Numerical Solution of Electromagnetic Scattering by Multiple Cylinders

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***Abstract* –** A numerical solution based on integral equation is derived for an electromagnetic scattering from M multiple parallel cylinders. The problem is two-dimensional and the integral equation is solved using the Nyström method. To validate the algorithm, we compare our numerical results with the semi-analytical ones obtained from multipole expansion method.

I. INTRODUCTION

Problems of multiple scattering are of significant importance in many areas of technology. Indeed, many wave propagation problems can be modeled as such. Examples include electromagnetic and optical communication, imaging, object characterization, electronic and optical components, etc. Hence, the development of efficient and accurate numerical simulation for such problems is highly desirable. In this paper we discuss an efficient computational algorithm for the problem of approximating the scattered electromagnetic field from two dimensional multiple parallel dielectrics of arbitrary cross-sections. For the sake of clarity, we only consider the TM polarization case. The method caries out easily for other polarizations as well. When solving this type of problems, there are two possible directions to follow. The first is to analytically treat a simplified model [1,2] that captures the relevant properties of the actual problem. Although analytic solutions are rarely possible for the structures of arbitrary realistic complexity, they provide a closed mathematical description, and in most cases a better understanding, of the solution. The second class of algorithms utilize numerical methods [3,4] to treat more realistic descriptions of the underlying physics. However, in exchange, it can be difficult to find fast and accurate computational model. Since, for many applications, the assumption of simple geometry is far from warranted we develop in this paper an efficient algorithm that can handle complex geometries. In particular, our method is based on boundary element method (BEM). Usually, for BEM approximations, the implementation is based on either Green's theorem in each dielectric objet [5,6] or the use of single and/or double layer potentials [7]. In the case of one dielectric object, both methods lead to a pair of integral equations for a pair of unknowns. We deduce that, by using these approaches for multiple dielectric scatterers, for M interfaces we have $2M$ unknown functions to determine. For one dielectric object a single integral equation involving one unknown function was obtained [8] by using a hybrid of integral equation and Green's theorem. It is also possible to obtain single integral equations by using the extended boundary condition method [5]. But this later method suffers from the choice of the boundary as well as ill-posedness. The purpose of this paper is to obtain an efficient numerical solution of single Fredholm type integral equations on each interface for multiple dielectric scattering by the use of boundary layers and Green's formula. The method, which reduces the number of unknowns by half, converges very fast and is accurate. The numerical computation is implemented by using the Nyström method. Our results are validated by numerical examples for circular cylinders where analytic solution is found by using the multipole expansion method.

II. THE MATHEMATICAL FORMULATION OF THE PROBLEM

Let Ω_l, $l = 1, 2, \cdots M$, be the cross-sections of M parallel cylinders, describing the scatterers (Fig. 1) and

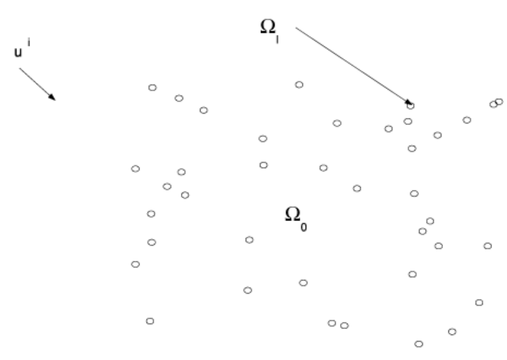

Fig. 1. Multiple scattering of a plane wave u^i by many cylinders.

Let Γ_l be the boundary of Ω_l. The unit outward normal ν to Γ_l is assumed to be directed towards the exterior. We denote the field outside (in the air) as Ω_0, i,e. $\Omega_0 = \mathbf{R}^2 \backslash \cup_{l=1}^{M} (\overline{\Omega}_l)$.

For simplicity we consider an s-polarized field incident upon the dielectric (nonmagnetic) cylinders of cross-sections Ω_l, with the electric field parallel to the x_3-axis. But generalization to other polarizations and materials does not present any difficulty.

