Analysis of Dielectric Loaded Scalar Horn Radiators

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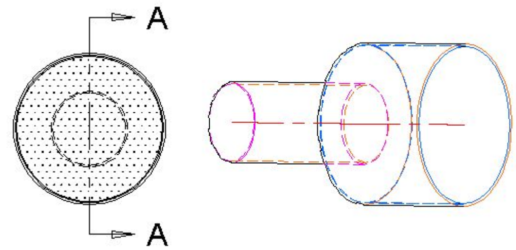
***Abstract* –** The dielectric loaded horn radiators are commonly used in various applications due to their distinguished features, such as low cross-polarization, pattern symmetry and simple production. The analysis of this kind of horn, mode matching (MM) and integral equation methods have been preferred in the literature. In the present study, the radiation of plane harmonic scalar waves from a dielectric loaded circular horn radiator is treated by using the mode matching method in conjunction with theWiener-Hopf technique. The solution is exact but formal since infinite series of unknowns and some branch-cut integrals with unknown integrands are involved. Approximation procedures based on rigorous asymptotic are used and the approximate solution to the Wiener-Hopf equations are derived in terms of infinite series of unknowns, which are determined from infinite systems of linear algebraic equations. Numerical solution of these systems is obtained for various values of the parameters, of the problem. Their effect is presented on the directivity of the circular feed horn.

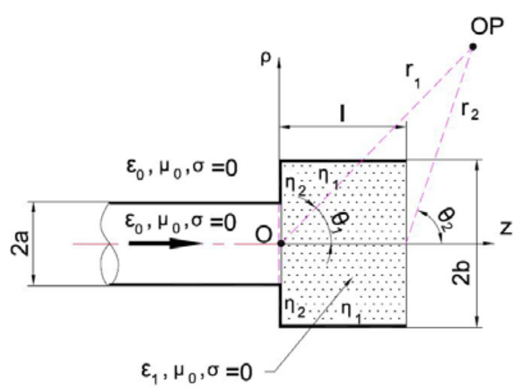
***Key words* –** Dielectric loaded wide angle scalar horn radiator, Wiener-Hopf Technique, integral equations, circular waveguide, step discontinuity.

I. INTRODUCTION

In the recent years, scalar feed horns are commonly used widespread applications such as feeds in reflector radiator systems used in microwave and acoustics, because of their well-known properties of pattern symmetry and zero or low cross-polarization. To analyze the performance of such feeds, one needs to know accurately their near- and far-field patterns. The aperture fields of a pure-mode horn are generated by a single mode, which is the dominant mode in the waveguide. These horns use "hybrid" modes where there is a single mode, which is composed of hybrid combination of two other modes. The scalar feed is circular horn antenna with grooves, perpendicular to the wall of the horn. The grooves change the fields so as to provide desirable properties of axial beam symmetry, low side lobes and cross-polarization. This means that the horn produces an aperture field in which the field's distributions are approximately linear. The very low cross-polarization means that the field in the aperture are essentially scalar and for this reason, the corrugated horn is sometimes referred as scalar horn [1]. The radiation characteristics of circular waveguides and horns have been the subject to several previous investigations [2 - 5]. Some of the approximate and computational methods such as surface integral methods; hybird MM/ finite element (FE)/ method of moment (MoM)/ finite difference (FD) methods have been presented for the analysis of horns [6]. The analysis reported in [7] is recently generalized [8] to the case where the aperture's inner surface and the intersection area with the flange of the waveguide horn are treated as different impedance materials. The aim of the present work is to produce an analysis of the case where the aperture of the waveguide horn is loaded as different dielectric materials, as shown in Fig. 1.



a. Dielectric loaded circular horn radiator.



b. Geometry of the problem.

Fig. 1. a. Dielectric loaded circular horn radiator, b. geometry of the problem.

The aperture region of the scalar horn is loaded by a simple dielectric material (non-magnetic and non-conducting-dielectric rod) having the permitivity

ε_1. The variables η_1 and η_2 are the complex admittance of the aperture's inner surface of the horn and the intersection area with the flange of the waveguide, respectively. To this end we consider the problem of dominant modes in the circular waveguides propagating out of semi-infinite duct, via another coaxial cylindrical duct of finite length and bigger radius, and the issuing into free space.

In the progress of the radiation pattern analysis of dielectric loaded scalar feed horn, attention has been given to consider the propagation of plane waves by circular structures, because the complexity of these structures is not always possible to obtain rigorous analytical solutions to radiation problems. The Wiener-Hopf Technique is applicable to open and closed structures.

The method adopted here is similar to the one employed in [8] and consists of expressing the total field in the waveguide region in terms of normal waveguide modes and using the Fourier transform elsewhere. To this end, by introducing the Fourier transform for the scattered field and applying the boundary conditions in the transform domain, the problem is reduced into a modified Wiener-Hopf equation. Using the mode matching method in conjunction with the Wiener-Hopf technique the radiation of plane harmonic scalar waves from a scalar feed horn were treated. The solution is exact but formal since infinite series of unknowns and some branch-cut integrals with unknown integrands are involved. Approximated procedures based on rigorous asymptotic are used, and the approximate solution to the Wiener-Hopf equations are derived in terms of infinite series of unknowns, which are determined from infinite systems of linear algebraic equations. Numerical solution of these systems is obtained for various values of the parameters of the problem and their effect on the directivity of the scalar feed horn is presented. The time dependence is assumed to be $\exp(-i\omega t)$, with ω being the angular frequency, and is suppressed throughout the paper.

