviour, commonly known as the *stagnation problem*, arises when the constant $\alpha^{(n)}$ becomes close to machine–zero and causes both the current density $\mathbf{J}^{(n+1)}$ and field residual $\mathbf{F}^{(n+1)}$ to be non–incremental as the iteration step is increased (see Table 1). Furthermore, since $\mathbf{F}^{(n+1)}$ is then of approximately the same value as $\mathbf{F}^{(n)}$, the inner product $< \mathbf{F}^{(n+1)^*}, \mathbf{F}^{(n+1)} >$ is then also approaching

machine–zero and thus, no new direction vector can be created for the advancement of the unknown current vector. The algorithm then becomes stagnated and no meaningful solution can be obtained no matter how many more steps are added.

A simple remedy which involves restarting the iterative solution with a small perturbation of the zero initial estimate — commonly adopted in most FFT–based implementations— has been suggested by Smith, Peterson and Mittra [12]. However, tests performed by the authors using their *ad–hoc* procedure indicated that an errorenous solution may result if insufficient care is taken in choosing the amount of the required perturbation, although convergence is usually alleviated whenever the zero starting vector is slightly perturbed. The extent of perturbation, thus, plays an important role in ensuring both a convergent and correct solution, and since this parameter is not known in advance, general use of Smith *et. al.*'s procedure for scatterers other than a simple conducting strip is fraught with difficulties. For higher–dimensional problems, the perturbations in the different coordinate components at each discretization cell of the computational domain would best be related to some known quantity of the problem under consideration. At the start of the plate scattering simulation, the only known quantity is the incident field distribution $\mathbf{E}^i$ over the plate. Thus, it is logical that the initial estimate $\mathbf{J}^{(0)}$ should be taken as $\mathbf{E}^i$.

Fig. 2 shows the current distributions from a H–polarized grazing incidence achieved with this initial estimate procedure. Also included are the CGFFT solution at two different error tolerances. It is clear from this figure that the same behaviour is observed for this configuration as for the broadside case of Fig. 1, i.e. the Bi–CGFFT method requires a much larger error tolerances than its CGFFT counterpart when applied to the same problem. The numerical results associated with the CGFFT method at the tolerance sufficiently required by the Bi–CGFFT are, again, not recognizable — particularly in the $y-$component, where the spikes at the back corners are certainly non–physical. The final results for the Bi–CGFFT at $\epsilon = 0.3$ and the CGFFT at $\epsilon = 0.5 * 10^{-1}$ are virtually the same.