Each domain Ω_l $l = 0, 1, \cdots M$, has permittivity ϵ_l. The scatterers are assumed to be illuminated by an incident field E^i which is a plane wave with direction $\mathbf{d}$ and angle α, i.e. $\mathbf{d} = (\cos\alpha, \sin\alpha)$. With use of a time dependence in $e^{-j\omega t}$ (ω is the frequency and $j = \sqrt{-1}$), the incident electric field is given, for every point $\mathbf{x} = (x_1, x_2)$, by,

$$E^i(\mathbf{x}) = e^{j\omega\sqrt{\epsilon_0}\mathbf{x}\cdot d}.$$

Then (cf. [1]) we have to solve the Helmholtz equation in each dielectric object Ω_l, $l = 1, 2, \cdots M$ and in the outer region Ω_0,

$$(\nabla^2 + \kappa_l^2)E_l = 0 \quad \text{in} \quad \Omega_l, \quad l = 0, \cdots, M$$

where the wave numbers κ_l are given by $\kappa_l = \omega\sqrt{\epsilon_l}$. For $l = 1, 2, \cdots M$, the electric field E_l represents E in Ω_l, and in Ω_0 we have $E = E_0 + E^i$, where E_0 is the scattered electric field.

In addition, E_0 must satisfy the Sommerfeld radiation condition, i.e.,

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{1/2} \left(\frac{\partial E_0}{\partial |\mathbf{x}|} - j\kappa_0 E_0 \right) = 0..$$

We denote the fundamental solution to the Helmholtz equations (the free-space source) by,

$$\Phi_k(\mathbf{x}, \mathbf{y}) = -\frac{j}{2} H_0^{(1)}(\kappa_k |\mathbf{x} - \mathbf{y}|), \quad k = 0, 1, \cdots, M$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero. We use the factor $j/2$ (instead of the standard $j/4$) for convenience in the derivation of the integral equations below. In the sequel we shall assume that $\epsilon_0 = 1$.

III. THE INTEGRAL EQUATION APPROACH TO SOLVE THE PROBLEM

We would like to obtain a set of M equations with M unknowns on each boundary Γ_l of Ω_l, $l = 1, \cdots, M$. Now, for $k = 0, 1, \cdots M$, $l = 1, 2, \cdots, M$ and (density) functions ϕ_l, ψ_l, define the single and double layer potentials,

$$S_k^l \phi_l(\mathbf{x}) = \int_{\Gamma_l} \Phi_k(\mathbf{x}, \mathbf{y}) \phi_l(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^2 \backslash \Gamma_l$$

and

$$D_k^l \psi_l(\mathbf{x}) = \int_{\Gamma_l} \frac{\partial}{\partial \nu(\mathbf{y})} \Phi_k(\mathbf{x}, \mathbf{y}) \psi_l(\mathbf{y})\, ds(\mathbf{y}), \mathbf{x} \in \mathbf{R}^2 \backslash \Gamma_l,$$

respectively. Their normal derivatives at some point on a boundary Γ_m, $m \neq l$, are given by,

$$M_k^{l,m} \phi_l(\mathbf{x}) = \frac{\partial}{\partial \nu(\mathbf{x})} S_k^l \phi_l(\mathbf{x}), \quad \mathbf{x} \in \Gamma_m$$

and

$$N_k^{l,m} \psi_l(\mathbf{x}) = \frac{\partial}{\partial \nu(\mathbf{x})} D_k^l \psi_l(\mathbf{x}), \quad \mathbf{x} \in \Gamma_m.$$

Accordingly we shall denote $S_k^{l,m}\phi_l(\mathbf{x})$ and $D_k^{l,m}\psi_l(\mathbf{x})$ the values of $S_k^l \phi_l(\mathbf{x})$ and $D_k^l \psi_l(\mathbf{x})$ when $\mathbf{x}$ belongs to Γ_m, $m \neq l$.

It is known (cf. [9] Sections 2.4 and 2.5) that when $\mathbf{x}$ approaches Γ_l, S_k^l and N_k^l are continuous whereas D_k^l and M_k^l exhibit jumps. In particular,

$$S_k^l = \hat{S}_k^l, N_k^l = \hat{N}_k^l, D_k^l = \hat{D}_k^l \mp I, M_k^l = \hat{M}_k^l \pm I \quad (1)$$

where the upper (lower) sign corresponds to the limit when $\mathbf{x}$ approaches Γ_l from outside (inside) and I is the identity operator. The hats on the operators mean the case when $\mathbf{x} \in \Gamma_l$.

