MODELLING EDDY CURRENTS IN UNBOUNDED STRUCTURES USING THE IMPEDANCE METHOD

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Abstract

In this paper extensions to the impedance network method are presented. In particular the method has been applied to problems with boundaries extending to infinity. The infinite boundary condition can also be applied to lines of symmetry in the given geometry. Two dimensional surface models have been verified by comparison of numerical and experimental results in which the potential was measured along the edge of copper sheeting of various shapes located in a uniform, quasi-static magnetic field. The method has potential for modelling three dimensional structures including anisotropic earth planes, arbitrarily shaped buried objects, and both finite and infinitely long faults, dykes, pipes, cylinders and cracks.

I. Introduction

The three dimensional impedance method has been used to model eddy currents induced in heterogeneous human models by a time varying quasi-static magnetic field [1-4]. In this work, the object was modelled by a uniform cubic three dimensional mesh of impedance cells. The values of the impedances were determined from the size of the element and the conductivity of the material being modelled. Using standard circuit analysis the current was solved for each face of every cell.

Other recent approaches to the calculation of eddy currents using numerical techniques include an integral formulation using a set of R, L and C elements [5], the boundary element formulation [6], finite element modelling [7], edge element [8], the integral equation approach [9] and many others. There are considerable difficulties in modelling the eddy currents induced in conductive media when both the source field is non-uniform and the media is both inhomogeneous and has arbitrary shape or is of infinite extent in one or more directions. In this paper the impedance method [1] is extended to address these limitations. By discretising the two dimensional area or three dimensional volume in terms of a spatial array of impedance elements various structures can be represented. The individual parameters of each element and spatial variations in the applied field can also be modelled. Varying symmetries in the solution space together with prudent application of boundary conditions reduces the number of elements required to solve a given problem. The resultant matrix generated from this technique is used to solve for the current in each cell for a spatially varying time harmonic magnetic field. The technique is applied to the solution of eddy current problems in materials with a spatial variation in conductivity in the presence of a quasi-static magnetic field.

Until now calculations using this method have only dealt with finite sized bodies with air boundaries. This excludes the possibility of modelling a large range of problems where boundaries extend to infinity. For this reason an infinite boundary condition is desirable.

II. Impedance method theory

Maxwell's Equation in differential form (1) describes the electric field $\underset{\sim}{E}$ generated by the presence of a time varying magnetic field $\underset{\sim}{B}$.

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \qquad (1)$$

With a surface area $\underset{\sim}{S}$ enclosed by the boundary path C, one can apply Stoke's Theorem to equation (1) and obtain

$$\oint_C \underset{\sim}{E}.\underset{\sim}{dl} = -\int_S \frac{\partial \underset{\sim}{B}}{\partial t}.\underset{\sim}{ds} \qquad (2)$$

For the quasi-static case the left hand side of equation (2) represents the induced potential in the loop V, ie.

$$V = \oint_C \underset{\sim}{E}.\underset{\sim}{dl} \qquad (3)$$

and the right hand side of equation (2) is the negative of the time derivative of flux Φ crossing the surface (S), where

$$\Phi = \int_S \underset{\sim}{B}.\underset{\sim}{ds} \qquad (4)$$

For the time harmonic case we have the relation

$$-\frac{\partial\Phi}{\partial t} = \oint_{C} \underset{\sim}{E}.d\underset{\sim}{l} = -j\omega\Phi \quad (5)$$

where ω is the angular frequency of the magnetic field.

By dividing the media into a rectangular mesh of discrete impedance elements Z, we can apply Stoke's theorem to each cell individually providing that each cell is sufficiently small so that the applied magnetic field is uniform within its boundary. The value of the impedance on each side of the cell is directly related to the length of the side and the complex dielectric constant of the cell. The magnitude of the inducing magnetic flux is directly related to the area enclosed by the cell and the angle between the $\underset{\sim}{S}$ and $\underset{\sim}{B}$. For a two dimensional surface formulation the values of Z are expressed in terms of the cell length L, an arbitrary impedance element transverse thickness A (usually expressed as an area in the three dimensional model although it is the thickness in the two dimensional formulation) by the following relation [3].

$$Z = \frac{L}{(\sigma + j\omega\varepsilon_r\varepsilon_0).A} \quad (6)$$

where σ is the conductivity and ε_r is the relative permittivity of the material and ε_0 is the dielectric constant of free space.

