'H-ORIENTED' AND 'B-ORIENTED' METHODS IN A PROBLEM OF NONLINEAR MAGNETOSTATICS: SOME METHODOLOGICAL REMARKS

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ABSTRACT

Problems which depend on a small parameter in their formulation can often be studied by a perturbation approach. Whether the perturbation is "regular" or "singular" is important in many respects. In magnetostatics, due to some inherent duality, both kinds of perturbation may happen, depending on the chosen formulation ("b-oriented" vs. "h-oriented" methods). Singularly perturbed problems are numerically more difficult than regularly perturbed ones. We suggest that this might explain why, as some recent numerical observations seem to suggest, b-oriented methods should give better accuracy in a specific class of nonlinear magnetostatic problems at high permeability.

INTRODUCTION

This paper is a case study, which butts on a methodological question of such generality that it seems almost preposterous to address it in written form: how should the particularities of a physical problem (here, the presence of "small parameters") guide the modeler in the selection of a numerical method? This is a basic subject, one about which everybody has definite opinions (and even, sometimes, strong feelings), but also an elusive one, difficult to treat in a comprehensive way. (This is why there are relatively so few books or paper collections devoted to mathematical modelling, like e.g., [1, 2, 3, 6, 7, 9, 11, 13, 15, 22]. Why most of them seem so remote from the kind of mathematical modelling *we* do is a puzzling question.)

One might say that, after all, this is quite normal. Why should "tricks of the trade" be honored with formal dissertations? But a moment of reflection will show that some of the most powerful of such tricks deserve to be thus treated, that they are, and that it is eventually beneficial for all those concerned. Take Fourier analysis, for instance. On one level, it is an efficient *dimensionality-reduction* device, which helps replace the numerical solution of a fully three-dimensional problem with a series of simpler ones, in one or two dimensions. On another level, it is the practical and usable by-product of a majestic mathematical theory: harmonic analysis [12, 17]. Is this a coincidence? Probably not. The development of the theory was stirred by the efficiency of what was at the time a modelling trick, the use of trigonometric series in the study of heat transfer, and it has payed back largely, as far as mathematical modelling is concerned. This is the classical example of a practical modelling tool—a trade trick—backed by a strong mathematical theory, and of the dialectics of their historical development. Other examples will come to

mind (the "least squares" trick in relation with control and identification theory, some aspects of Geometrical Diffraction Theory, some statistical tests . . .).

The small parameter issue (an instance of which will be presented in a moment) might deserve such a status. Here there is, no doubt, a trick: use dimensional analysis to expose "small parameters", identify the corresponding terms in the equations, drop them. There is also a grandiose theory, or rather, theories: asymptotic analysis, perturbation theory. But the theory appears incomplete, it lacks unity, and the trick itself is far from being fail-safe: when your equation is $-\varepsilon u'' + u = f$, or $-u'' + \varepsilon u = f$, should you drop the ε-term, or not? (Think of ε as related, for instance, with the skin-depth.) The answer to this particular question is known, but lots of similar ones are still to be addressed. Here is a rich field of study, full of prospects for both intellectually stimulating mathematical theories and usable practical recipes.

Yet, case studies may prove more effective than comprehensive studies of large scope, at this stage. One such study comes from the experience of a small electromagnetism community (the "TEAM workshop", [22, 23]) whose emphasis is on low-frequency and static situations. The test-case, described below in general terms, is a non-linear magnetostatics problem on which the same observation has been made by several independent investigators: numerical methods based on the use of the vector potential tend to perform better than those using a scalar potential. This calls for an explanation, which is the theme of the present paper. In short, there are two "small parameters" in this problem, the ratio of the airgap to some characteristic length, and the ratio of permeabilities in the air and in the iron. Their interplay results in the possibility of formulating the problem as a perturbation with respect to some similar, much simpler problem. This is a standard idea. What is new here is that, due to some symmetry inherent in Maxwell equations, there are *two* ways to do this, based respectively on the use of the scalar and of the vector potential, and that the nature of the perturbation is not the same for both: it's "regular" perturbation in the former case, "singular" perturbation in the latter. An attempt is made to link these facts with the numerical observations.

