TECHNICAL FEATURE ARTICLE

POWER SERIES ANALYSIS OF WEAKLY NONLINEAR CIRCUITS

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ABSTRACT

This is the third in a series of articles that explores the analysis and modeling of nonlinear behaviors in circuits, devices, and receiver systems. Analytic and numerical methods can be developed to readily analyze complex nonlinearities from elemental formulations such as the weakly nonlinear series. The topics discussed are quite general and have application to such diverse areas as automatic control, broadcasting, cable television, communications, EMC, electronic devices, instrumentation, signal processing, and systems theory. The previous articles in this series discussed the nonlinear effects of intermodulation, spurious responses, desensitization, cross modulation, gain compression/expansion as well as the concepts of average power, available power and/or exchangeable power [1-3]. In this article, we discuss in greater depth the various nonlinear modes and mechanisms that may arise in practical systems and components that incorporate nonlinear devices.

INTRODUCTION

All circuits containing electronic components are inherently nonlinear. Nevertheless, the preponderance of analyses dealing with electronic circuits assumes linear behavior. This paradox exists because (1) linear circuits are characterized by linear equations that are relatively easy to solve, (2) many nonlinear circuits can be adequately approximated by equivalent linear circuits provided the input signals are sufficiently small, and (3) closed-form analytical solutions of nonlinear equations are not ordinarily possible.

One model of a nonlinear circuit that is readily analyzed is shown in Figure 1. Observe that this model consists of a zero-memory nonlinearity preceded and followed by isolated linear filters. Use of this model is referred to as the power series approach. The nonlinearity is characterized in the time domain by its power series coefficients $\{a_1, a_2, \ldots, a_N\}$ and is said to be weakly nonlinear when only the first few terms of the power series are needed to represent the nonlinear behavior. Typically the linear filters that model the linear circuits preceding and following the nonlinear portion of the electronic device are characterized in the frequency domain by their linear transfer functions H(f) and K(f).

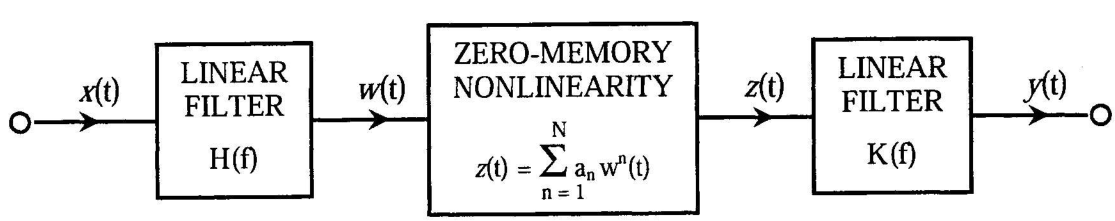

Figure 1. Power Series Model for a Nonlinear System With Memory

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A system is said to have memory when the output at time *t* depends upon values of the input prior to time *t*. The nonlinearity in Figure 1 has zero memory because its output at a specific instant of time depends upon its input only at the same instant. Circuits containing energy storage elements have memory while purely resistive circuits have zero memory. Since the linear filters in Figure 1 are intended to be frequency selective, they contain energy storage elements. As a result, their outputs depend upon the past history of their inputs and the nonlinear system, as a whole, possesses memory.

The power series model is readily analyzed because the individual blocks shown in Figure 1 can be treated as isolated segments. Specifically, given the input $x(t)$, the output $w(t)$ of the first linear filter is readily obtained using conventional linear analysis. The output $z(t)$ of the zero-memory nonlinearity is then determined by substitution of $w(t)$ into the power series representation of the nonlinearity. Finally, the circuit output $y(t)$ is easily obtained as the response of the second linear filter to the known input $z(t)$. Thus, analysis of the power series model readily proceeds from input to output.