For a good conductor ($\sigma >> \omega\varepsilon_r\varepsilon_0$) and using a quasi-static assumption, equation (6) becomes

$$R = \frac{L}{\sigma A} \quad (7)$$

We restrict the formulation to non-magnetic materials in time harmonic fields. A single rectangular cell is shown in Fig. 1(a), together with the electrical analog (b).

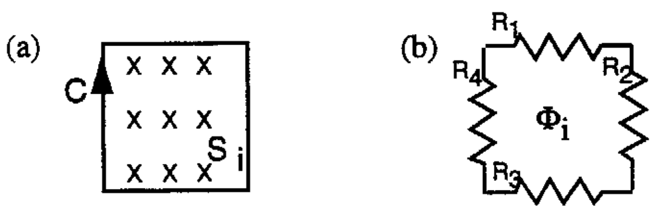

Figure 1: *An arbitrarily enclosed region of space and its discretized equivalent cell.*

In terms of the discretization process, it is assumed that the magnetic field B_i across the element is uniform and so the flux Φ_i within each cell is directly related to the area of the cell S_i. For an arbitrary four sided cell, we can write

$$j\omega\Phi_i = \sum_{j=1}^{4} V_{Rj} \quad (8)$$

where V_{Rj} is the voltage across the j^{th} resistor in the boundary path surrounding the i^{th} cell.

From (8) using Ohm's Law we can solve for the current in a single loop I_{Ci}. The relation then becomes

$$j\omega\Phi_i = \sum_{j=1}^{4} I_{Ci} R_j \; ; \quad (9)$$

where I_{Ci} = cell i loop current.

The full model consists of many interconnected cell loops in a mesh. A two dimensional representation is shown in Fig. 2

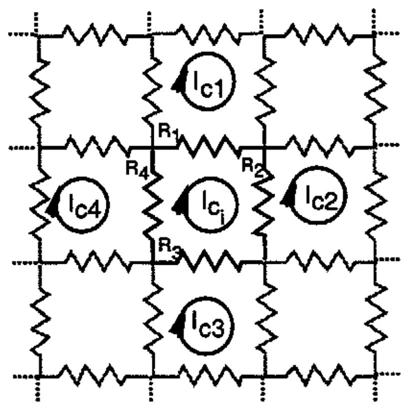

Figure 2: *A portion of the two dimensional resistive element mesh.*

Hence the relationship between a one cell I_{Ci} and its nearest neighbour cell currents I_{Cj} is

$$j\omega\Phi_{Ci} = \sum_{j=1}^{4} (I_{Ci} - I_{Cj}).R_j \quad (10)$$

For a conductive sheet consisting of n cells, the system can be represented as

$$[R]_{nxn} . [I]_{1xn} = j\omega[\Phi]_{1xn} \quad (11)$$

where R_{ij} forms the resistance network, I is a column matrix of unknown currents and Φ is a row matrix of magnetic flux through each element.

From (11) it is possible to calculate the loop current in each cell and from this, calculate the net current distribution throughout the solution space. Thus, we can determine the net current through each resistive element R by summing the contributing currents from adjoining cells. For the two dimensional surface formulation there are a maximum of two component currents per resistive element, however for a three dimensional model there are potentially four component currents flowing through a resistive element. For example, if the resistive element lies in the x direction then there are two loop currents in the XOY plane and two in the XOZ plane [4].

Note that the applied field flux [Φ] can be spatially variable for each element and so non-uniform fields can be modelled. The flux generated by induced currents in a particular cell will be in the opposite direction to the applied flux and extends radially from the cell. If we consider the four immediately adjacent cells then to a first approximation we would expect less than 25% of this flux will flow through each adjacent cell. Since the field falls away as r^{-3} contributions to other cells will be even less significant. Therefore re-radiated flux generated from the induced eddy currents is not considered in this method. This assumption is also used by others [1-4].