II. ANALYSIS

Consider the radiation of a time harmonic plane wave propagating along the positive z direction from a rigid cylindrical horn is defined by, $\{\rho = a, z \in (-\infty, 0)\} \cup \{\rho \in (a,b), z = 0\} \cup \{\rho = b, z \in (0,l)\}$ where (ρ, ϕ, z) denotes the usual cylindrical polar coordinates (Fig. 1). From the symmetry of the geometry of the problem, and of the incident field, the scalar field everywhere will be independent of ϕ.

Assuming the incident field is given by

$$u^i = \exp(ikz) \tag{1}$$

where $k = \omega/c$ denotes the wave number. For the sake of analytical convenience we will assume that the surrounding medium is slightly lossy and k has a small positive imaginary part. The lossless case can be obtained by letting $Im k \to 0$ at the end of the analysis.

The total field $u^T(\rho, z)$ can be written as,

$$u^T(\rho,z) = \begin{cases} u_1(\rho,z) & ;\ \rho > b, \quad z \in (-\infty,\infty) \\ u_2(\rho,z) & ;\ \rho \in (a,b),\ z < 0 \\ u_3(\rho,z) + u^i(\rho,z) & ;\ \rho \in (0,a),\ z < 0 \\ u_4(\rho,z) & ;\ \rho \in (0,b),\ z \in (0,l) \\ u_5(\rho,z) & ;\ \rho \in (0,b),\ z > l \quad . \end{cases} \tag{2}$$

By considering k_1 as the wave number of dielectric region, $u_j(\rho,z)$, $j = 1-5$ denote the scattered fields $u_j(\rho,z)$, $j = 1-5$, which satisfy the Helmholtz equation,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]u_j(\rho,z) = 0, \quad j = 1,2,3,5\,, \tag{3a}$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k_1^2\right]u_j(\rho,z) = 0, \quad j = 4 \tag{3b}$$

is the expression to be determined with the help of well known boundary, edge, and radiation conditions for the perfectly conducting structures. The boundary condition on the internal surfaces of the horn yield $\frac{\partial u}{\partial n} + ik\eta u = 0$, where n is the normal pointing outward the lining, and η is the complex specific admittance of the surfaces,

$$u_1(b,z) = u_2(b,z)\,,\ z < 0\,, \tag{4a}$$

$$\frac{\partial u_1}{\partial\rho}(b,z) = \frac{\partial u_2}{\partial\rho}(b,z)\,,\ z < 0\,, \tag{4b}$$

$$\frac{\partial}{\partial\rho}u_2(a,z) = 0\ ,\ z < 0\,, \tag{4c}$$

$$\frac{\partial}{\partial\rho}u_3(a,z) = 0\ ,\ z < 0\,, \tag{4d}$$

$$u_1(b,z) = u_5(b,z)\ ,\ z > l\,, \tag{4e}$$

$$\frac{\partial u_1}{\partial\rho}(b,z) = \frac{\partial u_5}{\partial\rho}(b,z) \quad z > l\,, \tag{4f}$$

$$\frac{\partial}{\partial\rho}u_1(b,z) = 0\,,\ z \in (0,l)\,, \tag{4g}$$

$$(ik_1 n_1 - \frac{\partial}{\partial\rho})u_4(b,z) = 0 \quad z \in (0,l)\,, \tag{4h}$$

$$u_3(\rho,0) + u^i\ ,\ \rho \in (0,a)\ , \tag{4i}$$

$$\frac{\partial}{\partial z}u_3(\rho,0) + \frac{\partial}{\partial z}u^i\,, \rho \in (0,a)\,, \tag{4j}$$

$$u_4(\rho,l) = u_5(\rho,l)\,, \rho \in (0,b)\ , \tag{4k}$$

$$\frac{\partial u_4}{\partial z}(\rho,l) = \frac{\partial u_5}{\partial z}(\rho,l)\,, \rho \in (0,b)\,, \tag{4l}$$

$$\frac{\partial u_2}{\partial z}(\rho,0)=0\,,\ \rho\in(a,b)\,, \tag{4m}$$

$$(ik_1n_2+\frac{\partial u_4}{\partial z})(\rho,0)=0\,,\ \rho\in(a,b)\,. \tag{4n}$$

To ensure the uniqueness of the mixed boundary-value problem, one has to take into account the following radiation and edge conditions,

$$u\approx\frac{e^{ikr}}{r},\ r=\sqrt{\rho^2+z^2}\,, \tag{4o}$$

$$u^T(b+0,z)=O\,,\ z\to -0\,, \tag{4p}$$

$$\frac{\partial}{\partial\rho}u^T(b+0,z)=O(z^{-1/3})\,,\ z\to -0\,, \tag{4q}$$

$$u^T(b,z)=O\,,\ z\to l+0\,, \tag{4r}$$

$$\frac{\partial}{\partial\rho}u^T(b,z)=O((z-l)^{-1/2})\,,\ z\to l+0\,. \tag{4s}$$