Although the power series model is an adequate representation for many electronic circuits, the reader is cautioned that this model is not universally applicable. A more general model is based upon the Volterra series or nonlinear transfer function approach [1]. However, the power series model does provide insight into the many nonlinear effects that occur in weakly nonlinear circuits [2].

Response of First Linear Filter

Allowing for the presence of interfering signals in addition to the desired signal, assume Q sinusoidal signals to be present at the input to the power series model shown in Figure 1. Hence,

$$x(t) = \sum_{q=1}^{Q} |E_q| \cos(2\pi f_q t + \theta_q). \tag{1}$$

By introducing the complex amplitude

$$E_q = |E_q|\, e^{j\theta_q} \tag{2}$$

and defining

$$E_{-q} = E_q^*,\ E_0 = 0,\ f_{-q} = -f_q, \tag{3}$$

the excitation can be expressed as

$$x(t) = 1/2 \sum_{q=1}^{Q} \left[E_q e^{j2\pi f_q t} + E_q^* e^{-j2\pi f_q t}\right]$$

$$= 1/2 \sum_{q=-Q}^{Q} E_q e^{j2\pi f_q t}. \tag{4}$$

The transfer function of the first linear filter in Figure 1 is denoted by

$$H(f) = |H(f)| e^{j\psi(f)}. \tag{5}$$

Because the filter is linear, superposition applies. Consequently, its response is a sum of sinusoids at the same frequencies as those contained in the input. Each sinusoidal output magnitude at a particular frequency equals the corresponding input magnitude multiplied by the magnitude of the transfer function at that frequency while each output phase angle equals the corresponding input angle plus the phase angle of the transfer function at that frequency. In particular,

$$w(t) = \sum_{q=1}^{Q} |E_q|\, |H(f_q)| \cos\left[2\pi f_q t + \theta_q + \psi(f_q)\right] \tag{6}$$

For real circuits the transfer function has the property that

$$H(-f) = H^*(f) = |H(f)|e^{-j\psi(f)}. \qquad (7)$$

As a result, $w(t)$ can be written as

$$w(t) = 1/2 \sum_{q=1}^{Q} [E_q H(f_q) e^{j2\pi f_q t} + E_q^* H^*(f_q) e^{-j2\pi f_q t}]$$

$$= 1/2 \sum_{q=-Q}^{Q} E_q H(f_q) e^{j2\pi f_q t}. \qquad (8)$$

Response of Zero-Memory Nonlinearity

The output of the zero-memory nonlinearity is

$$z(t) = \sum_{n=1}^{N} a_n w^n(t). \qquad (9)$$

The n^{th} term in this sum is said to be of n^{th} degree because it involves $w(t)$ raised to the n^{th} power. Focusing on the n^{th}-degree portion of $z(t)$,

$$a_n w^n(t) = a_n/2^n \left[\sum_{q=-Q}^{Q} E_q H(f_q) e^{j2\pi f_q t} \right]^n =$$

$$a_n/2^n \sum_{q_1=-Q}^{Q} \cdots \sum_{q_n=-Q}^{Q} E_{q_1} \cdots E_{q_n} H(f_{q_1}) \cdots H(f_{q_n}) e^{j2\pi (f_{q_1}+\ldots+f_{q_n})t}. \qquad (10)$$

Consequently, Equation (9) can be written as

$$z(t) = \sum_{n=1}^{N} \sum_{q_1=-Q}^{Q} \cdots \sum_{q_n=-Q}^{Q} A_n(q_1,\ldots,q_n) e^{j2\pi(f_{q_1}+\ldots+f_{q_n})t} \qquad (11)$$

where

$$A_n(q_1,\ldots,q_n) = a_n/2^n E_{q_1} \cdots E_{q_n} H(f_{q_1}) \cdots H(f_{q_n}). \qquad (12)$$