To arrive at the desired integral equation we define a layer ansatz (a combination of single and double layer potentials) in Ω_l, $l = 1, 2, \cdots, M$, and apply Green's theorem in Ω_0. So, for $l = 1, 2, \cdots, M$, let,

$$E_l = \left(-j\rho_l S_l^l + D_l^l\right) \phi_l(\mathbf{x}) \quad \mathbf{x} \in \Omega_l$$

where ρ_l are arbitrary nonzero complex numbers.

We have, by jump relations (1),

$$\begin{cases} E_l = P_l^l \phi_l \\ \frac{\partial}{\partial \nu} E_l = Q_l^l \phi_l \end{cases} \quad \text{on} \quad \Gamma_l \quad (2)$$

where $P_l^l = -j\rho_l \hat{S}_l^l + (I + \hat{D}_l^l)$ and $Q_l^l = -j\rho_l(-I + \hat{M}_l^l) + \hat{N}_l^l$.

In the exterior region, we use Green's theorem to obtain (cf. [9] pp. 68-70),

$$\begin{cases} E_0(\mathbf{x}) = \sum_{l=1}^{M} \left(S_0^l \frac{\partial}{\partial \nu} E(\mathbf{x}) - D_0^l E(\mathbf{x}) \right), \mathbf{x} \in \Omega_0, \\ f(\mathbf{x}) = \sum_{l=1}^{M} \left(D_0^l E(\mathbf{x}) - S_0^l \frac{\partial}{\partial \nu} E(\mathbf{x}) \right), \mathbf{x} \in \mathbf{R}^2 \backslash \overline{\Omega}_0, \end{cases} \quad (3)$$

where $f = 2E^i$.

Now, using the jump relations (1), we obtain the second equation in the system (3) on Γ_l, $l = 1, 2, \cdots M$. Using the boundary conditions, and substituting E_l and $\partial E_l / \partial \nu$ (given in equation (2)) into these equations we arrive at a set of M integral equations with M unknowns ϕ_l on Γ_l, $l = 1, 2, \cdots, M$,

$$f = \hat{A}_0^l \phi_l - \sum_{m=1, m \neq l}^{M} A_0^{m,l} \phi_m \quad \text{on } \Gamma_l \quad (4)$$

where

$$\hat{A}_0^l = \left((\hat{D}_0^l - I) P_l^l - \hat{S}_0^l Q_l^l \right),$$

and

$$A_0^{m,l} = \sum_{m=1, m \neq l}^{M} \left((D_0^{l,m} - I) P_l^l - S_0^{l,m} Q_l^l \right).$$

This problem is discretized using the Nyström method [7,10]. The resulting matrix equation, that involves many matrix-vector multiplications resulted from the multiplications of layer potentials and/or their derivatives, is solved by a two-grid iterative method [11,12]. The matrix vector multiplications can be done quickly by FMM routines [13].

IV. NUMERICAL VALIDATION AND RESULTS

It is well known that E_0 has the following asymptotic behavior [9],

$$E_0(\mathbf{x}) = \frac{e^{j\kappa_0|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \left\{ E_\infty(\frac{\mathbf{x}}{|\mathbf{x}|}) + O\left(\frac{1}{|\mathbf{x}|}\right) \right\} \quad |\mathbf{x}| \to \infty$$

where E_∞ is known as the far (scattered) field. It is related with the intensity at infinity $\mathbf{I}_\infty$(or the bistatic differential cross section) as,

$$\mathbf{I}_\infty = 2\pi|E_\infty|^2.$$

We wish to compute an approximation of the far field E_∞. We use θ to denote the observation angle, i.e., $\mathbf{x} = |\mathbf{x}|(\cos(\theta), \sin(\theta))$. Unless otherwise stated we use $\omega = 1$, and $n_l = 1.5$, $l = 1, 2, \cdots, M$.

For validating the algorithm we start with the computation of the far field for circular cylinders. A quasi-analytical solution (QAS) can be obtained in this case [1].

