COMPUTATION OF STATIC AND QUASISTATIC ELECTROMAGNETIC FIELDS USING ASYMPTOTIC BOUNDARY CONDITIONS

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***Abstract* — This paper presents the computation of static and quasistatic electromagnetic fields using asymptotic boundary conditions (ABC). Asymptotic boundary conditions for eddy current problems due to external field excitations are derived. For electrostatic fields, ABC-s are used in conjunction with Laplace's equation while for quasistatic magnetic fields, ABC-s are employed in conjunction with the integrodifferential finite element method. The effect of outer boundary locations on the accuracy of the simulation results is examined. This study shows that in these cases, ABC-s can improve the computation accuracy compared to the usual truncation of outer boundaries.**

1. INTRODUCTION

The finite element method (FEM) is a powerful method for the computation of electromagnetic fields. However, special techniques must be used when the solution domain is infinite, since the exterior region must be properly represented. Researchers sometimes use a simple approach in which the outer boundary is truncated with a *Dirichlet* or *Neumann* boundary condition. In addition, there are a number of well-known techniques to modify the finite element method to accommodate the open regions [1]. Examples of such modifications are ballooning [2], infinitesimal scaling [3], spatial transformations [4], infinite elements [5] and others. Unfortunately, these modifications are limited by various shortcomings. An alternative approach is to combine the finite element method with the integral equation method or Green's function approach to account for the open region. Good examples of such combination are the hybrid finite element - boundary integral equation method [6] and the integrodifferential finite element - Green's function method [7]. Usually, hybrid approaches destroy the sparsity of the finite element matrices. The measured equation of invariance (MEI) method, which was presented in [8], uses the assumed charge distributions on the conductors to determine the relationships of the unknown potentials around outer boundaries. The relationships thus obtained are subsequently employed in the finite element or finite difference simulations. The MEI method can preserve the sparsity of the finite element or finite difference matrices. However, the proof of convergence for this method has yet to be found in the published papers.

Recently, absorbing and asymptotic boundary conditions have been used in conjunction with the finite element method [9]-[12]. The sudden popularity of the ABC is due to the fact that it is local as compared to hybrid approaches. This locality preserves the sparsity of finite element matrices. However, most of the studies are focussed on wave problems. This paper applies the asymptotic boundary conditions to the study of static and quasistatic problems. Although there are studies using the ABC for static or quasistatic problems [13]-[16], there are still unanswered questions such as the effects of outer boundary locations on the solution accuracy. In addition, the use of asymptotic boundary conditions for eddy current problems due to external field excitations has not been reported.

This paper investigates the employment of asymptotic boundary conditions in conjunction with finite elements for the computation of static electric and quasistatic magnetic field problems. For the electrostatic problem, the electric scalar potential and stored energy of two parallel infinitely long, circular cylindrical conductors are calculated. For the quasistatic magnetic field problem, the ABC-s due to external field excitations are derived and used for the calculation of the induced eddy current power losses of an infinitely long, circular cylindrical conductor, excited by a uniform transverse magnetic (TM) field.

The accuracy of ABC-s are compared with analytical and known numerical results, where applicable, as well as brute force truncations. The objective of this study is to evaluate the usefulness and limitations of asymptotic boundary conditions.

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2. FORMULATION OF ASYMPTOTIC BOUNDARY CONDITIONS

The essence of using the finite element method for the solution of unbounded field problems is the proper representation of the exterior region. The spectrum of the various techniques for such solutions are not without shortcomings. The recent flurry of work on absorbing and asymptotic boundary conditions underscores the need for and importance of an efficient technique suitable for the finite element implementation. The ABC-s resolve the difficulties associated with an infinite boundary by emulating the field behaviour at infinity on the finite boundary. The absorbing boundary conditions are used for wave propagation and scattering. In this paper, we are only concerned with asymptotic boundary conditions which are derived for static or quasistatic fields.

The outer boundary used in the asymptotic boundary conditions serves as an impedance junction to connect the region internal to it with the region external to it. Such a connection is facilitated by the surface integrals of the normal derivatives of the unknowns which represent the flux continuity conditions. As a result, the derivation of asymptotic boundary conditions centres on the representation of the normal derivatives of the unknown scalar potentials.