By taking the Fourier transform of $u(\rho,z)$ with respect to the variable z and considering also above mentioned boundary and continuity conditions in the transform domain α, the problem is reduced into the following modified Wiener-Hopf equation of the third kind, which is valid in the strip $Im(-k)<Im(\alpha)<Im(k)$,

$$\begin{aligned}&-\frac{b}{2}F_1(b,\alpha)+\frac{\dot{H}_-(b,\alpha)}{K^2(\alpha)}Q(\alpha)+\frac{e^{i\alpha l}\dot{H}_+(b,\alpha)}{K^2(\alpha)R(\alpha)}\\ &=\frac{i\alpha}{\pi}\sum_{m=0}^{\infty}\frac{J_1(Z_ma)}{J_1(Z_mb)}\frac{f_m}{Z_m}\frac{1}{\delta_m^2-\alpha^2}\\ &\quad+e^{i\alpha l}\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)}{\alpha_m^2-\alpha^2}[g_m-i\alpha h_m]\end{aligned} \tag{5a}$$

where

$$H_-(\rho,\alpha)=\int_{-\infty}^{0}u_1(\rho,z)e^{i\alpha z}dz\,, \tag{5b}$$

$$H_+(\rho,\alpha)=\int_{l}^{\infty}u_1(\rho,z)e^{i\alpha(z-l)}dz\,, \tag{5c}$$

$$H_1(\rho,\alpha)=\int_{0}^{l}u_1(\rho,z)e^{i\alpha z}dz\,, \tag{5d}$$

$$K(\alpha)=\sqrt{k^2-\alpha^2}\,, \tag{5e}$$

$$R(\alpha)=i\pi J_1(Kb)H_1^{(Kb)}\,, \tag{5f}$$

$$Q(\alpha)=\frac{H_1^{(1)}(Ka)}{\pi H_1^{(1)}(Kb)\left[J_1(Ka)Y_1(Kb)-J_1(Kb)Y_1(Ka)\right]}\,, \tag{5g}$$

$$Z_m=K(\delta_m),\qquad m=0,1,2,..\,, \tag{5h}$$

$$J_1(j_m)=0,\ m=0,1,2,...\,, \tag{5i}$$

$$\alpha_m=\sqrt{k^2-\left(j_m/b\right)^2},\qquad m=0,1,2,...\,, \tag{5j}$$

$$f_m=\frac{\pi^2}{2}\frac{J_1^2(Z_mb)Z_m^2}{J_1^2(Z_ma)-J_1^2(Z_mb)}\int_a^b f(t)\left[J_1(Z_ma)Y_0(Z_mt)-Y_1(Z_ma)J_0(Z_mt)\right]tdt \tag{5k}$$

$$g_m=\frac{2}{b^2J_0^2(j_m)}\int_0^b g(t)J_0(\frac{j_m}{b}t)tdtm\neq 0\,, \tag{5l}$$

and

$$h_m=\frac{2}{b^2J_0^2(j_m)}\int_0^b h(t)J_0(\frac{j_m}{b}t)tdt,m\neq 0\,. \tag{5m}$$

Using the factorization and the decomposition procedures together with the Liouville theorem, the modified Wiener-Hopf equation in (5a) can be reduced to the following system of Fredholm integral equations of the second kind,

$$\begin{aligned}&\frac{\dot{H}_+(b,\alpha)}{(k+\alpha)R_+(\alpha)}=\\ &-\frac{1}{2\pi i}\int_{L^+}\frac{\dot{H}_-(b,\tau)R_-(\tau)Q(\tau)e^{-i\tau l}}{(k+\tau)(\tau-\alpha)}d\tau\\ &+\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m+i\alpha_mh_m\right](k+\alpha_m)R_+(\alpha_m)}{2\alpha_m(\alpha+\alpha_m)}\,,\\ &-\frac{i}{2\pi}\sum_{m=0}^{\infty}\frac{J_1(Z_ma)}{J_1(Z_mb)}\frac{f_m}{Z_m}\frac{k+\delta_m}{\delta_m+\alpha}R_+(\delta_m)e^{i\delta_m l}\end{aligned} \tag{6a}$$

$$\begin{aligned}&\frac{\dot{H}_-(b,\alpha)Q_-(\alpha)}{(k-\alpha)}=\\ &\frac{1}{2\pi i}\int_{L^-}\frac{\dot{H}_+(b,\tau)e^{i\tau l}}{(k-\tau)R(\tau)Q_+(\tau)(\tau-\alpha)}d\tau\\ &-\frac{b}{2}\sum_{m=0}^{\infty}\frac{J_0(\xi_m)\left[g_m-i\alpha_mh_m\right](k+\alpha_m)e^{i\alpha_m l}}{2\alpha_m(\alpha-\alpha_m)Q_+(\alpha_m)}\\ &+\frac{i}{2\pi}\sum_{m=0}^{\infty}\frac{J_1(Z_ma)}{J_1(Z_mb)}\frac{f_m}{Z_m}\frac{k+\delta_m}{\delta_m-\alpha}\frac{1}{Q_+(\delta_m)}\end{aligned} \tag{6b}$$