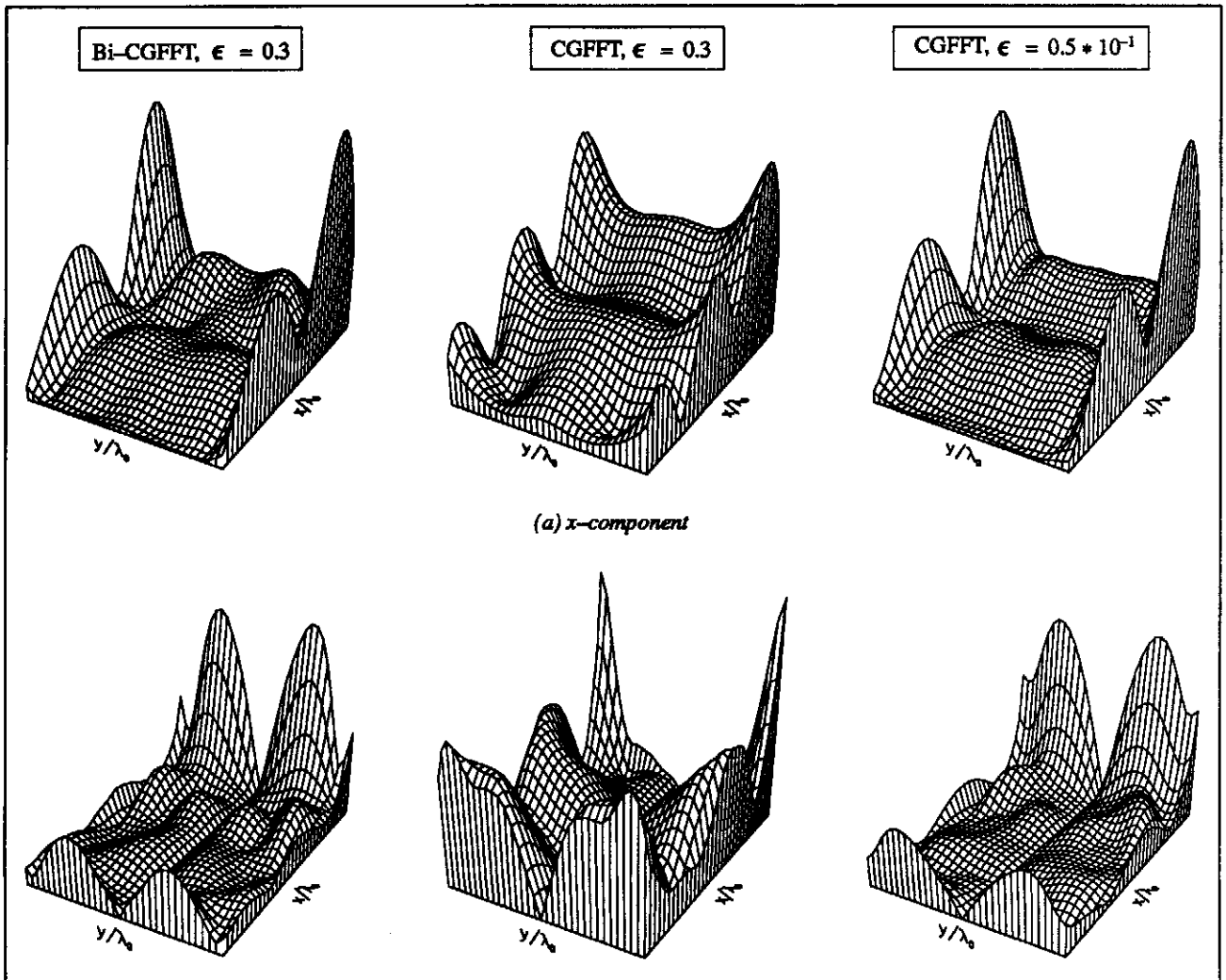
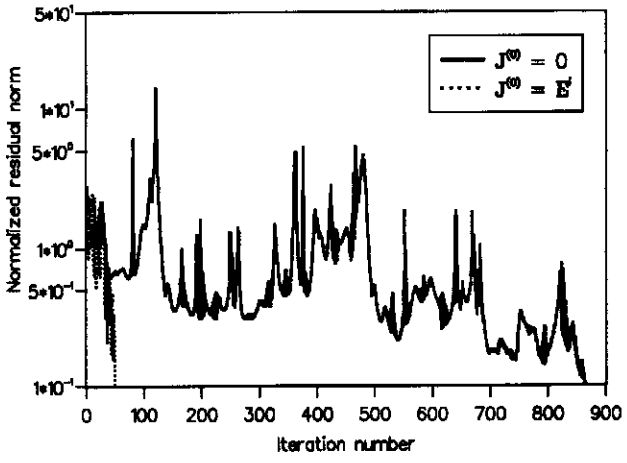


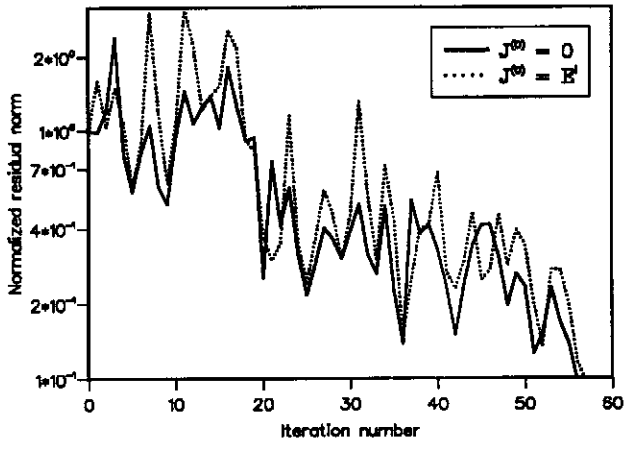
Fig. 2: H–polarized grazing incidence results by the CGFFT and Bi–CGFFT methods with different error tolerances.

C. Oblique incidences

Next, the use of $\mathbf{E}^i$ as an initial estimate for other angles of incidence is investigated. Fig. 3 shows the convergence rates of the Bi–CGFFT method when subject to two arbitrary oblique angles of incidence. The erratic behaviours of the two convergence curves are the consequences of the fact that successive residual norms in a Biconjugate Gradient formulation need not satisfy the inequality $\| \mathbf{F}^{(n)} \| < \| \mathbf{F}^{(n-1)} \|$ at all iteration steps as would be expected in a CGFFT formulation.



(a) $\theta_i = 30^0$ (E-polarized)



(b) $\theta_i = 45^0$ (H-polarized)

Fig. 3: Comparison of convergence rates between the two methods of estimating the initial unknown from a Bi–CGFFT solution.