The two dimensional surface formulation is sufficient to solve many problems and the method is easily adaptable to three dimensions and has been demonstrated in the literature [1]. The computation of Φ as it appears in equation (11) is still from the magnetic flux through each cell which must be calculated from the magnetic field component normal to the surface area of each cellface.

Boundary conditions

A variety of boundary conditions are required for the accurate solution of models of arbitrary shape or semi-infinite size. Application of the appropriate boundary conditions can reduce the number of cells required to solve a given problem and hence increase the computational efficiency. This allows larger problems to be solved on a given platform. A number of boundary conditions are discussed below.

(i) Open boundary/insulator boundary

In the case of an open boundary where a conductor meets an insulator, cells lying on the edge do not allow current to flow beyond the boundary, ie. there are no sources/sinks of current beyond the boundary. Hence for an open boundary the contributing current I_{Cj} in equation (8) is zero. Thus any element at this boundary is expressed in terms of its four bounding resistances and up to three nearest neighbouring current elements.

(ii) Infinite boundary

Where a uniform material extends to infinity and is subjected to a uniform magnetic field it can be classified as an infinite boundary. At infinity, adjoining cells in the direction of the boundary will have identical currents. If the infinite boundary is located at x = , then it follows that $\frac{dI}{dx} = 0$ and the net current flowing through the element on the boundary of these two cells will be zero (the summation of two equal and opposite component currents). This is a Neumann boundary condition. Equation (10) then becomes a summation of potentials j = 1..3. The location of the infinite boundary is chosen such that in the solution, the difference in current between cells adjacent to the boundary is less than (say) 5%.

With 3 or more infinite boundaries the cell currents are no longer constrained. This is reflected by an ill conditioned matrix [R] where no unique solution exists because the number of unknown quantities exceeds the number of knowns and the matrix equation cannot be solved.

(iii) Symmetry

Along a line (eg. parallel to the Y axis) or plane of symmetry (eg. in the YOZ-plane) then $\frac{dI}{dx} = 0$. This is also a Neumann boundary condition, identical to the infinite boundary condition.

Computational aspects

The computational model was developed and implemented on a Sun Sparc Server 10. Mesh generation and processing was implemented using custom software in 'C' with double precision variables. For a two dimensional surface formulation, mesh generation yields an n x n matrix for an n element problem. The matrix generated has a sparse diagonally dominant band structure. Because the problems exhibit a low condition number an exact rather than an iterative inversion routine was implemented. Standard LU decomposition as described in [10] was used. Matlab™ was used to create the net current vectors and voltage plots from the numerically obtained values of cell loop currents [I] in equation (11).

Non-rectangular cells

Equation (9) can be modified to calculate the induced current in one cell in terms of any number of nearest neighbour cells. It follows from this equation that the number of sides n, of a given cell determines the exact form of this equation (j=1..n). Triangular elements allow more accurate representation of many structures where normally highly detailed staircase approximations are required.

III The numerical results

To verify the method in two dimensions, a uniform, square, conductive sheet of dimensions 9cm x 9cm was modelled in a uniform quasi -static magnetic field. The numerical model was evaluated for many cell sizes. The voltages between points on one edge of the sheet were evaluated. This is equivalent to the spaced pickup experiment (described below). Note that a cell with dimensions of 3cm x 3cm allows the solution of only two data points whereas a cell size of 0.25cm x 0.25cm produces 18 data points. These results are shown in Fig. 3. It should be noted that there is good agreement between the numerical results obtained for all cell sizes although the spatial resolution is obviously limited by the cell size. The error for even the largest cell size is less that 5%.