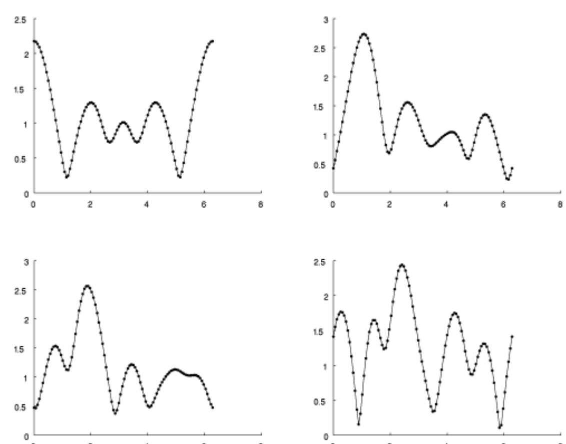

Fig. 2. The absolute value of the far field against the incident angle α for two circular cylinders of different radius using the BEM (solid line) and QAS (dots) algorithms. Here we use $\theta = 0$ (top, left), $\theta = \pi/4$ (top, right), $\theta = \pi/2$ (bottom, left), and $\theta = 3\pi/2$ (bottom, right). We have used $\epsilon_1 = 1.5$ and $\epsilon_2 = 2.3$.

In Fig. 2, for different observation angles θ, we plot the absolute value of the far field against the incident angle α for two circular cylinders of radii $r = 1$ and $r = 2$ using QAS (dots) and the BEM (solid line) described in this paper. We see a very good match of the two solutions. This is achieved for 8 grid points of the Nyström implementation.

Next we look at the case of more circular cylinders. To this end we add two more cylinders of radii $r = 0.5$ and $r = 0.25$. The positions of the four cylinders is the same as for the objects in Fig. 5. The result is given in Fig. 3 where we give similar computations as the case of the two circular cylinders. Like for the previous case we see an excellent match of the two methods. To see the exponential convergence of our integral equation method we plot the absolute value of the far field against the number of grid points in Fig. 4.

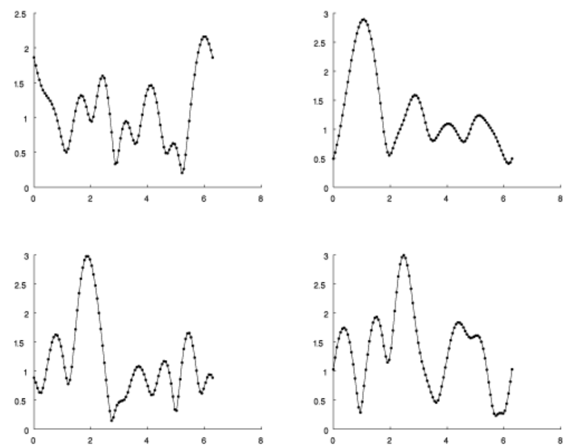

Fig. 3. The absolute value of the far field against the incident angle α for four circular cylinders of different radius using the BEM (solid line) and QAS (dots) algorithms. Here we use $\theta = 0$ (top, left), $\theta = \pi/4$ (top, right), $\theta = \pi/2$ (bottom, left), and $\theta = 3\pi/2$ (bottom, right). We have used $\epsilon_1 = 1.5$, $\epsilon_2 = 2.3$ $\epsilon_3 = 1.9$ and $\epsilon_4 = 0.5$.

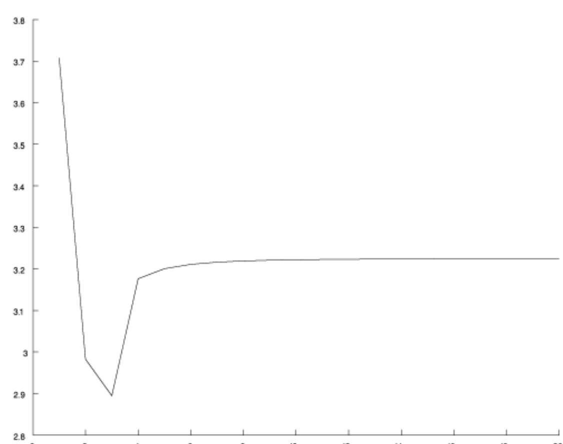

Fig. 4. The absolute value of the far field against the number of grid points for the case of four circular cylinders with different radii. Here we use $\theta = 0$ and $\alpha = 0$.