Similar situations where two small parameters are "competing", each one driving the problem toward different limits, seem to be common to other areas of electromagnetics, for example, skin effect, current flow around a crack, and even (for higher frequencies) surface impedance.

'H-ORIENTED' AND 'B-ORIENTED' METHODS IN MAGNETOSTATICS

We shall consider the following general situation (Fig. 1): a coil C (sustaining a given, permanent, current-density j^g), and a ferromagnetic piece M. The whole space is E, and $A = E - C - M$ is the air-region. A non-linear behavior law $b = \Gamma(h)$ is given (no

hysteresis). For the present discussion, we assume a Γ such that vectors b(x) and h(x) are collinear at all points x in M, and that b(x) vanishes if h(x) does. One is requested to compute the fields b and h. The relevant equations are

(1) $\quad \text{curl } h = j^g \text{ in E, } b = \Gamma(h), \text{ div } b = 0 \text{ in E.}$

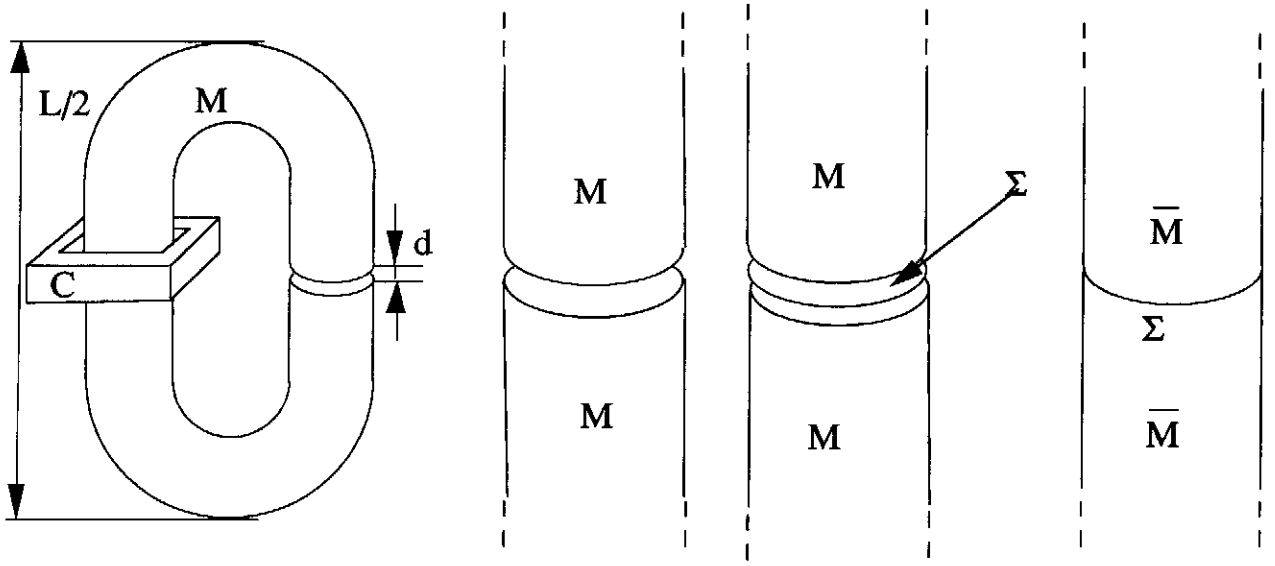

Figure 1. A typical problem (left). Definition of surface Σ and domain $\overline{M}$, to be used in the sequel (right).

We are only interested here in comparing the accuracy achievable with various methods, not in things like the computing time, the number of steps in iterative procedures, etc. So we may consider a *linear* problem with the same solution as (1), namely

(2) $\quad \text{curl } h = j^g \text{ in E, } b = \mu\, h, \text{ div } b = 0 \text{ in E,}$

where μ is the function $x \to |b(x)|/|h(x)|$, as computed from the actual solution $\{b, h\}$. Whatever can be said about the merits of various methods as regards (2) will thus be relevant to (1).

