A Genetic Algorithm Optimization Procedure for the Design of Uniformly Excited and Nonuniformly Spaced Broadband Low Sidelobe Arrays

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ABSTRACT. This paper presents a systematic methodology for designing uniformly excited broadband low sidelobe linear and planar antenna arrays by varying interelement spacings. In the past, attempts to develop a robust array broadbanding design technique have been only marginally successful because of the large number of possible spacing combinations involved, coupled with the theoretical limitations surrounding the problem. The genetic algorithm (GA) has recently proven to be a very effective design tool for nonuniformly spaced low sidelobe antenna arrays with uniform excitation intended for operation at a single frequency. This paper introduces an approach for extending previous applications of GA to include the design of optimal low sidelobe arrays that are operable over a band of frequencies. In addition, it will be demonstrated that designing for low sidelobe operation over a bandwidth adds significant array steerability that can be described by a simple mathematical relation. Finally, it will be shown that the GA objective function is no more complicated to evaluate for broadbanding purposes than it is in the single frequency case. Several examples of GA-designed broadband low sidelobe arrays will be presented and discussed.

1. Introduction

In recent years, genetic algorithms have found a fairly strong presence in electromagnetics optimization problems involving antenna design. The difficulty in solving many antenna design problems is that very often there are many parameters and no practical analytical methods available to optimally determine them. Such difficulties make robust search strategies, like genetic algorithms, very important. The main advantages of using the GA over other search strategies are: 1) the GA can search from any number of random points to find a solution, 2) the GA works with a coding of the parameters and not the actual parameters, 3) GA's use random, not deterministic, transition rules, and 4) the GA does not require the evaluation of derivatives [1]. Several books have been written which discuss genetic algorithms and demonstrate many useful applications [2-4]. Among the first applications of genetic algorithms in antenna design was the thinning of large arrays [1]. Some other varieties of antenna arrays to which the GA has been applied include planar arrays [1,5], multiple beam arrays [6], and Yagi-Uda arrays [7]. There have also been several excellent review articles and books written about GA's and their application to solving complex engineering electromagnetics problems [4], [8-11].

The capability of GA's to produce optimal low sidelobe designs for linear arrays of uniformly excited isotropic sources (at a single frequency) by allowing only the interelement spacings to vary was first demonstrated in the pioneering work of [8]. Interelement spacings were decided by using a 3 bit parameter such that they could vary in increments of $\lambda/8$ with a minimum interelement spacing of $\lambda/4$. In this paper, we will demonstrate that GA's are also an extremely useful tool for broad-banding of uniformly excited, unequally spaced antenna arrays. There are three major advantages of the technique employed in this paper when compared to previously published methods, such as those described in [8]. These advantages are that 1) a much finer discretization ($\approx \pm 0.01 \lambda$) will be used, 2) the GA-designed arrays will have minimal sidelobes over a band of frequencies instead of at just one frequency, and 3) these arrays will typically have a much wider angular region over which the main beam can be steered compared to those optimized for low sidelobe performance at a single frequency. The steady-state genetic algorithm with uniform crossover [12] was chosen for use in optimizing the array designs discussed in this paper.

Although many traditional analytical techniques exist for placing elements in unequally spaced arrays for broad-banding purposes, *viz.* [13-15], none of these methods are capable of producing significantly low sidelobe levels over the entire band. The focus of many of these methods is to place elements in an array such that the minimum separation between elements is greater than or even much greater than a wavelength. The advantage of such large interelement spacings is that a larger bandwidth can be achieved because a lower minimum frequency is possible. The disadvantage, however, is that a theoretical lower bound exists on the sidelobe level when average interelement separations exceed a wavelength [16]. This theoretical minimum is usually not low enough for practical applications. Keeping in mind this theoretical limitation, a design optimization technique will be introduced in this paper which attempts to place elements such that the average interelement spacing in the array is always less than a wavelength.