Finally we look at the case of non-convex boundaries where analytical results can not be obtained. In particular,

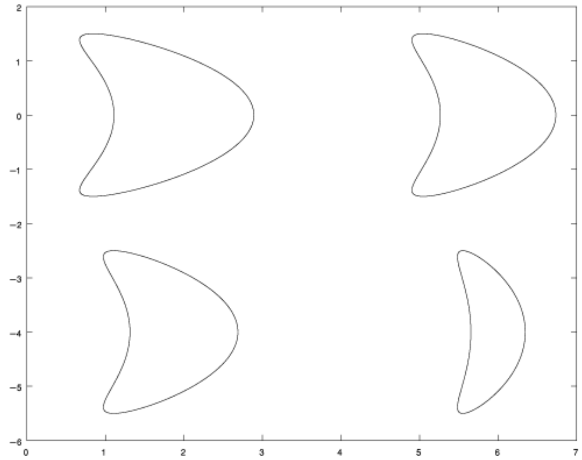

Fig. 5. Four non convex dielectric objects of different sizes.

consider the case of four such boundaries as given in Fig. 5. They have the following parametric formula for $0 \leq t \leq 2\pi$,

$$(x, y) = \gamma_j \left(\cos(t) + 0.65cos(2t) - 0.65, 1.5sin(t)\right)$$

where γ_j, $j = 1, \cdots, 4$, are random real numbers.

When we analyze the convergence in Fig. 6 we see, as in the case of circular cylinders, a very fast convergence.

In Fig. 7 we give the result of the far field against the incidence angle for various number of grid points.

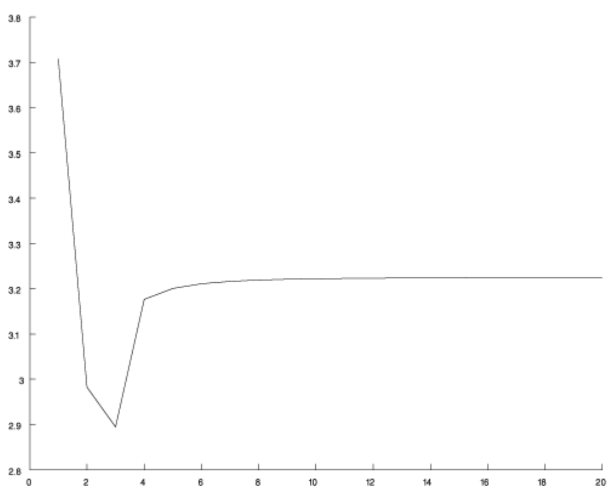

Fig. 6. The absolute value of the far field against the number of grid points for Fig. 5. Here we use $\theta = 0$ and $\alpha = 0$.

V. CONCLUSION

We have developed an efficient numerical algorithm for the computation of scattered fields for two dimensional parallel dielectrics. The numerical simulations show very good results compared to existing methods. Our future work will be to apply this method for analyzing photonic bandgaps and to the three dimensional objects.

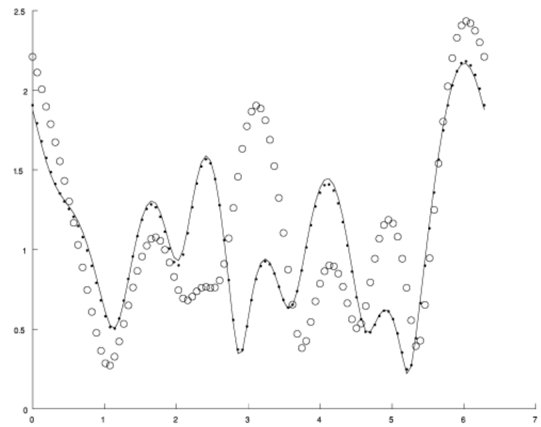

Fig. 7. The absolute value of the far field against the incident angle α for the geometry in Figure 5 using the BEM for two grid points ('o'), six grid points (dots) and eight grid points (solid line). Here we use $\theta = 0$, $\epsilon_1 = 1.5$, $\epsilon_2 = 2.3$ $\epsilon_3 = 1.9$ and $\epsilon_4 = 0.5$.

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