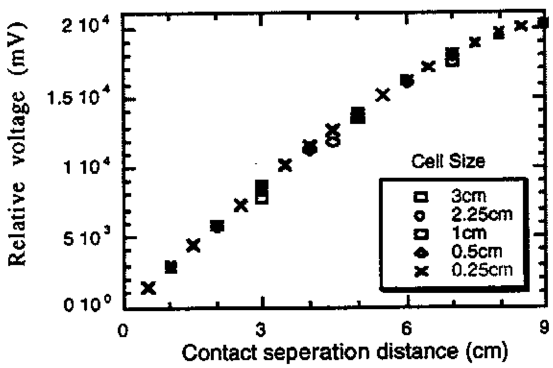

Figure 3: *Model accuracy for a variety of cell sizes ($f=1kHz$, $\sigma = 5.8x10^7 S/m$).*

To model an infinitely long strip about a small region of interest infinite boundary conditions are used. The conductor/air interface is an open boundary since the currents are confined to the conductor with little current injection to/from the air. Fig. 4 shows an 18 x 18 cell model of 1cm x 1cm cells with R = 1 /m with two open and two infinite boundary conditions applied.

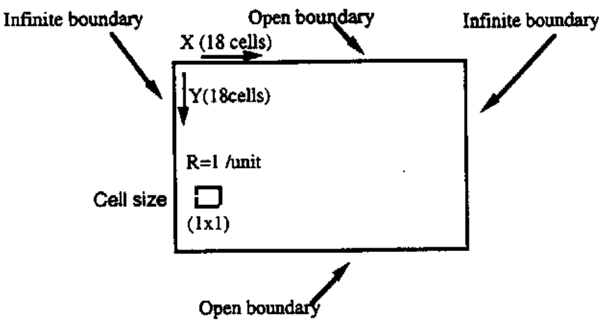

Figure 4: *Model of an infinitely long conductive strip.*

IV Experimental setup

Experimental measurements were made to provide a direct comparison with results from the numerical model. A split shielded loop of diameter 50cm was placed around the test area (Fig. 5). The loop was driven with a signal generator at 1kHz to create a uniform magnetic field normal to and inside the loop. A typical field strength of 5μT was generated. Using a small search coil, the magnetic field inside the area bounded by the coil was found to be constant to within less than 5%. Copper foil of various shapes were placed in this region normal to the magnetic field.

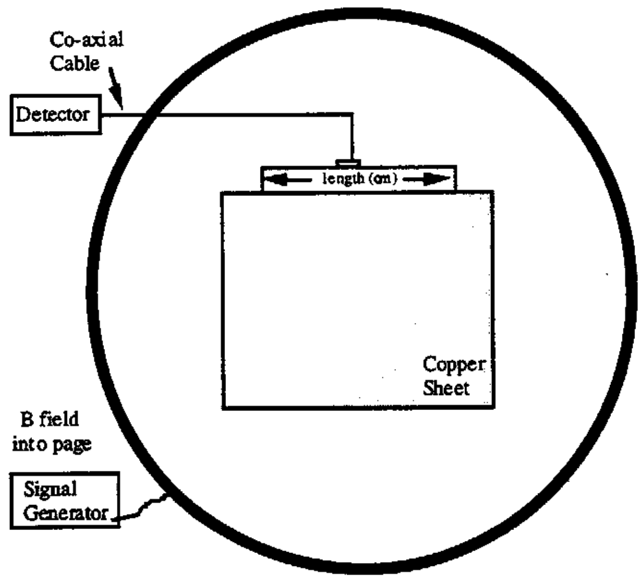

Figure 5: *Experimental setup.*

Voltage probes with electrical connections to the edge of the foil were connected to a detector using co-axial cable. The detector circuit had an input impedance of $10^{12}\Omega$. This was created by using commercially available JFET operational amplifiers in an instrumentation amplifier configuration. Detector electronics was housed in a small aluminium box (5cm x 7cm x 10cm) which was battery powered with a LCD display. There was no earth reference to the detector. The electronics consisted of a band pass filter tuned to 1kHz, amplifier stages, an RMS to DC detector and DMM (digital multimeter) module. The instrument was calibrated by varying the field strength stepwise and recording the probe output.

The objective in this design was to minimise any electric field generation from the coil and magnetic or electric field induction in the detector electronics. When the earth connection to the shield of the excitation coil was removed, the signal level of a test coil placed within the test area showed poor directionality indicating electric field coupling. With the earth connection in place, the detector output was zero when the wire test coil was oriented parallel to the magnetic field.