Another important consideration in the design of antenna arrays is their steerability. Broad-band arrays have the property that they may exhibit perfect steerability at lower frequencies of operation, but steerability is reduced when moving to higher frequencies. The fact that steerability changes with frequency can be quantified by the bandwidth-steerability product of the array [16].

A useful conversion factor will be introduced in Section 2 that permits design tradeoffs to be made between bandwidth and (minimum) element separation. Steerability issues will also be briefly discussed in Section 2. Section 3 begins by considering an example of an optimized low-sidelobe array design intended for operation at a single frequency. Following this, four examples of genetic algorithm produced broadband low sidelobe array designs are presented and discussed– two linear arrays (Section 3) and two planar arrays (Section 4). In addition, the GA objective function used to produce each design is given in the respective sections. All array designs considered in this paper were specified to have a maximum possible bandwidth with a minimum element separation of λ/4 and the lowest possible sidelobe level throughout the band.

2. Some Considerations for Broad-Banding Arrays

In designing a broadband array for low sidelobe performance, it is sufficient to design for the highest desired frequency of operation f_2. Having done this, the frequency may then be varied from f_2 to any f_1, provided $f_1 \le f_2$, without the appearance of any higher sidelobes. The bandwidth for such an array is defined to be $B = f_2 / f_1$. Furthermore, we note that if a minimum separation between two elements exists at the lowest design frequency f_1 that is considered too small for practical purposes, then that spacing can be made larger at the expense of a smaller bandwidth $B' < B$ (i.e., $f_2' < f_2$). This property is best illustrated by the following useful transformation:

$$s' = \left(\frac{f_2}{f_2'}\right)s = \left(\frac{B}{B'}\right)s \tag{1}$$

where

$B' = f_2' / f_1$

s = the set of original element locations $\{s_n = d_n\lambda_1 : n = 1,2,\ldots,N\}$

s' = the set of new element locations $\{s_n' = d_n'\lambda_1 : n = 1,2,\ldots,N\}$

Hence, the array configuration need only be optimized for a desired maximum bandwidth B, subject to some specified tolerance on the minimum element separation. Once this optimal array design has been found using the GA then, if desired, the transformation given in (1) may be employed to find modified designs which tradeoff larger element separations for smaller operating bandwidths.

Another notable characteristic of broadband arrays is how steerability is affected with increasing bandwidth. It can be shown that the bandwidth and steerability of a linear array are related by the following formula, which is known as the bandwidth-steerability product [15]:

$$B(1+\cos\theta_0) = \frac{2w_0}{d_{ave} / d_{\min}} \tag{2}$$

where

B = bandwidth

θ_0 = steering angle

w_0 = the maximum value of $(d_{ave} / \lambda)\cos\theta$ that can be used before a sidelobe will exceed the desired sidelobe level

d_{ave} = average interelement separation in the array

d_{min} = smallest interelement separation in the array

The right-hand side of this equation is a constant, and is characteristic of the individual array. Note that at a 1:1 bandwidth, the right-hand side of the equation must be at least two to guarantee perfect steerability. When a bandwidth of larger than 1:1 is desired, the left-hand side of this equation limits steerability at some of the higher frequencies in the band. Thus, while a broadband array may exhibit perfect steerability at lower frequencies of operation, steerability may become limited at higher frequencies of operation. In addition, arrays designed to operate at only one frequency when interelement spacings are small may not exhibit any steerability.

3. Linear Broadband Array Designs

The array factor expression for the far-field radiation pattern of a symmetric linear array of isotropic sources can be written in the following form:

$$AF(\theta) = 2\sum_{n=1}^{N} I_n \cos[2\pi(f/f_1)d_n \cos\theta + \alpha_n] \quad (3)$$

where

$$\alpha_n = -2\pi(f/f_1)d_n \cos\theta_0 \quad (4)$$

and

$2N$ = the total number of elements in the array

I_n = excitation current amplitude of the nth element in the array

α_n = excitation current phase of the nth element in the array

$s_n = d_n \lambda_1$ = total distance of the nth element from the origin (note that the parameter d_n is unitless)