For a source-free static field, the potential ϕ, subject to suitable boundary conditions, is governed by *Laplace*'s equation

$$\nabla^2\phi = 0 \tag{1}$$

For a two-dimensional problem, if the potential is zero at infinity, the solution in the infinite exterior region in the polar coordinates can be expressed as the following harmonic expansions:

$$\phi = \sum_{n=1}^{\infty} \frac{a_n}{r^n} \cos(n\theta+\alpha_n) \tag{2}$$

where a_n and α_n are the coefficient and phase angle of the nth harmonic, respectively.

Differentiation of (2) with respect to r leads to

$$\frac{\partial\phi}{\partial r} = -\sum_{n=1}^{\infty} n\frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n) \tag{3}$$

The division of (2) by r yields

$$\frac{\phi}{r} = \sum_{n=1}^{\infty} \frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n) \tag{4}$$

Summation of (3) and (4) produces

$$\frac{\partial\phi}{\partial r} + \frac{\phi}{r} = -\sum_{n=1}^{\infty} n\frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n)$$

$$+\sum_{n=1}^{\infty} \frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n) \tag{5}$$

$$\frac{\partial\phi}{\partial r} + \frac{\phi}{r} = \sum_{n=2}^{\infty} (1-n)\frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n) \tag{6}$$

If we omit the third and higher harmonics, the dominant error will be determined by the second harmonic and we have

$$\frac{\partial\phi}{\partial r} + \frac{\phi}{r} = -\frac{a_2}{r^3}\cos(2\theta+\alpha_2) \tag{7}$$

Equation (7) can be subsequently rewritten as

$$\frac{\partial\phi}{\partial r} + \frac{\phi}{r} = O(r^{-3}) \tag{8}$$

The first order asymptotic boundary operator is therefore

$$B_1 = \frac{\partial}{\partial r} + \frac{1}{r} \tag{9}$$

Consequently, the first order asymptotic boundary condition is

$$B_1(\phi) = 0 \tag{10}$$

If we let $u=B_1\phi$, then

$$u = \sum_{n=2}^{\infty} (1-n)\frac{a_n}{r^{n+1}} \cos(n\theta+\alpha_n) \tag{11}$$

The derivative of u with respect to r is

$$\frac{\partial u}{\partial r} = -\sum_{n=2}^{\infty} (1-n)(1+n)\frac{a_n}{r^{n+2}} \cos(n\theta+\alpha_n) \tag{12}$$

The ratio of u to r is given by

$$\frac{u}{r} = \sum_{n=2}^{\infty} (1-n)\frac{a_n}{r^{n+2}} \cos(n\theta+\alpha_n) \tag{13}$$

Therefore,

$$\frac{\partial u}{\partial r} + \frac{3u}{r} = \sum_{n=3}^{\infty} (n-2)(n-1)\frac{a_n}{r^{n+2}} \cos(n\theta+\alpha_n) \tag{14}$$

It is obvious that the dominant error is due to the third harmonic. Thus,

$$\frac{\partial u}{\partial r} + \frac{3u}{r} = 2\frac{a_2}{r^5} \cos(5\theta+\alpha_5) \tag{15}$$

Equation (15) can be represented by

$$\frac{\partial u}{\partial r} + \frac{3u}{r} = O(r^{-5}) \tag{16}$$

The second order asymptotic boundary operator is therefore

$$B_2 = (\frac{\partial}{\partial r} + \frac{3}{r})(\frac{\partial}{\partial r} + \frac{1}{r}) = O(r^{-5}) \tag{17}$$

Due to the presence of the second order radial derivative $\partial^2\phi/\partial r^2$ in (17), the second order asymptotic boundary operator cannot be used directly for the finite element implementation. To overcome this difficulty, we rewrite (17) as

$$\frac{\partial^2\phi}{\partial r^2} + \frac{4}{r}\frac{\partial\phi}{\partial r} + \frac{2\phi}{r^2} = 0 \tag{18}$$

