Minimizing The 3D Solenoidal Basis Set in Method of Moments Based Volume Integral Equation

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Abstract **─** A method for completely minimizing a 3D-solenoidal basis set, and identifying its linearly independent basis functions in method of moments (MoM) based volume integral equation (VIE) is presented. The method uses the connecting information of the geometric mesh and the properties of the 3D solenoidal basis function to remove the null space of the basis set. Consequently, the approach is not prone to numerical inaccuracies due to finite machine precision resulting from matrix manipulations. In addition, our method is not restricted to simply connected or contiguous mesh regions; it is applicable to a wide variety of complicated mesh regions featuring holes and voids. Finally, an expression for determining the minimum number of linearly independent solenoidal basis functions in a given tetrahedral mesh is presented.

Index Terms – Divergence-free, method of moments (MoM), and solenoidal edge basis function.

I. INTRODUCTION

When applying VIE based method of moments (MoM) formulations in electromagnetic problems involving arbitrary shaped three-dimensional (3D) bodies, the solution domain is best discretized using a number of tetrahedral elements [1]. In each tetrahedral element, a basis function is defined to best approximate the properties of the electromagnetic quantity of interest. It has been shown that the choice of an appropriate basis function is of critical importance when applying MoM formulations [2-4]. The most widely used basis function for 3D VIE modeling is the Schaubert-Wilton-Glisson (SWG) basis function defined in [1]. It is well suited for modeling the electric flux density **D** as it enforces the boundary condition of a continuous normal component of **D** on the faces of the tetrahedra where it is defined. However, the SWG basis function does not have the property of zero-divergence, which is demanded of **D** in a dielectric. This has encouraged a number of authors to use a divergence-free or solenoidal basis function that more accurately describes the physics within a dielectric [5, 6]. One such basis function is the 3D solenoidal basis introduced by de Carvalho*et al* [7, 8] for modeling the electromagnetic scattering of inhomogeneous dielectrics. Kulkani*et al* [2] demonstrated considerably better performance of the 3D-solenoidal basis function compared with the SWG basis function, including the fact that there was a significant reduction (1.67 to 2 times) of the number of unknowns for the same tetrahedral mesh. Although, their analysis was promising, they stressed the difficulty of implementing a preliminary conditioning operation to remove the null space associated with the solenoidal basis set.

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Several approaches were proposed and implemented in order to identify the linearly independent 3D solenoidal basis functions in a given tetrahedra mesh. In [2], the Gram or covariance is formed and reduced by row operations to an echelon form using Gauss-Jordan elimination with partial pivoting. This is a complex method involving matrix manipulations that are prone to numerical inaccuracies due to finite machine precision. As a result, methods utilizing only the connecting information in the geometric mesh have been developed by several authors. In one method, reference [5] constructed an undigraph using the nodes and edges in the geometric mesh, and created a generating tree linking the nodes and edges. The limitation of this method is that it is only applicable to simply connected mesh regions (or mesh regions without holes). In another method, reference [3] proposed a solution where the nodes of the mesh are counted and tested, one-by-one, with a set of established criteria. Unfortunately, this method like the previous one is not applicable to objects and regions with holes.

In this paper, we expand on the work reported in [3] and introduce a simple, yet robust method for determining the linearly independent 3D solenoidal basis functions in a given tetrahedral mesh. The proposed method is based on the connecting information in the geometric mesh and the properties of the 3D solenoidal basis function. As such, it is not prone to numerical inaccuracies resulting from complex matrix manipulations. It is also applicable to all mesh regions with multiple holes and voids. Specifically, it can be used to accurately determine the number of voids in a given tetrahedral mesh. In addition, an accurate expression for determining the minimum number of linearly independent basis functions in a given mesh is presented.

II. THEORY AND FORMULATION

A. Basis definition

The 3D-solenoidal basis function is defined within a tetrahedron of volume V as shown in Fig. 1. It is a constant vector field that is perpendicular to the basis edge vector as depicted in Fig. 1. In this case, vector $\overrightarrow{\boldsymbol{CD}}$ denotes the basis edge vector while vector $\overrightarrow{\boldsymbol{AB}}$, represents the opposite edge vector parallel to the constant vector field. For any given point within the tetrahedron, the basis function with respect to the corresponding basis edge vector can be expressed mathematically as,

$$\boldsymbol{f}(\mathbf{r}) = \frac{\mathbf{e}}{3\boldsymbol{V}}\mathbf{u}(\mathbf{r}) \tag{1}$$

where

$$\mathbf{u}(\mathbf{r}) = \begin{cases} \mathbf{1} & \mathbf{r} \in \boldsymbol{V} \\ \mathbf{0} & \mathbf{r} \notin \boldsymbol{V} \end{cases}. \tag{2}$$