where the paths of integration L^+ and L^- are depicted in [7]. Here, $R_+(\alpha),Q_+(\alpha)$ and $R_-(\alpha)=R_+(-\alpha)$, $Q_-(\alpha)=Q_+(-\alpha)$ are the split functions [8] regular and free of zeros in the upper $\left(Im\alpha>Im(-k)\right)$ and lower $\left(Im\alpha<Imk\right)$ halves of the complex $\alpha-$ plane, respectively, resulting from the Wiener-Hopf factorization of $R(\alpha)$ and $Q(\alpha)$, which are given by (5f) and (5g), in the following form,

$$R(\alpha)=R_+(\alpha)R_-(\alpha)\,, \tag{7a}$$

$$Q(\alpha)=Q_+(\alpha)Q_-(\alpha)\,. \tag{7b}$$

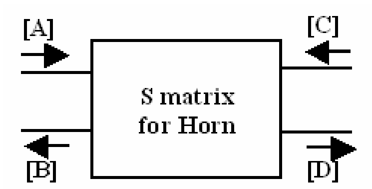
The explicit expressions for $R_+(\alpha)$ and $Q_+(\alpha)$ can be obtained by using the results of [9], [10]. For $kl>>1$, the coupled system of Fredholm integral equations of the second kind in (6a) and (6b), are susceptible to a treatment by iterations

$$\dot{H}_+(b,\alpha)_-=\dot{H}_+^{(1)}(b,\alpha)+\dot{H}_+^{(2)}(b,\alpha)+\cdots\,, \tag{8a}$$

$$\dot{H}_-(b,\alpha)_-=\dot{H}_-^{(1)}(b,\alpha)+\dot{H}_-^{(2)}(b,\alpha)+\cdots\,. \tag{8b}$$

III. MODAL MATCHING TECHNIQUE: DETERMINATION OF THE EXPANSION COEFFICIENTS

Modal matching technique (MMT) is a powerful numeric method of analyzing horn radiators in which the actual profile of the horn is replaced by a series of uniform waveguide sections. The MMT can be considered as a method of obtaining the overall transmission and reflection properties of a horn. The horn is represented as a box as shown in Fig. 2, where [A] and [B] are column matrices containing the forward and reflection coefficients of all the modes looking into the horn from source side. Similarly, [C] and [D] represent column matrices containing the forward and reflection coefficients of all the modes looking into the aperture of the horn from outside [11 - 13].



$$\begin{bmatrix}[B]\\ [D]\end{bmatrix} = [S]\begin{bmatrix}[A]\\ [C]\end{bmatrix} \quad [S] = \begin{bmatrix}[s_{11}]\,[s_{12}]\\ [s_{21}]\,[s_{22}]\end{bmatrix}$$

Fig. 2. Horn represented as a scattering matrix [S].

The field in the cavity can be expressed in terms of the waveguide normal modes as follow,

$$u_3(\rho,z) = \sum_{n=0}^{\infty} c_n e^{-i\beta_n z} J_0(j_n \frac{\rho}{a}) \tag{9a}$$

with

$$\beta_n = \sqrt{k^2 - \frac{j_n^2}{a^2}}, \qquad n = 0,1,2\ldots\ldots \tag{9b}$$

Here ξ_n 's are the roots of the characteristic equation

$$J_1(j_n) = 0, \qquad n = 0,1,2\ldots\ldots \tag{9c}$$

Similarly, in the region $0 < \rho < b$, $0 < z < l$, $u_4(\rho,z)$ can be expressed in terms of the following normal waveguide modes,

$$u_4(\rho,z) = \sum_{n=0}^{\infty}\left(p_n e^{i\nu_n z} + q_n e^{-i\nu_n z}\right) J_0(\xi_n \frac{\rho}{b}), \tag{10a}$$

$$ik_1 b\eta_1 J_0(\xi_n) + \xi_n J_1(\xi_n) = 0, \qquad n = 0,1,2\ldots, \tag{10b}$$

$$\nu_n = \sqrt{k_1^2 - \frac{\xi_n^2}{b^2}}, \qquad n = 0,1,2\ldots. \tag{10c}$$

Now, from the continuity relations we get

$$\frac{\partial}{\partial z}u_4(\rho,0) = \begin{cases} \frac{\partial}{\partial z}u_3(\rho,0) + ik, & \rho\in(0,a) \\ -ik_1\eta_2 u_4(\rho,0), & \rho\in(a,b) \end{cases} \tag{11a}$$

$$u_4(\rho,0) = u_3(\rho,0) + 1; \qquad \rho\in(0,a), \tag{11b}$$

$$\frac{\partial u_4}{\partial z}(\rho,l) = g(\rho) = \sum_{m=0}^{\infty} g_m J_0(\xi_m \frac{\rho}{b});\ \rho\in(0,b), \tag{11c}$$

and

$$u_4(\rho,l) = h(\rho) = \sum_{m=0}^{\infty} h_m J_0(\xi_m \frac{\rho}{b});\ \rho\in(0,b). \tag{11d}$$