θ = angle measured from the line passing through antenna elements

θ_0 = steering angle

f_1 = base (minimum) frequency of operation

f = desired frequency of operation

The objective function used by the GA in this paper is based on the array factor expression given in (3), where the desired goal is to minimize the maximum relative sidelobe level (RSLL) of the array over some prescribed bandwidth. In other words, each gene has an associated RSLL calculated from

$$F(\theta) = \max\left|2\sum_{n=1}^{N} I_n \frac{\cos[2\pi B d_n \cos\theta]}{AF_{max}(\theta)}\right| \quad (5)$$

where

$AF_{max}(\theta)$ = peak of the main beam (for normalization)

$B = f_2/f_1$ = desired bandwidth of the array ($f_2 \geq f_1$)

The parameters d_n were selected by the GA to minimize the maximum sidelobe level with I_n set to unity for all values of n. The discretization of d_n was made relatively fine, such that it could be varied in increments of approximately ±0.01 between zero and some maximum selected value. The use of any finer discretization was found to yield little improvement in the overall results. It was also found that, in the case of broadband array optimization (*B>1*), the GA objective function need not be any more complicated to evaluate than it is for optimization of array performance at a single frequency (*B=1*). This is one of the attractive features of the technique presented here, since it means that the overall design optimization time required by the GA will be essentially the same regardless of whether single-frequency or broadband array configurations are being considered.

Previous attempts to design low sidelobe linear antenna arrays using the GA have been limited to operation at a single frequency (i.e., for *B=1*) [1,5]. The GA approach introduced in this paper is also able to produce low sidelobe designs for *B=1* as a special case of a more general procedure which is valid for *B>1*. For example, given a uniformly excited 40 element array with a minimum element separation requirement of a quarter-wavelength (i.e., $\Delta_n = (d_{n+1} - d_n) \geq 0.25 \;\; \forall\, n = 1,2,\ldots,N-1$), the GA was able to generate an array with maximum sidelobe levels as low as –28.86 dB (see Figure 1a). Figure 1b shows the array factor of the same array with the main beam steered from 90^0 (broadside) to 91^0. Notice that steering the beam by even such a small amount as 1^0 in this case causes sidelobes to rise above the broadside maximum sidelobe level of –28.86 dB. This property is a direct consequence of the fact that the array is not designed to operate over a significant bandwidth, as predicted by the bandwidth-steerability product (2). It will be demonstrated in this paper, however, that significant steerability is possible for broadband arrays where *B>1*.

The first broadband design that will be considered is also a uniformly excited 40 element array. The minimum element separation requirement will be a quarter-wavelength at the lowest design frequency ($f = f_1$). In this case, the GA was able to optimize interelement spacings so that a bandwidth of $B = 3.5$ was possible for broadside operation with a maximum sidelobe level of -19.41dB throughout the entire band (see Table 1). Figures 2a–2c show plots of the array factor at the low-band ($f = f_1$), mid-band ($f = (f_1 + f_2)/2$), and high-band ($f = f_2$) design frequencies. The radiation patterns of an un-optimized, uniformly spaced 40 element array at the same three frequencies are shown in Figures 3a-3c for comparison purposes. Note that, as expected, the maximum sidelobe level under these conditions is about –12.5 dB. In addition, Figures 4a-4c show the optimized 40 element array from Figures 2a-2c with the main-beam steered to 60^0 also at low-band, mid-band, and high-band design frequencies. These figures demonstrate that, for the low-band ($f = f_1$) and mid-band ($f = 2.25 f_1$) frequencies, it is possible to steer the main-beam to 60^0 without any increase in the synthesized sidelobe level. However, for the high-band frequency ($f = 3.5 f_1$), we see that the synthesized sidelobe level can no longer be maintained when the beam is steered to 60^0. Further investigation reveals, as predicted by (2), that there is almost no steerability for this array when $f = f_2 = 3.5 f_1$. The maximum frequency at which this array exhibits perfect steerability was found to be $f = 1.75 f_1$.