The ratio $\mu(x)/\mu_0$ is a feature of the solution (and as such, it depends on the imposed current j^g). It is usually high (about 2000, typically), but tends to decrease when the driving current j^g is increased (*saturation* phenomenon). Let us call ε the average of $\mu_0/\mu(x)$ over M. We define, for all x in M, $\mu_1(x) = \varepsilon\, \mu(x)$. Note that μ_1 is close to μ_0, except perhaps in saturated zones. This ε is one of the small parameters alluded to in the Introduction. It will play an important role in what follows. The other small parameter is

the ratio d/L of the gap-width d to the average length L of the magnetic circuit M. We shall assume that d/L, though small, is still large with respect to ε: thus, the reluctance of the whole device is mainly due to the air gap. This is the assumption used in the TEAM workshop study, and is the rule in problems of this kind. (But the assumption is a crucial one: if the reluctance was mainly due to the iron core, this would modify our conclusions.)

Although the discussion will be limited to linear problems, we are not losing anything significant in generality, because one can always tackle (1) by iterating on μ, Newton-Raphson style, a linear system being solved at each step. In fact, almost all methods in actual use in nonlinear magnetostatics seem to rely on this approach. They fall into two main families: 'b-oriented' methods, which yield b (for instance by computing the magnetic vector potential), and 'h-oriented' methods, which aim at h (for instance by computing the magnetic scalar potential). For the sake of definiteness, let us formalize this:

Definition 1. *A vector field* h *[resp.* b*] is said "curl-conformal" [resp. "div-conformal"] if its tangential [resp. normal] part is continuous across all surfaces in its domain of definition.*

Definition 2. *A method will qualify as "h-oriented" [resp. as "b-oriented"] if it computes* h *[resp.* b*] in such a way that its tangential [resp. normal] continuity is exactly enforced, i.e., if it yields a curl-conformal* h *[resp. a div-conformal* b*].*

Let us develop an example of b-oriented method. A tetrahedral mesh is first designed, large enough to cover M, C, and enough of the air-region to justify neglecting what happens outside the meshed volume. Let $\mathcal{N}$ be the set of nodes. Let λ_n be the "hat-function" associated with node n (i.e., the unique continuous, piecewise affine, function equal to 1 at node n and to 0 at all other nodes). Call IP^1 the set of all vector fields of the form

$$(3) \qquad a = \Sigma_{n \in \mathcal{N}} \mathbf{a}_n \lambda_n ,$$

where the degrees of freedom (DoF) $\mathbf{a}_n$ are vectors, one per node. Now look for a in IP^1 such that

$$(4) \qquad \int_E \mu^{-1} \operatorname{curl} a \,.\, \operatorname{curl} a' = \int_E j^g \,.\, a' \quad \forall\ a' \in IP^1$$

(the so-called "weak form" of the equation $\operatorname{curl}(\mu^{-1} \operatorname{curl} a) = j^g$). This is a linear system in terms of the 3 x N components of the $\mathbf{a}_n$s (N, the number of nodes in the mesh). There exists a solution (because $\operatorname{div} j^g = 0$), perhaps not unique, but anyway, all solutions will have the same curl. (Numerical difficulties that one may encounter in solving (4) are not our concern.) This is a b-oriented method, because it gives b = curl a, thus enforcing the

essential requirement about the continuity of the normal component of b across all surfaces, including the faces of the mesh. The field $h = \mu^{-1}$ curl a fails to have the symmetric property of continuity of its tangential part through surfaces. This is so because the relation curl $h = j^g$ is only *weakly* enforced by (4).

On the other hand, the well-known "single magnetic potential" method is h-oriented. Its principle is to look for the field h in the form $h = \text{grad } \varphi + h^g$, where h^g is a field that satisfies rot $h^g = j^g$ and div $h^g = 0$. (Such a field can be constructed from j^g by Biot-Savart integration.) The unknown in this problem is thus the potential φ. According to the general finite element approach, one thus has to find a function φ of the form

$$(5) \qquad \varphi = \Sigma_{n \in \mathcal{N}} \varphi_n \lambda_n$$

such that

$$(6) \qquad \int_E \mu \, (\text{grad } \varphi + h^g) \, . \, \text{grad } \varphi' = 0$$

for all test-functions φ' themselves of the form (5). (This is a way to enforce the condition $\text{div}(\mu h) = 0$, in weak form.) Here the φ_ns are *scalar* degrees of freedom. This time h has the required tangential continuity, but $b = \mu h$ fails to have normal continuity, because div $b = 0$ is only weakly enforced: the method is "h-oriented".