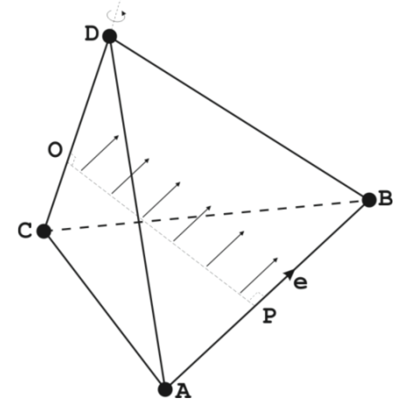


Fig. 1. Definition of the 3D solenoidal basis function in a tetrahedron.

The basis function $\boldsymbol{f}(\mathbf{r})$ defined in equation (1) is such that its divergence within the tetrahedron is zero, i.e., $\boldsymbol{f}(\mathbf{r})$ is solenoidal. Also, $\boldsymbol{f}(\mathbf{r})$ has the desired property that the normal component of its flux is continuous across any internal face boundary, and the total flux of its normal component through any face that is not parallel to $\boldsymbol{f}(\mathbf{r})$ is equal to one. In a typical tetrahedral mesh region consisting of multiple interconnecting tetrahedra, equation (1) can be modified to include the contributions from all neighbouring tetrahedra that share a common basis edge vector giving,

$$\mathbf{F}_{\boldsymbol{i}}(\mathbf{r}) = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{N_T}} \frac{\mathbf{e}_{\boldsymbol{k}}}{3\boldsymbol{V}_{\boldsymbol{k}}}\mathbf{u}_{\boldsymbol{k}}(\mathbf{r}) \tag{3}$$

where N_T is the total number of tetrahedra connected to the i^{th} edge. With the definition of equation (3), any physical electromagnetic quantity that is solenoidal can be approximated in the discretized region using the 3D solenoidal basis function. Examples of physical quantities include the electric flux density $\mathbf{D}$ in a pure dielectric, the magnetic flux density $\mathbf{B}$, the curl of the magnetic field $(\nabla \times \mathbf{H})$ or the total volumetric current density, and the curl of the electric field $(\nabla \times \mathbf{E})$.

B. Size of the basis set

It was shown in [2] that the size of the 3D-solenoidal basis set is far smaller than the number of edges in the mesh region. The number of linearly independent solenoidal basis functions is given by,

$$\boldsymbol{N_{Basis} \leq N_{Faces} - N_{Tetrahedra}} \tag{4}$$

where N_{Basis} is the number of linearly independent solenoidal basis functions, N_{Faces} is the number of faces, and $N_{Tetrahedra}$ is the number of tetrahedra in the mesh. When the meshregion is not simply connected, i.e., is made up of holes and voids, equation (4) can be modified into,

$$\boldsymbol{N_{Basis} = N_{Faces} - N_{Tetrahedra} - N_{Voids}} \tag{5}$$

where N_{Voids} is the number of voids in the mesh. Equation (5) can be explained by considering a mesh region with no voids. A simple void can be created by removing an internal tetrahedron from the mesh. In this case, the previous number of independent solenoidal basis function will not change since the normal component of flux from the remaining neighbouring tetrahedra into the void is unchanged. Consequently, the number of voids has to be accounted for as in equation (5) since there is no change in the number of faces. The simple void can be made larger by the continuous removal of one tetrahedron and the faces it introduced in the region bounding the simple void. In this case, equation (5) is unaltered because each time a tetrahedron is removed, the exact number of faces it introduced in the mesh is also removed. Using a similar argument, the inclusion of a hole has no effect on equations (4) or (5). This can be explained by considering a simple hole that is created by removing exactly one tetrahedron and the faces it introduced in the mesh region. This simple hole can be extended by the continuous removal of exactly one tetrahedron and the faces it introduced in the region bounding the simple hole. In this case, the number of independent solenoidal basis function is still given by equations (4) or (5), since the number of tetrahedra and faces removed remain balanced.

C. Minimization Algorithm

Consider a mesh region containing a single tetrahedron as shown in Fig. 2. Formally, there are six defined solenoidal basis functions corresponding to the six labeled edges. In $\Re^3$ space, only three of these functions are linearly independent, and a more natural choice of these functions would be $\mathbf{F}_4(\mathbf{r})$, $\mathbf{F}_5(\mathbf{r})$, and $\mathbf{F}_6(\mathbf{r})$ defined on edges 4, 5, and 6 as shown in Fig. 2. In addition, edges 4, 5, and 6 form the basis function face of the tetrahedra. We define the seed node N_1 as the node that is opposite to the basis function face associated with the tetrahedron. Also, we describe the neighbouring edges of node N_1 as those edges that contain node N_1: edges 1, 2, and 3 respectively. A typical mesh region can be constructed by the addition of more tetrahedra to the single tetrahedron structure of Fig. 2. When tetrahedra are added, they will share nodes, edges or faces depending on the mesh geometry.