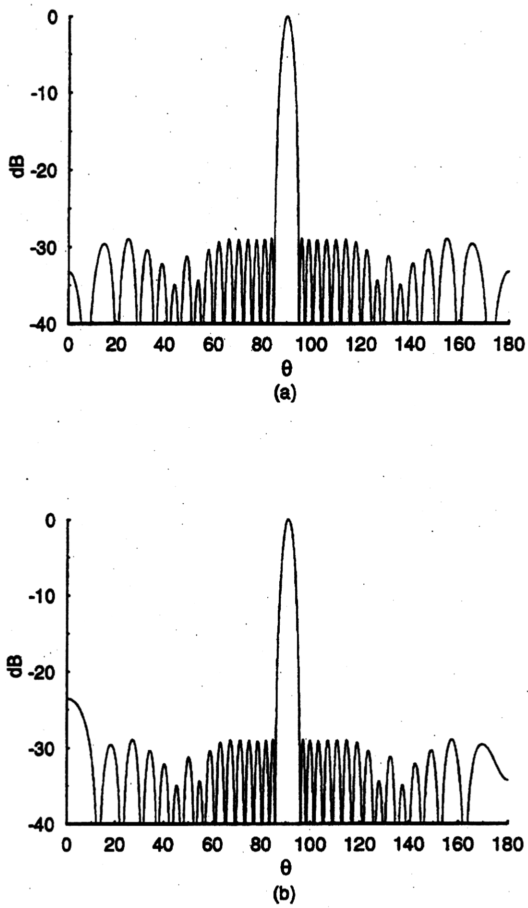

Figure 1. Array factor for a uniformly excited and nonuniformly spaced 40 element linear array of isotropic sources with (a) a broadside mainbeam and (b) the mainbeam steered to 91^0 (one degree from broadside). The maximum sidelobe level at broadside is –28.86 dB with a bandwidth of $B = f_2 / f_1 = 1$

Table 1. Element separations at $f = f_1$ for the GA-optimized 40 element linear array (see Figures 2a-2c).

Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)
1	0.125	6	1.375	11	2.625	16	3.955
2	0.375	7	1.625	12	2.885	17	4.215
3	0.625	8	1.875	13	3.135	18	4.475
4	0.875	9	2.125	14	3.435	19	5.625
5	1.125	10	2.375	15	3.695	20	6.485

Table 2. Element separations at $f = f_1$ for the GA-optimized 100 element linear array (see Figure 5).

Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)	Element Number (n)	Element Separation (s_n / λ_1)
1	0.125	14	3.375	27	6.865	40	11.355
2	0.375	15	3.625	28	7.115	41	11.615
3	0.625	16	3.875	29	7.365	42	12.105
4	0.875	17	4.125	30	7.615	43	12.355
5	1.125	18	4.375	31	7.865	44	13.115
6	1.375	19	4.625	32	8.115	45	13.365
7	1.625	20	4.875	33	8.365	46	14.095
8	1.875	21	5.365	34	9.095	47	14.345
9	2.125	22	5.615	35	9.345	48	14.595
10	2.375	23	5.865	36	9.595	49	15.085
11	2.625	24	6.115	37	9.845	50	15.585
12	2.875	25	6.365	38	10.105		
13	3.125	26	6.615	39	10.865		

Larger sized arrays were found to be capable of producing wider bandwidths. For example, using $\lambda_1 / 4$ as the minimum nterelement separation for a 100 element array, it was possible to optimize the array configuration using the GA to yield a bandwidth of $B = 3.97$ and a maximum sidelobe level of -20.32 dB (see Figure 5 and Table 2). The maximum frequency at which this array is steerable to 60^0 is at $f = 2.64 f_1$, and the maximum frequency at which this array exhibits perfect steerability is at $f = 1.98 f_1$ (see Figure 6). If larger interelement spacings are desired, then (1) may be used to determine the corresponding reduction in bandwidth that would result. For instance, increasing the $\lambda_1 / 4$ minimum interelement separation to 0.49625 λ_1 reduces the bandwidth from 3.97 down to 2. On the other hand, the bandwidth of the array can be doubled to $B = 7.94$ by allowing the minimum element separations to be as small as $\lambda_1 / 8$. Reducing the minimum element separation to $\lambda_1 / 8$ also doubles the bandwidth over which the array is steerable – i.e., in this case the above array would be perfectly steerable over a bandwidth of $B = 3.96$ with a maximum sidelobe level of –20.32 dB.