Inserting the series expansions of $g(\rho)$ and $h(\rho)$ [14] given in equations (5l) and (5m) into equations (11c) and (11d), respectively, and using equations (9a) and (10a) we get,

$$-\sum_{n=0}^{\infty} i\nu_n\left[p_n - q_n\right] J_0(\xi_n \frac{\rho}{b}) = \begin{cases} \sum_{m=0}^{\infty} i\beta_m c_m J_0(j_m \frac{\rho}{a}) - ik, & \rho\in(0,a) \\ ik_1 n_2 u_4(\rho,0), & \rho\in(a,b), \end{cases} \tag{12a}$$

$$\sum_{n=0}^{\infty}\left[p_n + q_n\right] J_0(\xi_n \frac{\rho}{b}) = \sum_{m=0}^{\infty} c_m J_0(j_m \frac{\rho}{a}) + 1, \quad \rho\in(0,a), \tag{12b}$$

$$\sum_{n=0}^{\infty} i\nu_n\left[p_n e^{i\nu_n l} - q_n e^{-i\nu_n l}\right] J_0(\xi_n \frac{\rho}{b}) = \sum_{m=0}^{\infty} g_m J_0(\xi_m \frac{\rho}{b}), \tag{12c}$$

and

$$\sum_{n=0}^{\infty}\left[p_n e^{i\nu_n l} + q_n e^{-i\nu_n l}\right] J_0(\xi_n \frac{\rho}{b}) = \sum_{m=0}^{\infty} h_m J_0(\xi_m \frac{\rho}{b}). \tag{12d}$$

Multiplying both sides of equations (12a) and (12b) by $\rho J_0(\xi_l \frac{\rho}{b})$ and by $J_0(\xi_l \frac{\rho}{a})$, respectively, and integrating from 0 to b and from 0 to a, respectively, we obtain the following system of linear algebraic equations (13a)-(13f),

$$\left.\begin{aligned} &\frac{a}{b}\sum_{m=0}^{\infty}\beta_m c_m \frac{J_0(j_m)}{(j_m/a)^2 - (\xi_l/b)^2}\times \\ &\quad \xi_l J_1(\xi_l \frac{a}{b}) - \frac{kab}{\xi_l} J_1(\xi_l \frac{a}{b}) = 0 \\ &k_1\eta_2 \sum_{m=0}^{\infty}(p_n + q_n)\frac{ab}{\xi_m^2 - \xi_l^2}\times \\ &\quad \left[\xi_l J_0(\xi_n \frac{a}{b}) J_1(\xi_l \frac{a}{b}) - \xi_n J_1(\xi_n \frac{a}{b}) J_0(\xi_l \frac{a}{b})\right] = 0 \end{aligned}\right\} n \neq l, \tag{13a}$$

$$\nu_n(p_n - q_n)\frac{b^2}{2}\frac{J_0^2(\xi_n)}{\xi_n^2}\left[\xi_n^2 - (k_1 b\eta_1)^2\right] = \left\{\begin{aligned} &\frac{a}{b}\sum_{m=0}^{\infty}\beta_m c_m \frac{J_0(j_m)}{(j_m/a)^2 - (\xi_l/b)^2}\times \\ &\quad \xi_l J_1(\xi_l \frac{a}{b}) + \frac{kab}{\xi_l} J_1(\xi_l \frac{a}{b}) \\ &-k_1\eta_2(p_n + q_n)\left\{\begin{aligned} &\frac{b^2}{2}\frac{J_0^2(\xi_n)}{\xi_n^2}\times \\ &\left[\xi_n^2 - (k_1 b\eta_1)^2\right] - \\ &\frac{a^2}{2}\left[J_0^2(\xi_n \frac{a}{b}) + J_1^2(\xi_n \frac{a}{b})\right] \end{aligned}\right\} \end{aligned}\right. , n = l, \tag{13b}$$

$$c_0 = \sum_{n=0}^{\infty}(p_n + q_n)\frac{2b}{a\xi_n} J_1(\xi_n \frac{a}{b}) - 1,\ m = 0, \tag{13c}$$

$$c_m = \frac{2}{abJ_0(j_m)} \times \sum_{n=0}^{\infty} \left[\begin{array}{l} (p_n + q_n) \dfrac{\xi_n}{(\xi_n/b)^2 - (j_m/a)^2} \\ \times J_1(\xi_n \dfrac{a}{b}) \end{array} \right], \quad m = 1, 2, ..., \tag{13d}$$

$$g_m = i\alpha_m \left[p_m e^{i\alpha_m l} - q_m e^{-i\alpha_m l} \right], \quad m = 0,1,2,...., \tag{13e}$$

$$h_m = p_m e^{i\alpha_m l} + q_m e^{-i\alpha_m l}, \quad m = 0,1,2,..... \tag{13f}$$