To eliminate the second order radial derivative, we substitute (18) back into the *Laplace*'s equation in the polar coordinates. The resultant second order asymptotic boundary expression is

$$\frac{\partial\phi}{\partial r} = -\frac{2}{3r}\phi + \frac{1}{3r}\frac{\partial^2\phi}{\partial\theta^2} \tag{19}$$

As a result, the first and second order asymptotic boundary conditions can be subsequently expressed as

$$\frac{\partial\phi}{\partial r} = a(r)\phi + b(r)\frac{\partial^2\phi}{\partial\theta^2} \tag{20}$$

where $a(r)$ and $b(r)$ are given as follows:

$a(r) = -\frac{1}{r};$ $\quad b(r) = 0$ $\quad$ for *first order ABC*

$a(r) = -\frac{2}{3r};$ $\quad b(r) = \frac{1}{3r}$ $\quad$ for *second order ABC*

3. FINITE ELEMENT IMPLEMENTATION

For a generalized *Helmholtz* equation of the form

$$\nabla\cdot(p\nabla\phi) + k^2 q\phi = 0 \tag{21}$$

we can use *Galerkin*'s criterion to transform it into

$$\int_{\Gamma} [p\nabla W\cdot\nabla\phi - k^2 qW\phi]dv = \int_{\Gamma_i} pW\frac{\partial\phi}{\partial n}ds + \int_{\Gamma_o} pW\frac{\partial\phi}{\partial n}ds \tag{22}$$

where p and q are related to material properties and angular frequencies, W is the weighting function which is the same as the interpolation function of the finite elements, Γ_i is the regular *Neumann* boundary and Γ_o is the asymptotic boundary.

In the case of static electric fields, (21) is reduced to *Laplace*'s equation.

In the case of time-harmonic quasistatic magnetic fields, (21) can be transformed into the integrodifferential equation in the conductors [17] and *Laplace*'s equation outside the conductors with p corresponding to the reluctivity of the medium and k^2 representing $j\sigma$. These transformed equations are amenable to the use of asymptotic boundary conditions by the substitution of the appropriate asymptotic boundary expressions into the surface integrals involving the asymptotic boundaries.

4. APPLICATIONS

To illustrate the application of ABC-s, we study an electrostatic potential problem and a quasistatic magnetic field problem.

4.1 Electrostatic Problem

The electrostatic problem consists of two parallel infinitely long circular cylindrical conductors as shown in Fig. 1. The two conductors are at potentials of 1 and -1 volt, respectively. The potential distribution and stored energy are calculated and compared with analytic and published results [13]. Due to symmetry, only the upper region (above the line *CB*) is discretized.

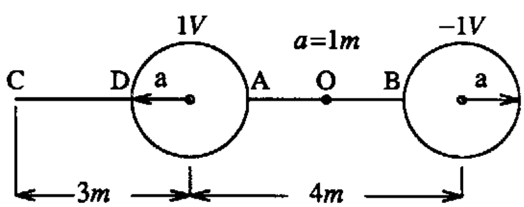

Fig. 1 Two circular cylindrical conductors with different potentials.

Figures 2 and 3 depict the potential distributions along the lines *AOB* and *CD*, respectively. Table 1 shows the stored electric energy. It is noted that the ABC improves the calculated results. While the homogeneous *Neumann* boundary condition (no-ABC FEM) can yield very accurate potentials along the line *AOB* (see Fig. 3), it produces much larger errors along the line *CD*. The asymptotic boundary condition, however, renders accurate solutions at both *AOB* and *CD*. The asymptotic boundary is at a radius of *6m* in Ref. [13]. In this paper, the asymptotic and the homogeneous *Neumann* boundaries are both half circles with a radius of *5m* and centred at *O*.