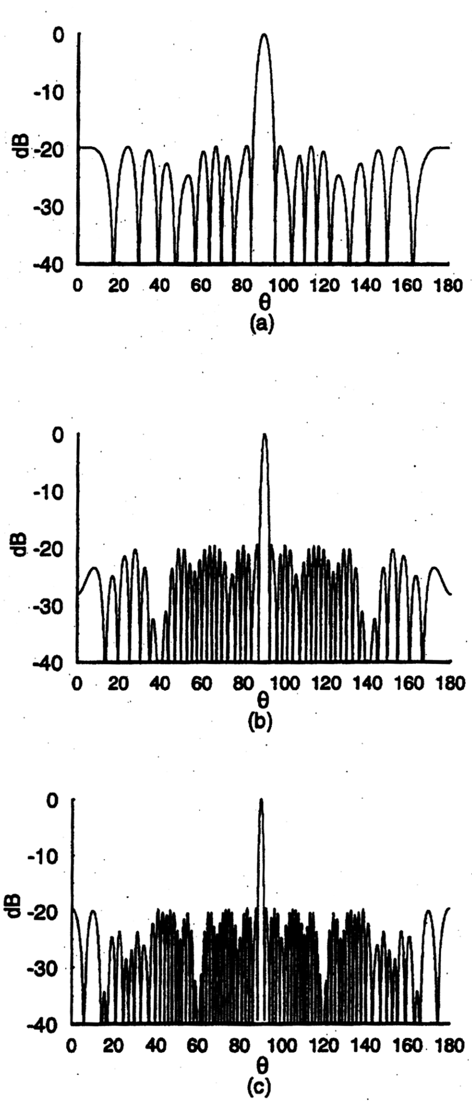

Figure 2. Plots of the array factor for an optimized broadside $(\theta_0 = 90^0)$ uniformly excited and nonuniformly spaced 40 element linear array of isotropic sources at (a) $f/f_1 = 1$, (b) $f/f_1 = 2.25$, and (c) $f/f_1 = 3.5$. The maximum bandwidth for this array is $B = f_2/f_1 = 3.5$.