This system of equations can be rearranged as,

$$g_m - i\alpha_m h_m = -2i\alpha_m q_m e^{-i\alpha_m l}, \quad m = 0,1,2,...., \tag{13g}$$

$$g_m + i\alpha_m h_m = 2i\alpha_m p_m e^{i\alpha_m l} \quad m = 0,1,2,.... , \tag{13h}$$

$$\begin{aligned} &\nu_r (p_r - q_r) \frac{b^2}{2} \frac{J_0^2(\xi_r)}{\xi_r^2} \left[\xi_r^2 - (k_1 b \eta_1)^2 \right] = \\ &\frac{2}{b^2} \sum_{m=0}^{\infty} \beta_m \left[\sum_{n=0}^{\infty} (p_n + q_n) \frac{\xi_n J_1(\xi_n \frac{a}{b})}{(\xi_n/b)^2 - (j_m/a)^2} \right] \\ &\times \frac{\xi_r J_1(\xi_r \frac{a}{b})}{(j_m/a)^2 - (\xi_r/b)^2} + 2\frac{kab}{\xi_r} J_1(\xi_r \frac{a}{b}) \\ &- k_1 \eta_2 (p_r + q_r) \left\{ \begin{array}{l} \dfrac{b^2}{2} \dfrac{J_0^2(\xi_r)}{\xi_r^2} \left[\xi_r^2 - (k_1 b \eta_1)^2 \right] \\ -\dfrac{a^2}{2} \left[J_0^2(\xi_r \dfrac{a}{b}) + J_1^2(\xi_r \dfrac{a}{b}) \right] \end{array} \right\}. \end{aligned} \tag{13i}$$

To obtain an approximate value for $\dot{H}_+(a,\alpha)$ and $\dot{H}_-(a,\alpha)$, we substitute $\alpha = k, \alpha_1, \alpha_2, ..., \alpha_N$ in equation (6a) and $\alpha = -\delta_1, -\delta_2, ..., -\delta_N$ in equation (6b). These equations together with equations (13g) to (13i) result in $3(N+1)$ equations for $3(N+1)$ unknowns. The solution of these simultaneous equations yields approximate solutions for $\dot{H}_+(b,k)$, $\dot{H}_+(b,\alpha_1)$, $\dot{H}_+(b,\alpha_2)$,... and $\dot{H}_-(b,-\delta_1)$, $\dot{H}_-(b,-\delta_2)$,... . Using equations (5k) to (5m) we obtain equations (14a) and (14b) together with equations (14c) to (14h),

$$\begin{aligned} &-\frac{b}{2} \frac{J_0(j_r)(g_r - i\alpha_r h_r)}{2(k+\alpha_r) R_+(\alpha_r)} = \\ &\frac{b}{2} \sum_{m=0}^{\infty} \frac{J_0(j_m)(k+\alpha_m)}{2\alpha_m} \left\{ \begin{array}{l} \dfrac{(g_m + i\alpha_m h_m) R_+(\alpha_m)}{\alpha_r + \alpha_m} \\ -\dfrac{(g_m - i\alpha_m h_m) e^{i\alpha_m l}}{Q_+(\alpha_m)} T_{rm}^1 \end{array} \right\} \\ &-\frac{i}{2\pi} \sum_{m=0}^{\infty} S_m (k+\delta_m) \times \left\{ \frac{R_+(\delta_m)}{\delta_m + \alpha_r} e^{i\delta_m l} - \frac{T_{rm}^2}{Q_+(\delta_m)} \right\} \end{aligned} \tag{14a}$$

$$\begin{aligned} &\frac{i\delta_r}{\pi Z_r} \frac{J_1^2(Z_r b) - J_1^2(Z_r a)}{J_1(Z_r a) J_1(Z_r b)} \frac{Q_+(\delta_r) f_r}{(k+\delta_r)} = \\ &\frac{b}{2} \sum_{m=0}^{\infty} \frac{J_0(j_m)(k+\alpha_m)}{2\alpha_m} \times \left\{ \begin{array}{l} (g_m + i\alpha_m h_m) R_+(\alpha_m) T_{rm}^3 \\ + \dfrac{(g_m - i\alpha_m h_m) e^{i\alpha_m l}}{(\alpha_m + \delta_r) Q_+(\alpha_m)} \end{array} \right\} \\ &-\frac{i}{2\pi} \sum_{m=0}^{\infty} S_m (k+\delta_m) \times \left\{ \begin{array}{l} R_+(\delta_m) e^{i\delta_m l} T_{rm}^4 - \\ \dfrac{1}{Q_+(\delta_m)(\delta_m + \delta_r)} \end{array} \right\}, \end{aligned} \tag{14b}$$

$$S_m = \frac{J_1(Z_m a)}{J_1(Z_m b)} \frac{f_m}{Z_m}, \tag{14c}$$

$$T_{rm}^1 = \left\{ \begin{array}{l} -(kb)^2 \dfrac{R_+(k)}{Q_+(k)} \dfrac{e^{ikl}}{(k+\alpha_m)} W_{-1/2}(-il(\alpha_r + k)) \\ + \dfrac{2k\pi b^2}{(a^2 - b^2)} \dfrac{R_+(k) e^{ikl}}{Q_+(k)(k+\alpha_r)(k+\alpha_m)} \times \\ \displaystyle\sum_{n=0}^{\infty} \frac{\dfrac{(k+\delta_n) R_+(\delta_n)}{(k-\delta_n) H_1^{(1)}(Z_n b) Q_+(\delta_n)} \times}{\dfrac{H_1^{(1)}(Z_n a) e^{i\delta_n l}}{(\delta_n + \alpha_m)(\delta_n + \alpha_r) \dot{M}(-\delta_n)}} \end{array} \right\}, \tag{14d}$$