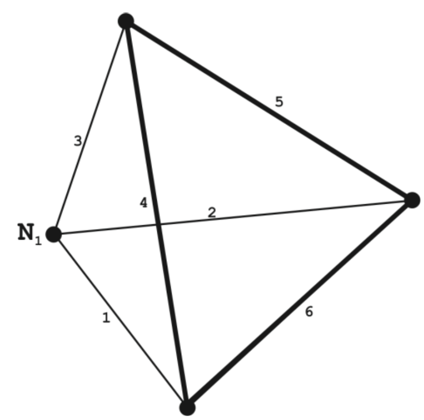


Fig. 2. 3D solenoidal basis functions defined as bold lines on the opposite face to seed node N_1.

According to [3], all tetrahedra having the same seed node are grouped together to form a structure very similar to an icosahedron (soccer ball structure [3]) where the seed node is at the center of the structure. The basis functions associated with all the tetrahedra in this structure are defined on the edges of the exterior faces. Subsequently, nodes on these exterior faces are used in turn as seed nodes to create more structures and this process continues as described in [3]. However, as more of these icosahedra like structures are formed, a situation will arise when a newly formed structure shares a node, an edge or a face with one or more previously formed structures. When this occurs, (5) is no longer satisfied,leading to an incorrect result. If the mesh region is simply connected or contiguous, then this problem can be avoided by appropriate seed node selection as discussed in [3]. In order to resolve this problem for non-contiguous mesh regions with multiple holes and voids, we introduce additional solenoidal basis functions in the structure when it shares one or more edges, or faces with one or more previously formed structures. These additional functions are determined by the nature of the contact between the structures. For simplicity, we will consider two possible scenarios for contact: edge contact and face contact between any two tetrahedra of the structure as shown in Fig. 3. Using equation (5) on the edge contact configuration shown in Fig. 3 (a) indicates that six and not five solenoidal basis functions are required. In order to find out which additional basis functions are required while still adhering to the icosahedra approach, we observe that basis function $\mathbf{F}_6(\mathbf{r})$ will be modified by the contact based on equation (3). Since the new tetrahedron shares an edge with the previous defined one, $\mathbf{F}_6(\mathbf{r})$ is modified accordingly into,

$$\mathbf{F_6}(\mathbf{r})_{\boldsymbol{New}} = \mathbf{F_6}(\mathbf{r})_{\boldsymbol{Previous}} + \frac{\mathbf{e_9}}{\mathbf{3}\boldsymbol{V_9}}\mathbf{u_9}(\mathbf{r})\,. \qquad (6)$$

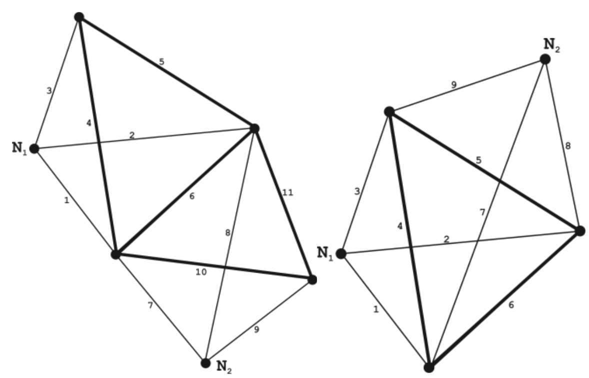
This modification will destabilize the previous choice of basis functions in the structure, and basis functions $\mathbf{F}_4(\mathbf{r})$, $\mathbf{F}_5(\mathbf{r})$, $\mathbf{F}_6(\mathbf{r})$, $\mathbf{F}_{10}(\mathbf{r})$, and $\mathbf{F}_{11}(\mathbf{r})$ no longer linearly combine to produce the dependent basis $\mathbf{F}_1(\mathbf{r})$, $\mathbf{F}_2(\mathbf{r})$, and $\mathbf{F}_3(\mathbf{r})$. However, since

$$\begin{aligned}\frac{\mathbf{e_9}}{\mathbf{3}\boldsymbol{V_9}}\mathbf{u_9}(\mathbf{r}) &= \mathbf{F_7}(\mathbf{r}) \pm \mathbf{F_{10}}(\mathbf{r}) \\ &= \mathbf{F_8}(\mathbf{r}) \pm \mathbf{F_{11}}(\mathbf{r})\,, \qquad (7)\end{aligned}$$