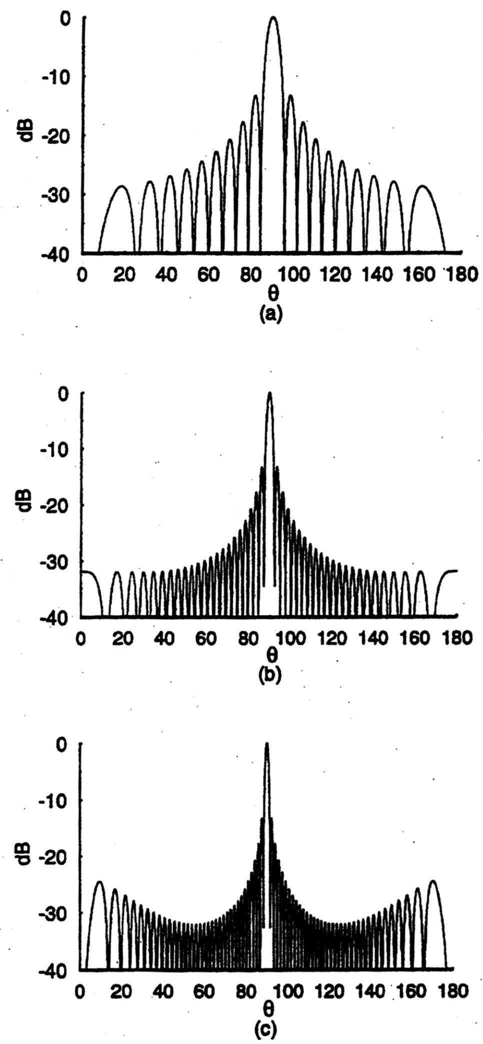

Figure 3. Plots of the array factor for a broadside $(\theta_0 = 90^0)$ uniformly excited and uniformly spaced 40 element linear array of isotropic sources at (a) $f/f_1 = 1$, (b) $f/f_1 = 2.25$, and (c) $f/f_1 = 3.5$.

4. Planar Broadband Array Designs

The GA optimization procedure described in the previous section for broadbanding linear arrays will be generalized in this section to include planar array configurations. In particular, the GA design approach will be developed for rectangular arrays as well as for concentric circular arrays with variable element spacings. The array factor for a non-uniformly spaced symmetric rectangular array of isotropic sources may be represented in the following form:

$$AF(\theta,\phi)=4\sum_{n=1}^{N}\sum_{m=1}^{M} I_{mn}\cos[2\pi d_{xn}(f/f_1)\sin\theta\cos\phi] \cdot\cos[2\pi d_{ym}(f/f_1)\sin\theta\sin\phi] \tag{6}$$

where

$2M$ = total number of elements in the y-direction
$2N$ = total number of elements in the x-direction
$s_{xn}=d_{xn}\lambda_1$ = element locations in the x-direction with respect to the origin
$s_{ym}=d_{ym}\lambda_1$ = element locations in the y-direction with respect to the origin

The corresponding RSLL in this case is calculated from

$$F(\theta,\phi)=\max\left|4\sum_{n=1}^{N}\sum_{m=1}^{M} I_{mn}\frac{\cos[2\pi d_{xn}B\sin\theta\cos\phi]\cos[2\pi d_{ym}B\sin\theta\sin\phi]}{AF_{\max}(\theta,\phi)}\right| \tag{7}$$

The GA uses (7) to determine the set of parameters d_{xn} and d_{ym} that yields the lowest possible sidelobe level over a specified bandwidth *B*, assuming that the array is uniformly excited (i.e., $I_{mn}=1$ for all values of m and n). In order to accomplish this, a spacing scheme was designed such that rows and columns were treated the same. For example, the GA selects a set of spacings $s=\{d_1,d_2,d_3,\ldots,d_n\}$, where n is the number of rows and columns in the array (i.e., $N=M=n$ where N and M are from (6)). The number d_i, $1\le i\le n$, then represents the interelement spacing between elements (i,j) and $(i-1,j)$ and the elements (j,i) and $(j,i-1)$ $\forall j \ni 1\le j\le n$, where the indices $(0,j)$ and $(j,0)$ are the y and x axes respectively. This scheme makes the objective function very simple to evaluate because the maximum sidelobe level will always be located in the $\phi=0^0$ and $\phi=90^0$ planes. Figures 7a-7c show radiation pattern cuts at $\phi=0^0$, $\phi=45^0$, and $\phi=90^0$, respectively, for a 4,096 element array which was designed to produced a bandwidth of $B=3.5$ and a maximum sidelobe level of –19.41dB with a minimum specified element separation of $\lambda_1/4$.

Next we will consider an alternative design optimization approach based on concentric circular arrays which results in a more spatially uniform distribution of sidelobes. The RSLL in this case is calculated from

$$F(\theta,\phi)=\max\left|\frac{\sum_{m=1}^{M}\sum_{n=1}^{N_m} I_{mn}\exp[j2\pi Ba_m\sin\theta\cos(\phi-\phi_{mn})+j\alpha_{mn}]}{AF_{\max}(\theta,\phi)}\right| \tag{8}$$

where

$$AF(\theta,\phi)=\sum_{m=1}^{M}\sum_{n=1}^{N_m} I_{mn} \cdot\exp[j2\pi(f/f_1)a_m\sin\theta\cos(\phi-\phi_{mn})+j\alpha_{mn}] \tag{9}$$

$$\alpha_{mn}=-2\pi(f/f_1)\sin\theta_0\cos(\phi_0-\phi_{mn}) \tag{10}$$

and

$r_m=a_m\lambda_1$ = radius of the m*th* ring array
M = total number of concentric ring arrays
N_m = total number of elements in the m*th* ring