Remark. Note that φ is single-valued here. There are variants in which a similar, but possibly *multivalued* magnetic potential is used [19], or more than one potential [18]. Clearly, such methods also are h-oriented.

Not all methods fall in one of the previous categories. A method can have both orientations. But in that case the behaviour law $b = \mu h$ will fail to be exactly satisfied, in general. (This point is developed in [5].)

Now the scene is set. We shall argue as follows: "b-oriented" methods should in general work better, because the smallness of the ratio μ_0/μ results in a *regularly* perturbed problem with such formulations, whereas "h-oriented" methods suffer from *singular* behavior with respect to this same parameter. Therefore, they are more sensitive to numerical error.

SINGULAR VS. REGULAR PERTURBATIONS

Let us first recall the difference between regular and singular perturbation, with the help of a simple example. Suppose we have to solve the two-points boundary-value problem

(7) $\quad -u'' + \varepsilon\, k(x)\, u = f,\ \ u'(0) = 0,\ \ u(1) = 0,$

for a numerically small value of ε, and a given positive function k. Note that this is equivalent to minimizing the functional $F_\varepsilon(v) = \int_{[0,\,1]} (|v'|^2 + \varepsilon\, k\, |v|^2 - 2\, f\, v)$ among all those in the set $V = \{v \in C^0[0, 1] : \int_{[0,\,1]} |v'|^2 < \infty,\ v(1) = 0\}$, which can easily be done by numerical methods. But instead, in order to take advantage of ε's smallness, one would rather make use of the "limit-problem", corresponding to $\varepsilon = 0$, whose solution u_0 is obtained directly, by quadratures, without having to solve a linear system. The idea is to expand u as $u_0 + \varepsilon\, u_1 + \ldots$, and to throw this into (7): again, u_1 can be obtained by straightforward integration, the same way u_0 was. And so on. Problem (7) is a perturbation of a simpler one (namely, to find u such that $-u'' = f,\ u'(0) = 0,\ u(1) = 0$), and therefore can be solved by a cascade of similar problems. Note that u_0 still achieves the minimum of some functional (namely, $F_0(v)$) over the same set V. This is what makes the perturbation "regular": the limit solution is solution to the limit problem, as obtained by setting $\varepsilon = 0$ in either (7) or its equivalent variational formulation, "minimize $F_\varepsilon(v)$ over V". Note also that u_ε *and* its first derivative both converge (in the sense of quadratic means) to u_0 and u'_0 respectively.

In contrast, "singular perturbation" occurs with

(8) $\quad -\varepsilon\, u'' + k(x)\, u = f,\ \ u'(0) = 0,\ \ u(1) = 0.$

If one wants to solve this numerically, the smallness of ε imposes a small discretization step, which is cumbersome, so making use of the limit problem seems again a good idea. But now two things go wrong. First, the would-be "limit-problem", i.e., (7) with $\varepsilon = 0$, or its variational version, "minimize $\int_{[0,\,1]} (|k\, |v|^2 - 2\, f\, v)$ over V", has no solution. Next, when one relaxes its requirements by dropping the boundary conditions, the solution u_0 of the relaxed problem, "minimize over $L^2([0, 1])$", which is $u_0 = f$, is not the limit of u_ε in the above sense: only u_ε converges towards u_0 in L^2, whereas u'_ε does not converge to u'_0 (Fig. 2).

Singular perturbation is thus characterized by the fact that, when passing to the limit, the solution "escapes" from the set where it naturally lives, and while it still does converge to something, it is in a weaker sense, and to an object which belongs to a larger universe. It does not mean that the (non-acceptable) "limit solution" u_0 is useless as a stepping stone to obtaining the true solution, u_ε, of (8). Indeed, u_0 is a term in some asymptotic expansion of u_ε, that can be derived via a moderately involved process called "matching asymptotic expansions". (Some references to these things are [10, 14, 16, 20, 21].) Quite often, a lot of numerical work can be avoided by such an asymptotic approach to the solution. This is so because, from the numerical point of view, singularly perturbed problems are tougher than regularly perturbed ones. In the case of (8) vs. (7), a glance at Fig. 2 shows why: the

step-size has to be very small, at least in the "boundary layers" near the ends of the interval.