we can choose either $\mathbf{F}_7(\mathbf{r})$ or $\mathbf{F}_8(\mathbf{r})$ as an additional basis function so that the dependent bases can now be determined. Similarly, it can be shown that for the contact configuration in Fig. 3 (b), two additional solenoidal basis functions must be chosen from the set $\{\mathbf{F}_7(\mathbf{r}), \mathbf{F}_8(\mathbf{r}), \mathbf{F}_9(\mathbf{r})\}$, since we have that,

$$\mathbf{F_7}(\mathbf{r}) \pm \mathbf{F_8}(\mathbf{r}) \pm \mathbf{F_9}(\mathbf{r}) = \mathbf{0}. \qquad (8)$$

Using the results from these configurations, we describe an algorithm to determine the linearly independent solenoidal basis functions from the set of all edges. A flow diagram of the algorithm is depicted in Fig. 4. The first process is to specify an arbitrary node as the head node and insert it into an empty linked list. Next, we pick the head or first node from the list as the first seed node, and we find all tetrahedra sharing this node. A two-step validation test is then performed on all tetrahedra found. The first step involves checking each tetrahedron found for one or more neighbouring edges that are not shared with any tetrahedra of a previous seed node, and one or more nodes on the basis function face that are already in the linked list. This is the mesh continuity test as defined in [3]. If no tetrahedron is found that satisfies these conditions then the test is successful. On the other hand, if one or more tetrahedra fail the test then we apply the next step of validation. The introduction of this second step is the key contribution of this paper that differentiates it from the work in [3]. The second step examines each tetrahedron that failed the mesh continuity test to determine how they touch the other tetrahedra from previously listed seed nodes. For this test to be successful, all tetrahedra must touch other tetrahedra from previously listed seed nodes as shown in either contact configuration of Fig. 3. If one or more tetrahedra touch in any other way, the seed node is removed and inserted at the end of the linked list and the process is repeated with a new seed node. Upon a successful outcome from validation testing, the appropriate edges of the basis function faces or neighbouring edges of all tetrahedra found are selected as basis function, and the nodes on the basis function faces are inserted into the linked list. This process is repeated until all nodes in the mesh are exhausted.



(a) (b)

Fig. 3. Possible contact configuration of two tetrahedra from seed nodes N_1 and N_2: (a) tetrahedra share an edge and (b) tetrahedra share a face.

III. EXAMPLES AND TESTING

In this section, we challenge our algorithm using different tetrahedra meshes of simple and complex geometrical structures containing multiple holes and voids. These meshes were created with the software package Netgen [9]; they are depicted in Fig. 5. The algorithm was implemented and executed on an Intel Core i5-2520M 2.5 GHz PC with 4.0 GB of RAM running

Open Suse 12.1. We tabulate the mesh parameters of each discretized region, as well as the output from the algorithm into Table I .In every case considered, the algorithm is able to identify the independent 3D solenoidal basis functions. Also, we observe that equation (5) is in agreement with the output from the algorithm. The algorithm is indeed very robust, and is applicable to a very broad class of complex meshes with multiple holes and voids, as seen. Moreover, the complete minimum number of independent solenoidal bases is always smaller than the number of edge bases in the discretization (typically by a factor ranging between 16% and 20%), resulting in further reduction in memory resources. The execution time for large meshes is in the order of a few seconds, and is less than a second for smaller meshes (fewer than 19,000 edges). These results are an indication of superior performance when compared to linear algebraic methods using the same PC hardware. Finally, the algorithm is very straight forward to implement, and does not suffer from numerical inaccuracies due to floating point matrix operations since it uses the connecting information in the mesh.