The spacing scheme was designed such that elements were placed on arcs spaced $\lambda_1/4$ apart, where λ_1 corresponds to the wavelength at the lowest design frequency f_1. In addition to this, the elements in each quadrant were assumed to be arranged symmetrically about their respective diagonal axis (e.g., the elements in the first quadrant are symmetric with respect to the $\phi=45^0$ axis). Figure 8 shows one quadrant of a concentric circular array in the equally spaced case that could be constructed using this spacing scheme. For this example, the minimum arc length between any two consecutive array elements was set to $\lambda_1/4$. Figure 9 shows the radiation pattern produced over the $\phi=45^0$ cut by a 308 element concentric circular array with element spacings optimized to yield a bandwidth of $B=3.5$ with a maximum sidelobe level of –21.91dB throughout the band.

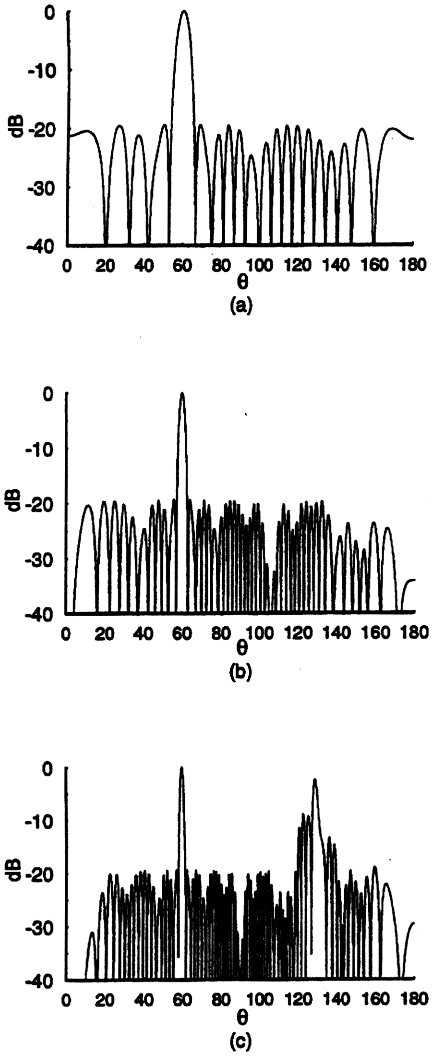

Figure 4. Plots of the array factor for an optimized uniformly excited and nonuniformly spaced 40 element linear array of isotropic sources with $\theta_0 = 60^0$ at (a) $f/f_1 = 1$, (b) $f/f_1 = 2.25$, and (c) $f/f_1 = 3.5$.

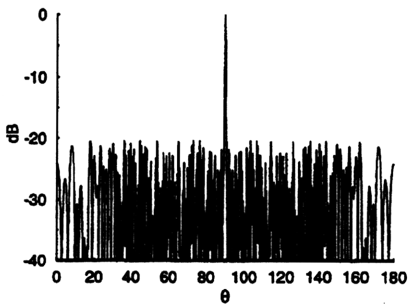

Figure 5. Array factor for an optimized uniformly excited and nonuniformly spaced 100 element linear array of isotropic sources with $f/f_1 = 3.97$.

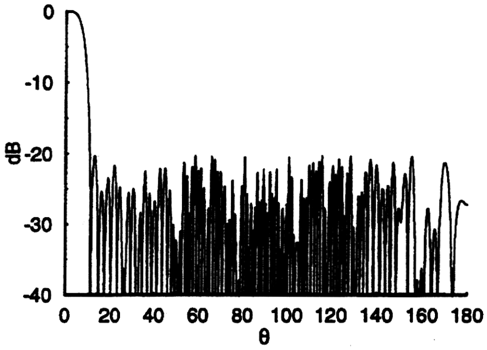

Figure 6. Array factor for an optimized uniformly excited and nonuniformly spaced 100 element linear array of isotropic sources with $\theta = \theta_0$ and $f/f_1 = 1.98$.