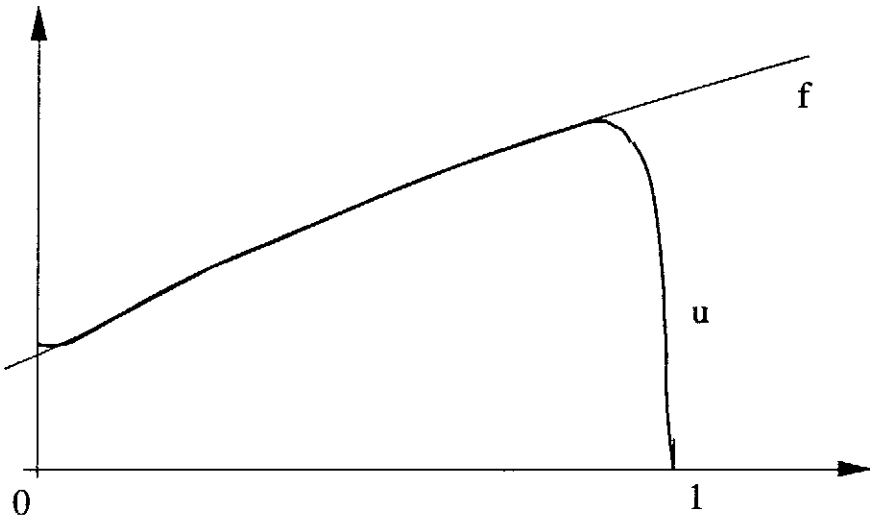

Figure 2. Solution of (7) for ε small (k = 1).

PROBLEM (2) AS A PERTUBED PROBLEM

For reference, let us write weak formulations analogous to (4) and (6), as they stand before any commitment to particular finite elements has been made. Call A (resp. Φ) the set of all square-integrable vector fields (resp. functions), with a square integrable curl (resp. gradient), over the whole space. The weak formulations are, for the (b-oriented) "a-method": *find* $a_\varepsilon \in A$ *such that*

$$\int_E \mu^{-1} \operatorname{curl} a_\varepsilon \,.\, \operatorname{curl} a' = \int_E j^g \,.\, a' \quad \forall\ a' \in A$$

(which can be rewritten as

(9) $$\int_{E-M} \mu_0^{-1} \operatorname{curl} a_\varepsilon \,.\, \operatorname{curl} a' + \varepsilon \int_M \mu_1^{-1} \operatorname{curl} a_\varepsilon \,.\, \operatorname{curl} a' = \int_E j^g \,.\, a' \quad \forall\ a' \in A),$$

and for the (h-oriented) φ-method: *find* $\varphi_\varepsilon \in \Phi$ *such that*

$$\int_E \mu\, (\operatorname{grad} \varphi_\varepsilon + h^g) \,.\, \operatorname{grad} \varphi' = 0 \quad \forall\ \varphi' \in \Phi$$

(which can be rewritten as

(10) $$\int_{E-M} \mu_0\, (\operatorname{grad} \varphi_\varepsilon + h^g) \,.\, \operatorname{grad} \varphi' + \varepsilon^{-1} \int_M \mu_1\, (\operatorname{grad} \varphi_\varepsilon + h^g) \,.\, \operatorname{grad} \varphi'$$

$$= 0 \quad \forall\ \varphi' \in \Phi).$$

Remark. one should compare (9) with the following:

$$\int \partial_x u_\varepsilon \, \partial_x u' + \varepsilon \int k \, u \, u' = \int f \, u' \quad \forall \; u' \in U,$$

which is equation (7) in weak formulation. The analogy we rely on is due to the presence of the small parameter ε in front of the second integral in both cases.

Let us now see in which way our problem (2), when formulated as (9) or as (10), can be considered a "perturbation" of some simpler one. A difficulty is that there are *two* small parameters in the picture: the above ratio ε ($\sim \mu_0/\mu$), and the ratio d/L of the gap-width to the circuit-length. The analysis in terms of two such "competing" small parameters is quite interesting; it leads—depending on which parameter dominates—to very different limit models (called "significant degenerations" in [10]), which do have meaningful interpretations within the present context. (See an application to skin-effect, and to the concept of "surface impedance" [8], in [4].) There is no space here to justify our main assumption that ε, not d/L, is the dominant small parameter in the present case: this would call for some lengthy dimensional analysis (whose principles, anyway, are familiar to all readers). We cannot either dwell on the Taylor expansions in terms of ε, and similar techniques, by which models (11) and (12) below are derived: although these are familiar and elementary mathematical tools, their application to the present case requires a heavy apparatus of functional analysis, that seems out of place. So we shall proceed more dogmatically: first, some notation, next a terse statement of the results, then some a posteriori justification.