$$\begin{aligned} T_{rm}^2 &= (kb)^2 \frac{R_+(k)}{Q_+(k)} \frac{e^{ikl}}{(k+\delta_m)} W_{-1/2}(-il(\alpha_r + k)) \\ &- \frac{2k\pi b^2}{(a^2 - b^2)} \frac{R_+(k) e^{ikl}}{Q_+(k)(k+\alpha_r)(k+\delta_m)} \times \\ &\frac{(k+\delta_n) R_+(\delta_n) H_1^{(1)}(Z_n a) e^{i\delta_n l}}{(\delta_m + \delta_n)(k-\delta_n) H_1^{(1)}(Z_n b) Q_+(\delta_n)(\delta_n + \alpha_r) \dot{M}(-\delta_n)}, \end{aligned} \tag{14e}$$

$$\begin{aligned} T_{rm}^3 &= (kb)^2 \frac{R_+(k)}{Q_+(k)(\alpha_m + k)} e^{ikl} W_{-1/2}(-il(\delta_r + k)) \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 e^{i\alpha_n l}}{\alpha_n (\delta_r + \alpha_n)(\alpha_n + \alpha_m) Q_+(\alpha_n)}, \end{aligned} \tag{14f}$$

$$\begin{aligned} T_{rm}^4 &= (kb)^2 \frac{R_+(k)}{Q_+(k)(\delta_m + k)} e^{ikl} W_{-1/2}(-il(\delta_r + k)) \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{R_+(\alpha_n)(k+\alpha_n)^2 e^{i\alpha_n l}}{\alpha_n (\delta_r + \alpha_n)(\alpha_n + \delta_m) Q_+(\alpha_n)}. \end{aligned} \tag{14g}$$

The function $W_{-1/2}(\xi)$ is related to the Whittaker function $W_{-1/2,0}(\xi)$ [15] by the relation (14h),

$$W_{-1/2}(\xi) = \exp(\xi/2) \xi^{-1/2} W_{-1/2,0}(\xi). \tag{14h}$$

By substituting equations (13g) and (13h) into equations (14a) and (14b) and also considering equation (13i), one can easily obtain the three infinite systems of linear algebraic equations with coefficients p_n , q_n and f_n .

IV. THE RADIATED FAR-FIELD AND COMPUTATIONAL RESULTS

The radiated field in the region $\rho > b$ can be obtained using,

$$u_1(\rho,z) = -\frac{1}{2\pi}\int_L \frac{H_0^{(1)}(K\rho)}{K(\alpha)H_1^{(1)}(Kb)}\times [\dot{H}_-(b,\alpha)+e^{i\alpha l}\dot{H}_+(b,\alpha)]e^{-i\alpha z}d\alpha \qquad (15a)$$

where L is a straight line parallel to the real α -axis, lying in the strip $Im(-k) < Im(\alpha) < Im(k)$. Utilizing the asymptotic expansion of $H_0^{(1)}(k\rho)$ as $k\rho \to \infty$

$$H_0^{(1)}(K\rho) = \sqrt{\frac{2}{\pi K\rho}}e^{i(K\rho-\pi/4)}. \qquad (15b)$$

The asymptotic evaluation of the integral in equation (15a) using the saddle point technique yields for the diffracted field for $k\sqrt{\rho^2+z^2} >> kl$,

$$u_1(\rho,z) \approx \frac{i}{\pi}\left\{ \begin{array}{l} \dfrac{e^{ikr_1}}{kr_1}\dfrac{\dot{H}_+(b,-k\cos\theta_1)}{\sin\theta_1 H_1^{(1)}(kb\sin\theta_1)} \\ +\dfrac{e^{ikr_2}}{kr_2}\dfrac{\dot{H}_-(b,-k\cos\theta_2)}{\sin\theta_2 H_1^{(1)}(kb\sin\theta_2)} \end{array} \right\} \qquad (16)$$

where $\dot{H}_+(b,\alpha)$ and $\dot{H}_-(b,\alpha)$ are given by equations (6a) and (6b), respectively. r_1,θ_1, and r_2,θ_2 are the spherical coordinates defined by

$$\rho = r_1\sin\theta_1 ,\ z = r_1\cos\theta_1 \qquad (17a)$$

and

$$\rho = r_2\sin\theta_2 ,\ z-l = r_2\cos\theta_2 . \qquad (17b)$$

In the far field region equation (16) reduces to

$$u_1(\rho,z) \approx \frac{i}{\pi}\left\{ \frac{\dot{H}_+(b,-k\cos\theta_1)+e^{-ikl\cos\theta_1}\dot{H}_-(b,-k\cos\theta_1)}{\sin\theta_1 H_1^{(1)}(kb\sin\theta_1)} \right\}\frac{e^{ikr_1}}{kr_1}. \qquad (18)$$

We can see that f_m and q_m decay exponentially with m so that the infinite algebraic systems converge very rapidly. Thus, they can be solved by truncating the infinite matrix and numerically inverting the resulting finite system. The value of the truncation number N is increased until the final physical quantities such as the amplitude of the radiated field or the reflection coefficients become insensitive up to desired digit after the decimal point.