Response of Second Linear Filter

Having obtained $z(t)$, the final step in the analysis is determination of $y(t)$, the output of the power series model. Note that this is also the response of the second linear filter whose transfer function is

$$K(f) = |K(f)|\, e^{j\phi(f)}. \qquad (13)$$

As was the case with the first linear filter, superposition applies and

$$y(t) = \sum_{n=1}^{N} \sum_{q_1=-Q}^{Q} \cdots \sum_{q_n=-Q}^{Q} B_n(q_1,\ldots,q_n) e^{j2\pi(f_{q_1}+\ldots+f_{q_n})t} \qquad (14)$$

where

$$B_n(q_1,\ldots,q_n) = A_n(q_1,\ldots,q_n) K(f_{q_1}+\ldots+f_{q_n}). \qquad (15)$$

Observe that the magnitude of $B_n(q_1,\ldots,q_n)$ is

$$|B_n(q_1,\ldots,q_n)| = a_n/2^n |E_{q_1}|\cdots|E_{q_n}|$$

$$|H(f_{q_1})|\cdots|H(f_{q_n})|\; |K(f_{q_1}+\ldots+f_{q_n})| \qquad (16)$$

while its angle is given by

$$\angle B_n(q_1,\ldots,q_n) =$$

$$\theta_{q_1}+\ldots+\theta_{q_n} + \psi(f_{q_1}) + \ldots + \psi(f_{q_n}) + \phi(f_{q_1}+\ldots+f_{q_n}). \qquad (17)$$

The most striking feature concerning the response, as given by Equation (14), is the presence of new frequencies not contained in the input. Terms involving these new frequencies are referred to as inter-modulation components. Their complex amplitudes depend upon the complex input amplitudes, the power series coefficients, and the linear transfer functions of the two filters evaluated at the appropriate frequencies.

Because $E_0 = 0$, each sum in the n-fold summation of Equation (10) contains 2Q terms. Consequently, the total number of terms in the n^{th}-degree portion of $z(t)$ is $(2Q)^n$. This number grows rapidly with increasing values of Q and n. The number of terms in $y(t)$ is identical to that of $z(t)$. As a result, evaluation of all of the terms in Equation (14) is extremely tedious. Simplification of this expression is discussed next.

Total Response for a Particular Frequency Mix

Intermodulation components whose frequencies fall well outside of the system passband are usually not troublesome because they are greatly attenuated by the frequency selectivity of the system. Thus, with regard to Equation (14), it is necessary to focus only on those terms whose frequencies fall either within or close to the system passband.

For example, consider a system tuned to 50 MHz with a 1 MHz bandwith. Assume the input to consist of two interfering sinusoids at $f_1 = 46$ MHz and $f_2 = 48$ MHz. If the system contains a nonlinearity, an intermodulation component at $2f_2 - f_1 = 50$ MHz may be generated. This falls at the tuned frequency and may cause significant interference with the desired signal when the amplitudes of the interfering tones are sufficiently large. On the other hand, the intermodulation component whose frequency is $2f_1 + f_2 = 140$ MHz falls will outside of the system passband and may be ignored.

Therefore, the first step in evaluating Equation (14) is to determine which intermodulation frequencies are of concern. Having done this, the second step is to determine the manner by which the pertinent intermodulation frequencies are generated. For this purpose, the concept of a frequency mix is introduced.

A frequency mix is characterized by the number of times the various frequencies appear in the frequency sum $(f_{q_1}+\ldots+f_{q_n})$ of Equation (14). For example, consider a power series model for which N = 5. Focus on the intermodulation frequency given by $2f_2 - f_1$. Corresponding to n=3, $2f_2 - f_1$ is produced by the single frequency mix $(f_2 + f_2 - f_1)$. Corresponding to n=5, $2f_2 - f_1$ is produced by the two frequency mixes $(f_2 + f_2 + f_2 - f_2 - f_1)$ and $(f_2 + f_2 + f_1 - f_1 - f_1)$.