Let us call $\overline{M}$ the union of M (cf. Fig. 1) and the gap, and Σ a surface congruent to the pole surfaces, located in the middle of the gap. Enlarge space Φ by accepting into it functions which are allowed to be *discontinuous* across Σ. Call $\overline{\Phi}$ the bigger space so obtained.

Now, the limit problems. The one in terms of the vector potential a splits into two successive problems: *find* $a_0 \in A$ *such that both conditions*

$$\text{(11)} \qquad \begin{cases} \int_{E-\overline{M}} \mu_0^{-1} \operatorname{curl} a_0 \,.\, \operatorname{curl} a' = \int_E j^g \,.\, a' \quad \forall \; a' \in A, \\ \\ \operatorname{curl}(\mu_1^{-1} \operatorname{curl} a_0) = 0 \quad \text{in } \overline{M}, \end{cases}$$

hold. The one in φ is *find* $\varphi_0 \in \overline{\Phi}$ *such that both conditions*

$$\text{(12)} \qquad \begin{cases} \int_{E-\overline{M}} \mu_0 \, (\operatorname{grad} \varphi_0 + h^g) \,.\, \operatorname{grad} \varphi' = 0 \quad \forall \; \varphi' \in \overline{\Phi}. \\ \\ \operatorname{grad} \varphi_0 + h^g = 0 \text{ in } \overline{M} - \Sigma, \end{cases}$$

hold. Then $b = \text{curl } a_0$ and $h = \text{grad } \varphi_0 + h^g$ are the limit solutions.

Why, on physical grounds? Imagine the air gap is shrunk to Σ, while the ratio $\varepsilon = \mu_0/\mu_t$ tends to zero. Then h tends to zero in the core, hence the second line of (12), and (since its circulation is held constant) tends to infinity in the air gap, hence the eventual discontinuity of the magnetic potential. As for b, one has $\text{div } b = 0$ and $\text{curl}(\mu_0^{-1} b) = j^g$ out of the core, and its flux lines impinge orthogonally to the surface of the core in the case of an infinite permeability: indeed, the first line of (11) implies all that. The second line of (11) describes what happens inside the core, and (since the tangential values of a are now known, from solving (11), first line) constitutes a well-posed problem. (All this, though it may not be obvious, relies on the above observation that d/L is large with respect to ε.)

Remark. Neither (11) or (12) are practical ways to solve the problem, if only because of the non-linearity: μ_t, of course, is not known in advance, so no useful information on the field in the core could be obtained that way. (The stray-field thus obtained, however, might be reasonably accurate.) We do not advocate the actual *use* of limit models in the specific case of Pb. 13. We just expect to get insight—about how difficult it will be to solve the *true* problem—from an examination of the *limit* one.◊

Now it should be clear that *the limit process from* (9) *to* (11) *is regular, while the one from* (10) *to* (12) *is singular:* for in the latter, there is this tendency of φ_ε to "escape from" space Φ (towards the larger one $\overline{\Phi}$), while nothing similar happens to a_ε, which stays in A.

The difference can be seen more concretely from what happens in the air at the boundary of Σ: a singularity of φ (actually, a jump-discontinuity), whereas a is regular (Fig. 3). Figure 3 is two-dimensional for convenience, but what is described here is a feature of the three-dimensional situation: the magnetic potential changes very rapidly across the airgap (Fig. 3, right). So, if one tries to model the situation by treating the airgap as a surface, across which φ can be discontinuous—which is precisely model (12)—the computed φ will look as on the left part of the figure, with an obvious singularity. Our point is that, although this limit model is not the one which is actually solved for, the latter is sufficiently closed to it for the singular character of the limit case to be felt, in various ways, when actually solving for φ.