V Results

In the first experiment a square sheet of copper foil (9cm x 9cm) was placed in the quasi-static magnetic field such that the field is normal to the copper sheet. This problem was solved numerically using a variety of cell sizes. With 18 x 18 cells the numerical solution for the quasi-static vector current distribution is displayed in Fig. 6. This current distribution shows a well known result where the induced current is circular and the greatest current magnitudes lie on the edge of the sheet. This was verified by experiment by monitoring the voltage drop across a spaced pickup at various points on the edge of the sheet. This current plot gives good agreement with a result generated using edge analysis [8].

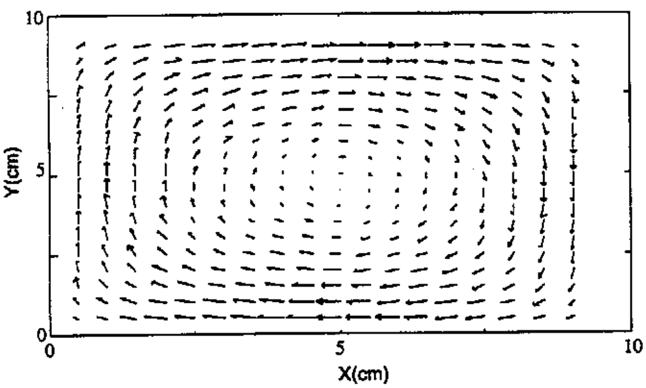

Figure 6: *Induced current distribution in a homogenous conductive square (9cm x 9cm) cell size (0.5cm x 0.5cm) calculated using the impedance method.*

Spaced pickup

In this set of measurements, separation between the measurement points was varied across the top of the sheet while remaining symmetrical about the position x = 4.5cm. The numerical model results are normalised to the experimental magnetic field strength by using a field strength probe. The model used had a cell size of 0.25cm. The model results were obtained by numerically integrating the currents between the ends of the pickup to obtain a measured potential. These results are displayed in Fig. 7. The difference between the model results and the experimentally measured data is less than 10%

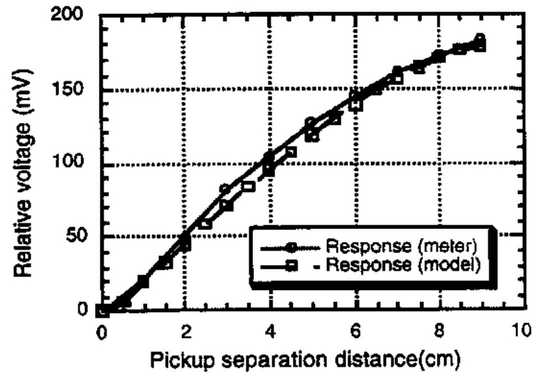

Figure 7: *Comparison between impedance method model results and experimental results for a spaced pickup.*

Length variations

In this experiment a rectangular (20cm x 9cm) sheet of copper foil was placed in a quasi-static magnetic field such that the field is normal to the surface of the copper sheet. The length of the sheet was progressively shortened from 20cm to 9cm in 0.5cm increments. The voltage difference was always measured between two corner points on one 9cm edge. The numerical model used had a cell size of 0.5cm x 0.5cm. Fig. 8 shows that the normalised response of the model compares well with the experimentally measured results. Clearly as the size of the sheet decreases, the response also decreases. There is however, a limiting sheet length (approximately 9cm) beyond which there is no significant increase in response.

It is clear that beyond a length of 9cm there is very little contribution to the measured voltage from additional copper material; ie. for short lengths the current is constrained by the y dimension and beyond the square, the current is constrained by the x direction.