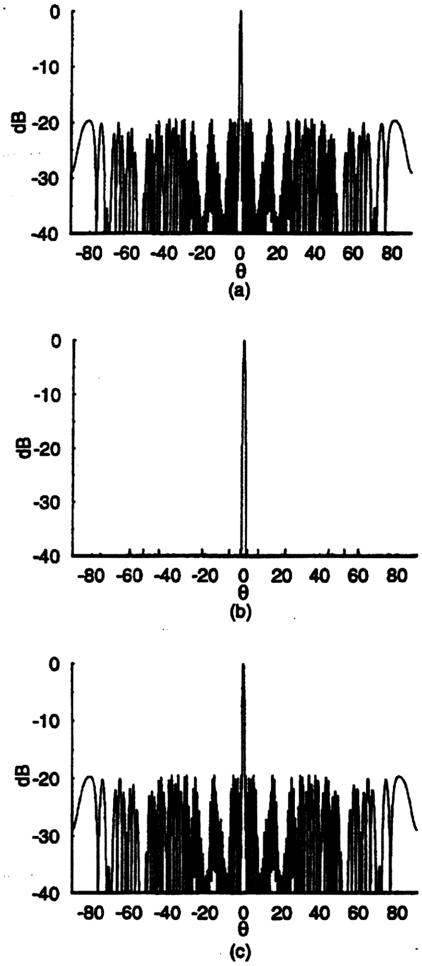

Figure 7. Radiation pattern cuts at $\phi = 0^0$ (Figure 7a), $\phi = 45^0$ (Figure 7b), and $\phi = 90^0$ (Figure 7c) of an optimized 4,096 element square planar array with $f/f_1 = 3.5$.

5. Conclusions

Uniformly excited array broad-banding has been achieved using a genetic algorithm optimization procedure with bandwidths as large as $B = 3.97$ for linear arrays and $B = 3.5$ for planar arrays with a minimum element separation of $\lambda_1/4$. Minimum element separation can easily be made larger to avoid mutual coupling effects, or it can be made smaller to increase bandwidth by using the convenient conversion factor given in (1). Array steerability issues have also been addressed in this paper. Steerability varies with operation frequency as predicted by (2) – it is greater at lower frequencies of operation and lesser at higher frequencies of operation. In addition, the bandwidth over which the array is steerable improves proportionally as (1) is used to increase bandwidth. It should also be noted that in order to include steerability within the optimization scenario (in the sense of a multi-objective constraints synthesis procedure), we could adopt a more general definition of the objective function that includes the right-hand side of (2). Finally, we point out that even lower sidelobe levels might be achieved in some cases by including the element pattern in the optimization scheme.

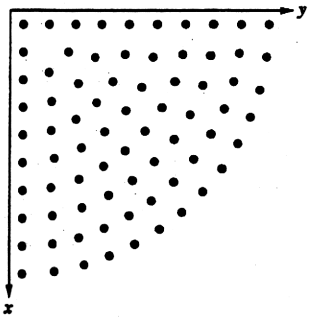

Figure 8. One quadrant of an equally spaced concentric circular ring array that is arranged symmetrically about its diagonal axis.

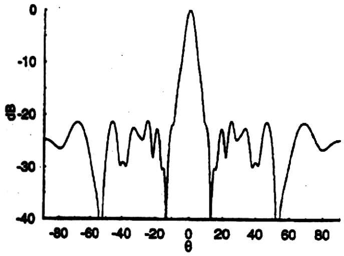

Figure 9. Radiation pattern cut at $\phi = 45^0$ for an optimized broadband concentric circular ring array with $f/f_1 = 3.5$.

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