The nature of the finite elements has not been a factor in this discussion. So we may conclude that regularity (resp. singularity) pertains to the whole family of a-methods (resp. φ-methods). Singularly perturbed problems are notoriously more difficult to solve numerically than regularly perturbed ones. Hence the tentative conclusion: b-oriented

methods *as a whole* should be considered with a favorable bias, for this particular problem. Of course, this should not be construed as a seal of approval for any *particular* b-oriented methods.

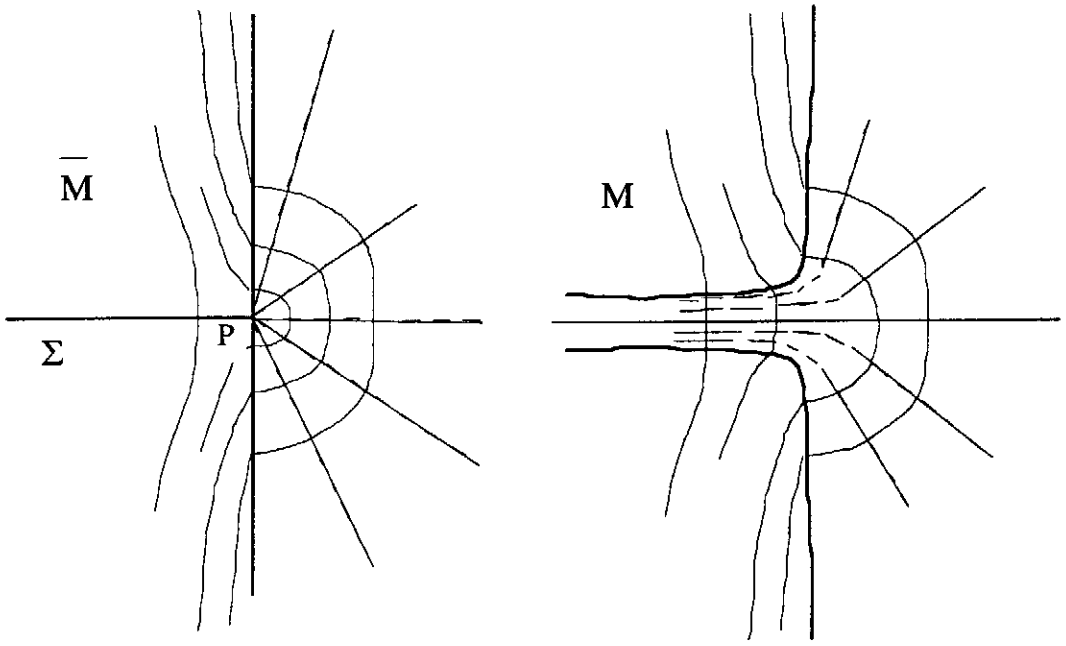

Figure 3. Flux lines in the vertical symmetry plane, near the edge of the air gap, showing the singularity of the field: left, as predicted by the limit models, right, as it really is. The situation is nearly two-dimensional. In that case, a is a scalar, whose isolines are precisely the flux-lines shown. Dotted lines are isovalues of φ, the magnetic potential. Clearly, a is continuous at point P, but φ is not, in the limit model.

CONCLUSION

Let us summarize the main line of our argumententation. The problem which is actually solved (the one with a small but non-zero ε) is closed to a singular limit when h-oriented methods are used, and to a regular limit when b-oriented methods are used. So in the critical region (near the edges of the poles), h-oriented methods are likely to behave in a nastier way than b-oriented ones. For instance, they may require a finer mesh if the same precision is to be achieved. But meshes, as a rule, are not what would be desirable, they are what computing resources allow. So we may expect, on the average, better precision from b-oriented methods in reported numerical experiments. This seems indeed to be the present trend, as regards this particular problem.

We have seen how an analysis of the nature of the perturbation (regular or singular) in a small parameter problem can be used to predict the relative success of a particular

family of numerical methods with respect to another. This was, at best, heuristics. Can the reasoning eventually be refined into one that would yield precise, provable, statements? Perhaps, but I tend to think efforts in this direction would be misdirected. In the present state of the art, it seems better to view the above explanation as a route towards a *conjecture*, to be confirmed or rejected on the basis of numerical experiments: that, for these problems of non-linear magnetostatics with high relative permeability, "b-oriented methods work better".

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