Cell shape variability

Two shapes were modelled experimentally and numerically. First the response from a symmetrical detector with a separation distance of 9cm across the top of a 9cm x 9cm square sheet of copper foil was measured. The foil was then cut from a top corner across the diagonal to a lower corner to form a triangular shape. The detector position remained unchanged. The response from this triangular shape was then measured. The two shapes were modelled using a square and triangular one element model respectively. The induced voltage of the triangular shape reduces to 66% of the square shape in both the numerical and experimental results. The measured and numerically modelled results were found to differ by 0.5%

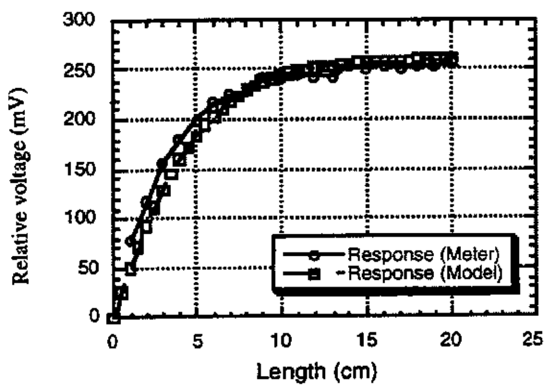

Figure 8: *Model and experiment results for copper rectangle length variations.*

Infinitely long homogenous conductive strip
In Fig. 4 the region of interest is divided into 18 x 18 cells. Cells on the top and bottom of this model are open boundaries and the cells on the left and right boundaries are infinite boundaries. Fig. 9 shows the vector currents obtained by the impedance method. Note that the boundary elements are included in columns 1 and 18 of the figure. The current through the model is parallel to the open boundary edges indicating that the infinite boundary condition is effective.

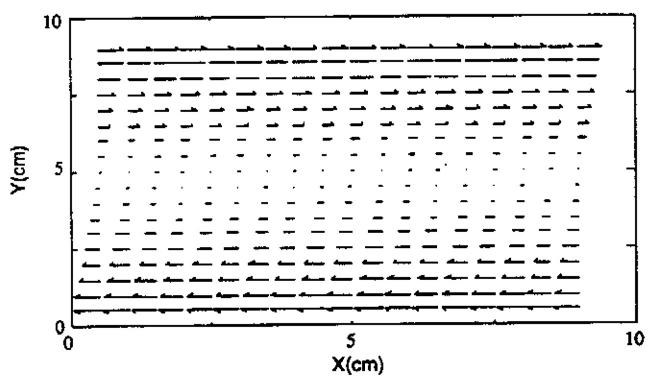

Figure 9: *Induced currents in an infinitely long conductive strip.*

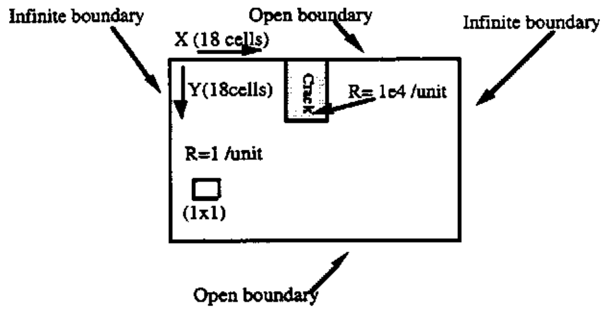

Figure 10: *A model of a surface crack in an infinitely long conductive strip.*

Surface crack modelling
Fig. 10 shows a resistive crack (1 x 5 cells, R = 1 x 10^4) in an infinitely long homogenous strip (R=1). The impedance method results are shown in Fig. 11. The induced currents flow around the resistive crack as expected.

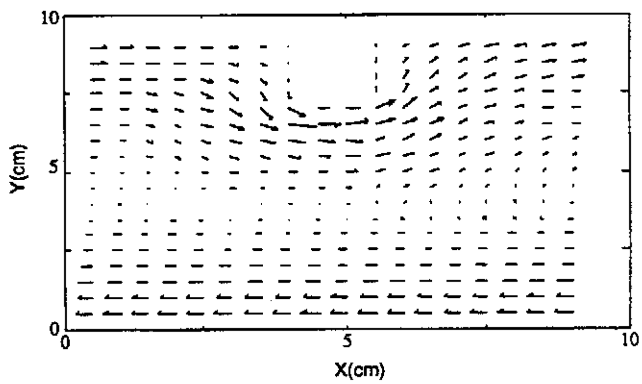

Figure 11: *Induced currents around a crack in an infinitely long conductive strip.*

Three dimensional modelling
The impedance method can be used to model three dimensional bodies in the presence of an applied magnetic field [3]. Computation time for a three dimensional model is dramatically increased (cubic rather than squared relationship) for given cell dimensions.