Table 1 Stored Electric Energy ($\times 10^{-10} J$)

ABC FEM	NO-ABC FEM	Ref. [12]
0.3855	0.3631	0.3946

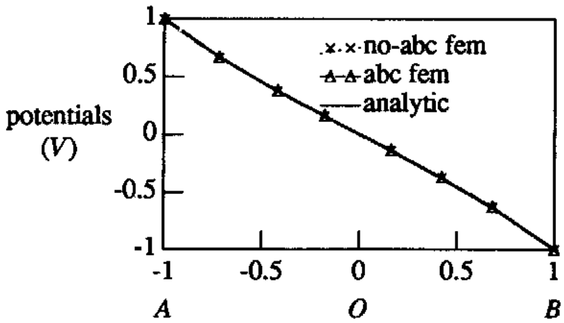

Fig. 2 Potential distribution along the line AOB.

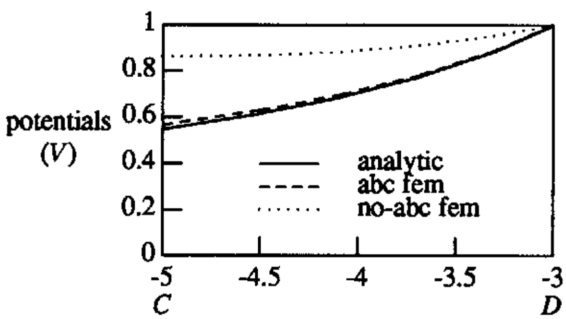

Fig. 3 Potential distribution along the line CD.

4.2 Quasistatic Magnetic Field Problem

The quasistatic magnetic field problem consists of an infinitely long circular cylindrical conductor excited by a uniform transverse magnetic field as shown in Fig. 4. The induced power loss in the conductor is calculated and compared with analytic results [17] and results obtained with the hybrid integrodifferential finite element - Green's function method [18].

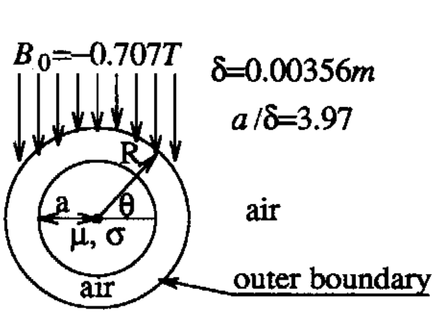

Fig. 4 A circular cylindrical conductor excited by a TM field.

It should be noted that in this eddy current problem, the z-component of the magnetic vector potential **A** is composed of two terms; one is the source term and the other is the reaction term. It is the reaction term, not the source term, that satisfies the ABC-s. Therefore, it is necessary to make appropriate transformations to accommodate this. In this 2D problem, the z-component of the magnetic vector potential **A** is denoted by A hereafter.

The integrodifferential equation governing eddy currents in the conductor is given in [7] as

$$\frac{1}{\mu}\nabla^2 A - j\omega\sigma A + j\omega\sigma\frac{\int \sigma A\, dS}{\int \sigma\, dS} = 0 \tag{23}$$

Let A be the total magnetic vector potential, it can be expressed as

$$A = A_e + A_s \tag{24}$$

where A_e is the reaction magnetic vector potential and A_s is the source magnetic vector potential. A_e satisfies the asymptotic boundary conditions. If the asymptotic boundary is circular, we have

$$\frac{\partial A}{\partial r} = \frac{\partial A_e}{\partial r} + \frac{\partial A_s}{\partial r} \tag{25}$$

From (20), we have

$$\frac{\partial A_e}{\partial r} = a(r)A_e + b(r)\frac{\partial^2 A_e}{\partial \theta^2} \tag{26}$$

Substituting for A_e and $\frac{\partial A_e}{\partial r}$ from (24) and (25) into (26) leads to

$$\frac{\partial A}{\partial r} = a(r)(A - A_s) + b(r)[\frac{\partial^2 A}{\partial \theta^2} - \frac{\partial^2 A_s}{\partial \theta^2}] + \frac{\partial A_s}{\partial r} \tag{27}$$

Rearranging of (27) produces the following expression

$$\frac{\partial A}{\partial r} = a(r) + b(r)\frac{\partial^2 A}{\partial \theta^2} + M_s \tag{28}$$

where M_s is given by

$$M_s = -a(r)A_s - b(r)\frac{\partial^2 A_s}{\partial \theta^2} + \frac{\partial A_s}{\partial r} \tag{29}$$

Substituting (28) and (29) into the integrodifferential finite element equation accomplishes the asymptotic boundary formulation for eddy current problems due to external field excitations.