It is worth emphasizing, however, that, barring the case of non–convergence for certain grazing incidences, the theoretical finite–termination property of a Biconjugate Gradient approach is still obtainable as with a Conjugate Gradient formulation, albeit with infinite–precision arithmetics [9]. However, as Fig. 3 (a) clearly shows, using $\mathbf{E}^i$ as an initial estimate can substantially reduced the computational cost required to achieve convergence as compared with simply using zero. The increase in iteration count by using $\mathbf{J}^{(0)} = 0$ in this case is a staggering 1680%. A similar increase rate has also been recorded for the near–grazing angle of 89°, where although convergence is achieved, the huge computational cost makes it necessary to use $\mathbf{E}^i$ as an initial estimate instead of zero. On the other hand, the detrimental effect of using $\mathbf{J}^{(0)} = \mathbf{E}^i$ at $\theta_i = 45^0$ in Fig. 3 (b) is minimal.

The convergence behaviours of the two angles in Fig. 3 are typical of a Bi–CGFFT solution when used with two different initial estimate procedures for other angles of incidence in both polarizations: either the convergence rates are considerably accelerated when $\mathbf{J}^{(0)} = 0$ is replaced by $\mathbf{J}^{(0)} = \mathbf{E}^i$, or only neglible decrease in convergence rate is observed [14]. In most cases, the significant speedup ratios indicate that the incident electric field $\mathbf{E}^i$ *should* be used as an initial estimate for the unknown current distribution in a Bi–CGFFT method.

D. The effect of using the incident electric field as an initial estimate in a CGFFT formulation

Using $\mathbf{E}^i$ as an initial estimate in a CGFFT formulation is, however, not beneficial to its resulting convergence rate as compared to the usual procedure of simply using $\mathbf{J}^{(0)} = 0$. Fig. 4 shows the convergence rates associated with three arbitrary angles of incidence in a CGFFT solution. The differences in convergence rates associated with the two starting procedures appear to be smaller as the angle of incidence increases. At grazing incidence, where $\theta_i = 90^0$, the discrepancy is minimal, whereas at broadside incidence ($\theta_i = 0^0$), it is best to use zero as a starting estimate. The study in this section shows clearly that no advantages can be gained by using $\mathbf{E}^i$ as an initial estimate for the unknown current distribution in a CGFFT application and zero distribution should be adopted as a starting estimate whenever a CGFFT approach is used to solve the plate scattering problem.

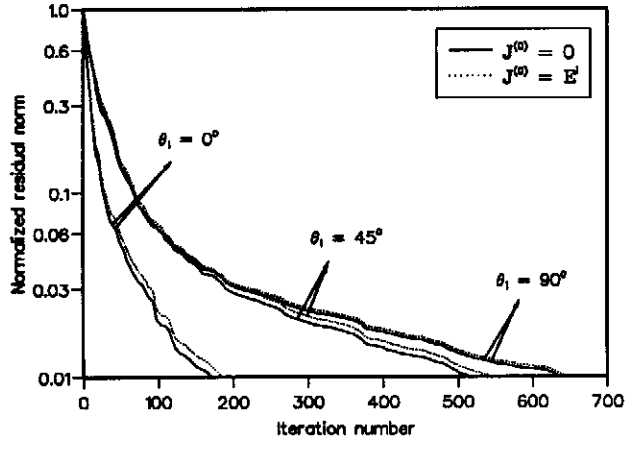


Fig. 4: Comparison of convergence rates of the two methods of estimating the initial unknown with a CGFFT approach on a $1\lambda_0 * 1\lambda_0$ plate.

I. CONCLUSIONS

In this paper, the applicability of the Biconjugate Gradient FFT applied to a pulse–basis formulation for the conducting plate problem has been demonstrated. The computational cost associated with the Bi–CGFFT is much less than a conventional CGFFT approach due to the reduction of a matrix–vector product per iteration step and a much larger error tolerance to achieve a similar well–behaved result as compared to that obtained by its CGFFT counterpart. The difficulty with non–convergence of the Bi–CGFFT at grazing incidence was alleviated by using the incident electric field $\mathbf{E}^i$ as an initial estimate. Furthermore, it was also demonstrated that using this starting guess procedure also accelerates the convergence rates associated with other angles of incidence.

The use of $\mathbf{E}^i$ as an initial estimate is recently shown to overcome the non–convergence problem associated with the application of the Bi–CGFFT method to 2– and 3–D dielectric bodies [15]. Although the cause of non–convergence is different from that associated with a thin conducting plate, the fact that using $\mathbf{J}^{(0)} = \mathbf{E}^i$ can induce convergence is a particularly pleasing aspect. Although no justification is available to explain for the improvement in convergence rates, this non *ad–hoc* initial starting procedure appears to be mandatory in the successful implementation of a Biconjugate Gradient FFT method.

The Biconjugate Gradient FFT has been increasingly used as an efficient algorithm to solve for the discrete–convolutional system which arises either from a FFT–based formulation similar to that considered in this paper [16], or as part of a hybrid method [17]. The incorporation of the starting guess procedure proposed by the authors in this paper may well help to alleviate any non–convergence problem associated with the application of the Bi–CGFFT to other problems of interest and allay the doubt concerning its consistent performance for computational electromagnetics.

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