The two dimensional surface formulation is adequate for the two dimensional cross section modelling of three dimensional structures on three conditions:

(a) The magnetic flux [Φ] is unaffected by the nearest adjacent plane perpendicular to the direction of the applied field. (There is no significant magnetic field attenuation if the distance is less than 10% of one skin depth).

(b) The conductivity distribution in the adjacent plane is not significantly different to that of the plane being modelled. (There will be negligible cross planar currents generated which give rise to additional out of plane current sources and sinks.)

(c) The magnetic field is always directed parallel to the surface vector of each cell. (If this is not the case then additional current sources and sinks will be present within the model.)

VI Conclusions

The impedance method can be used to solve eddy current problems in both bounded and unbounded regions. This formulation has the advantage of using a variety of cell shapes (for example triangular and rectangular) to best represent the structure in the model. In addition, there is no requirement for a uniform mesh. Different sizes and shapes can be used within the one model. The impedance method has been verified using copper sheet in a uniform magnetic field. The experimental measurements lie within 10% of the numerical model results for most measurements.

By applying appropriate boundary conditions, small regions of interest within large structures can be modelled. This enables large structures to be modelled with only a small number of elements if the region of interest is small. This technique allows for significant CPU savings and hence greater resolution .

Three dimensional modelling has great potential to model complex features. If the features of interest lie only in the plane orthogonal to the applied field then a two dimensional model is sufficient since there will be no interaction between the layers. The method has potential for the non-destructive detection of sub-surface flaws in metallic regions eg. aircraft wings and generator housings.

References

[1] O.P. Gandhi, J.F. DeFord and H. Kanai, "Impedance method for calculation of power deposition patterns in magnetically induced hyperthermia", *IEEE Trans. Biomed. Eng.*, vol. BME-31, no. 10, pp. 644-651, 1984.

[2] J. F. DeFord and O.P. Gandhi, "An impedance method to calculate currents induced in biological bodies exposed to quasi-static electromagnetic fields", *IEEE Trans. Electromagn. Compat.*, vol. EMC-27, no. 3, pp. 168-173, 1985.

3] N. Orcutt and O.P. Gandhi, "A 3-D impedance method to calculate power deposition in biological bodies subjected to time varying magnetic fields", *IEEE Trans. Biomed. Eng.*, vol. BME-35, pp. 577-583, 1988.

4] W. Xi and M.A. Stuchly, "Induced electric currents in models of man and rodents from 60Hz magnetic fields", *IEEE Trans. Biomed. Eng.*, vol. BME-41, no. 11, pp. 1018-1023, 1994.

[5] M. Gimignami, A. Musolino and M. Raugi, "Integral formulation for nonlinear magnetostatic and eddy currents analysis", *IEEE. Trans. Mag.* , vol. MAG-30, pp 3024-3027, 1994.

[6] I.D. Mayergoyz, "Boundary integral equations of minimum order for the calculation of three dimensional eddy current problems", *IEEE Trans. Mag.*, vol. MAG-18, no. 2, pp. 536-539, 1982.

[7] A. Konrad, M.V.K. Chari and Z.J. Csendes, "New finite element techniques for skin depth problems", *IEEE Trans. Mag.*, vol. MAG-18, no. 2, pp. 450-455, 1982.

[8] H.Tsuboi, F. Kobayashi and T.Misaki, "Computation of electric current distribution on thin conducting plate using edge element", *IEEE Trans. Mag.*, vol MAG-32, no. 3, pp. 745-748, 1996.

[9] L. Turner, "An integral equation approach to eddy-current calculations", *IEEE Trans. Mag.*, vol MAG-13, no. 5, pp. 1119-1121, 1977.

[10] W. Press, *Numerical recipes in C : The art of scientific computing*, 2nd ed., Cambridge University Press, 1992.