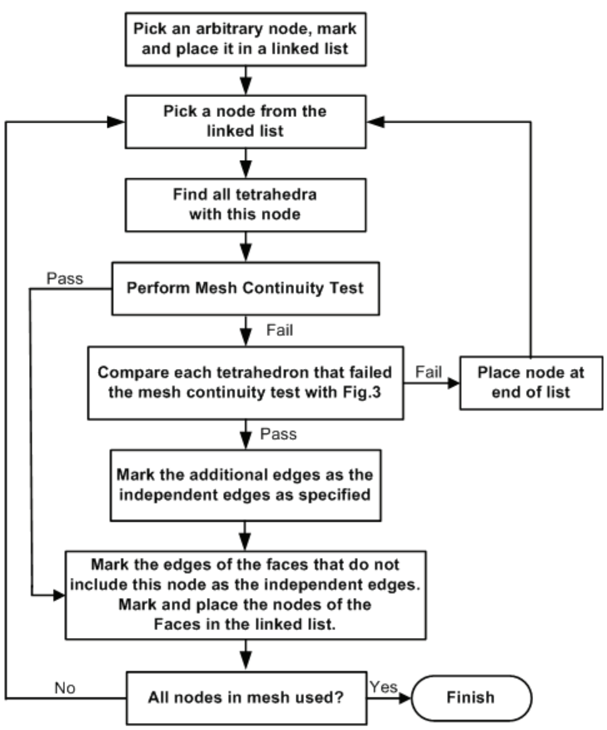


Fig. 4. Algorithm for identifying linearly independent basis functions.

Table I. Output from the algorithm for the mesh regions shown in Fig. 5.

Mesh Region	N_{Edges}	N_{Faces}- $N_{Tetrahedra}$– N_{Voids}	N_{Basis} (Algorithm output)	**Time** (s)
Fig.5(a)	12864	10723	10723	< 1
Fig.5(b)	11065	9253	9253	< 1
Fig.5(c)	187547	156001	156001	199
Fig.5(d)	24177	19455	19455	1
Fig.5(e)	34957	28240	28240	4
Fig.5(f)	13001	10544	10544	< 1

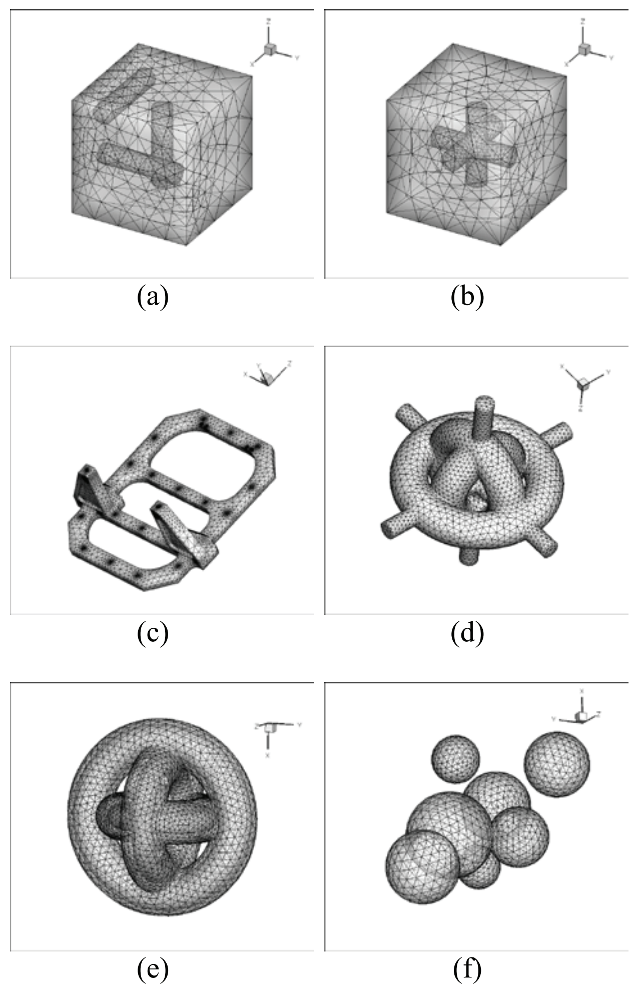


Fig. 5. Mesh regions generated with NETGEN: (a) cube with three cylindrical voids, (b) cube with a single void of merged cylinders, (c) steel frame with multiple holes, (d) gyroscope, (e) three connected tori with single connecting void, and (f) group of five connecting spheres and two separated spheres.

IV. CONCLUSION

In this paper, we presented a novel method for the determination of the minimum number of 3D-solenoidal basis functions in a given tetrahedral discretization. The method was derived using the connecting information in the mesh structure and the properties of the 3D-solenoidal basis function. The method is able to identify the independent basis functions even in the presence of holes and voids within the mesh structure. To our knowledge, this is the only algorithm that is insensitive to the presence of holes and voids in the mesh discretization. In addition, it provides significant savings in computational resources when compared to algorithms involving complex matrix manipulations. Finally, it is worthy of note that the contact configurations of Fig. 3 can be expanded to accommodate other ways tetrahedra from two or more seed nodes can come into contact. This will provide an improvement in the speed of the algorithm at the expense of an increase in the complexity of identifying the type of contact, and the additional basis edges resulting from that contact. The chosen contact configuration is simple and sufficient to guarantee proper functioning of the algorithm in any region.

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