If the outer boundary is at a sufficiently large distance from the eddy current conductor, the normal derivative of the reaction magnetic vector potential can be assumed to be zero.

Therefore, (25) can be rewritten as

$$\frac{\partial A}{\partial r} = \frac{\partial A_s}{\partial r} \tag{30}$$

Substitution of (30) into (22) leads to the inhomogeneous *Neumann* boundary condition integrodifferential finite elements.

In this eddy current problem, the uniform transverse magnetic field can be represented by the source magnetic vector potential as follows [19]:

$$A_s = -B_0 r \cos\theta \tag{31}$$

where B_0 is the source flux density and θ is as shown in Fig. 4.

Fig. 5 shows calculated power loss errors using first and second order ABC integrodifferential finite elements. Fig. 6 illustrates the loss errors employing the inhomogeneous *Neumann* boundary condition integrodifferential finite elements. The power loss error using the hybrid integrodifferential finite element - Green's function method is 0.12% [19]. In these plots, the abscissa is the ratio of the radius R of the outer boundary and the conductor skin depth δ.

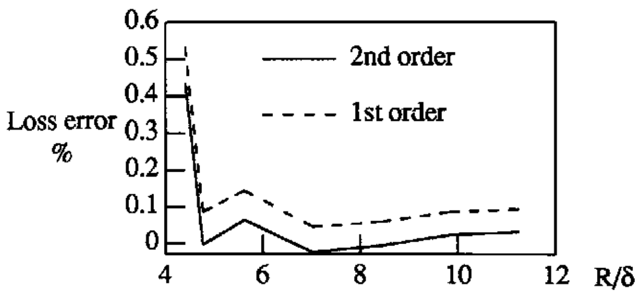

Fig. 5 Power loss errors using first and second order ABC integrodifferential finite elements as a function of the locations of the outer boundary.

It is noted that ABC-s using integrodifferential finite elements, like the hybrid integrodifferential finite element - Green's function method, can provide very accurate results. On the other hand, inhomogeneous *Neumann* boundary condition can only yield accurate solutions when the outer boundary is sufficiently far away from the conductors (R/δ is greater than 45). It is also observed that the second order ABC is generally better than the first order ABC and the ABC boundary need not be placed far from the conductor to obtain accurate results. It is sufficient to place the outer boundary half a skin depth away from the conductor surface. The mesh generation of the outer region also bears impact on the solution accuracy. The meshes should not be severely

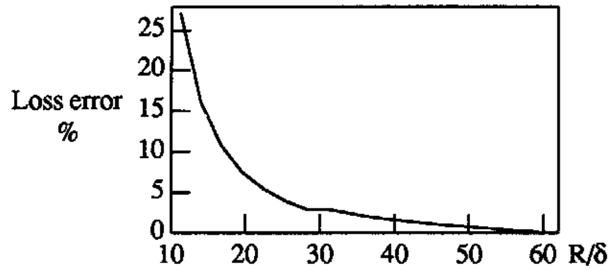

Fig. 6 Power loss errors with the inhomogeneous *Neumann* boundary condition as a function of the locations of the outer boundary.

unequilateral.

Since ABC with finite elements preserves the sparsity of the finite element matrices, it is a better choice compared to hybrid approaches in terms of computer memory and programming complexities.

5. CONCLUSIONS

This paper presents a study of the computation of static and quasistatic electromagnetic fields using finite elements with the asymptotic boundary conditions. Asymptotic boundary conditions for eddy current problems due to external field excitations are also derived. The effect of outer boundary locations on the solution accuracy is investigated. The study reveals that the employment of asymptotic boundary conditions improves the calculation results compared to the use of homogeneous and inhomogeneous *Neumann* boundary conditions. The accuracy of finite elements with ABC-s, like hybrid approaches, is very satisfactory. For eddy current problems, it is sufficient to place the outer boundary half a skin depth away from the conductor surface.

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