The reflection coefficient is calculated by using hybrid mode-matching (hMM)/ method-of-moment (MoM) technique presented in the waveguide synthesis program for waveguide networks WASP-Net [16]. The reflection coefficient calculating by WH is very close to hMM-MoM. The discrepancy between WH and hMM-MoM is %0.23 at the dominant mode propagation of the waveguide. The amplitude of the reflection coefficient is reduced by increasing the radius of the waveguide (*ka*) and the length of the aperture (*kl*) while *kb* is fixed. It is observed that the relative errors are reduced for higher frequencies by increasing number truncation number N.

Showing numerically can make another effective check of the analysis that the continuity relation in equation (12b) is satisfied. The absolute error is less than %1.02 for $N \geq 14$.

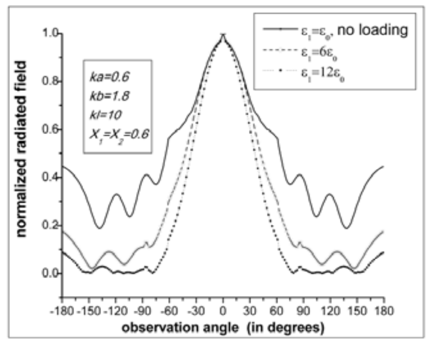


Fig. 3. Normalized radiated field versus the observation angle for different values of the k_1 $(\eta_1 = iX_1, \eta_2 = iX_2, X_1, X_2 > 0)$.

Figure 3 shows the variation of the normalized diffracted field amplitude $\left|u_1(r_1,\theta_1)/u_1(r_1,0)\right|$ versus the observation angle θ_1, for different values of k_1 when ka, kb and kl is fixed. Note that the directivity of the horn increases with increasing values of the dielectric material. Also it has been noted side lobe level is decrease explicitly with increasing values of the dielectric material.

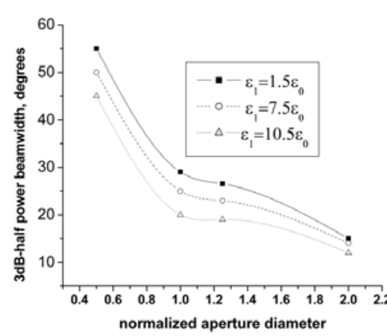


Fig. 4. 3dB beam-width to aperture diameter ($2b/\lambda$) ($X_1 = X_2 = 0.1, a/\lambda = 0.6, l = 1.5\lambda$).

Figure 4 shows the variation of the -3dB beamwidth versus the observation angle for different values of normalized aperture diameter. The 3-dB beamwidth decrease with the increasing values of $2b/\lambda$. Also note that the 3-dB beamwidth of the horn decreases with increasing values of the dielectric material.

Finally, Fig. 5 display the amplitude of the relative power level obtained in the present work for $a/\lambda = 0.0875$, $b/\lambda = 0.5$, $l/\lambda = 1.6$, the numerical results calculated by using MoM programmed by [17]. We can see that the results obtained in this work approach the numerical solution for $\eta_1 = \eta_2$ and fit quite well along the observation angle.

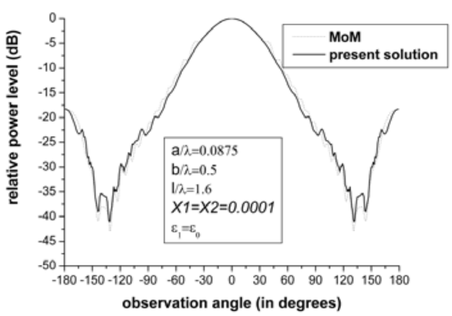


Fig. 5. Relative power level versus the observation angle (comparison with the MoM solution).

V. CONCLUSION

The radiation of plane harmonic scalar waves from a dielectric loaded using the mode matching method in conjunction with the Wiener-Hopf technique treats scalar feed horn. The solution is exact but formal since infinite series of unknowns and some branch-cut integrals with unknown integrands are involved. Approximation procedures based on rigorous asymptotic are used and the approximate solution to the Wiener-Hopf equations are derived in terms of infinite series of unknowns, which are determined from infinite systems of linear algebraic equations. The advantage of the WH Technique over other methods is that it is rigorous in the sense that the edge condition is explicitly incorporated in the analysis and that it has the potential of providing accurate and reliable results over broad frequency ranges. Furthermore, contrary to some numerical techniques, which are efficient only when the problem involves finite boundaries of limited length, the WH method does not suffer from restrictions. Numerical solution of these systems is obtained for various values of the dielectric materials of the problem and their effect on the directivity of the circular feed horn is presented in the scope of this work. By dielectric loading, it is possible to narrowing of the beamwidth and can provide low levels of the side lobes.

ACKNOWLEDGEMENT

The author is indebted to the referees for their constructive critics, which led to improving this work.

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