As far as frequency mixes are concerned, the order in which the frequencies appear is unimportant. For example, $(f_2 - f_1 + f_2)$ represents the same mix as does $(-f_1 + f_2 + f_2)$ and $(f_2 + f_2 - f_1)$. What is important is the number of times each frequency appears in the mix. Note that $(f_2 + f_2 - f_1)$ involves $-f_1$ once and f_2 twice, $(f_2 + f_2 + f_2 - f_2 - f_1)$ involves $-f_2$ once, $-f_1$ once, and f_2 three times, while $(f_2 + f_2 + f_1 - f_1 - f_1)$ involves $-f_1$ twice, f_1 once, and f_2 twice.

The concept of a frequency mix has been introduced in order to clarify the manner by which an intermodulation frequency is produced. To avoid confusion, frequency mixes, such as $(f_2 + f_2 + f_1 - f_1 - f_1)$, are enclosed in parentheses while intermodulation frequencies, such as $2f_2 - f_1$, are not.

To aid in the representation of a frequency mix, let the number of times that the frequency f_k appears be denoted by m_k. Considering negative frequencies, recall that $f_{-k} = -f_k$. Therefore, for an excitation consisting of Q sinusoidal tones, as given by Equations (1) and (4), the input frequencies are $f_{-Q},\ldots,f_{-1}, f_1,\ldots,f_Q$. It follows that any possible frequency mix can be represented by the frequency mix vector

$$\underline{m} = (m_{-Q},\ldots,m_{-1}, m_1,\ldots,m_Q). \qquad (18)$$

By way of example, assume Q = 2 corresponding to the four input frequencies f_{-2}, f_{-1}, f_1, f_2 and the frequency mix vector

$$\underline{m} = (m_{-2}, m_{-1}, m_1, m_2). \qquad (19)$$

The frequency mix $(f_2 + f_2 - f_1)$ is represented by $\underline{m}$ = (0, 1, 0, 2) while $\underline{m}$ = (1, 1, 0, 3) represents the frequency mix $(f_2 + f_2 + f_2 - f_2 - f_1)$ and $\underline{m}$ = (0, 2, 1, 2) represents the frequency mix $(f_2 + f_2 + f_1 - f_1 - f_1)$.

In general, the nth-order frequency mix $(f_{q_1}+\ldots+f_{q_n})$, as appears in Equation (14), can be expressed as

$$f_{\underline{m}} = \sum_{\substack{k=-Q \\ k\neq 0}}^{Q} m_k f_k$$

$$= m_{-Q}f_{-Q}+\ldots+m_{-1}f_{-1} + m_1f_1+\ldots+m_Qf_Q. \qquad (20)$$

Since exactly n frequencies are involved in an nth-order frequency mix, it follows that

$$\sum_{\substack{k=-Q \\ k\neq 0}}^{Q} m_k = m_{-Q}+\ldots+m_{-1} + m_1+\ldots+m_Q = n. \quad (21)$$

With regard to the n indices $q_1,\ldots,q_n$ of Equation (14), observe that $B_n(q_1,\ldots,q_n)$ and $(f_{q_1}+\ldots+f_{q_n})$ are unchanged by a permutation of the indices. Consequently, many of the terms in Equation (14) are identical. Corresponding to a particular frequency mix vector $\underline{m}$, it can be shown that the number of identical terms is given by the multinomial coefficient

$$(n; \underline{m}) = (n!)/[(m_{-Q}!)\ldots(m_{-1}!)(m_1!)\ldots(m_Q!)]. \quad (22)$$

For example, consider the frequency mix $(f_2 + f_2 + f_2 - f_2 - f_1)$ for which n = 5 and $\underline{m}$ = (1, 1, 0, 3). Using Equation (22), the number of identical terms in Equation (14) contributing to this mix is

$$(5; 1, 1, 0, 3) =$$

$$(5!)/[(1!)\ (1!)(0!)(3!)] = 20. \qquad (23)$$

Combining all of these terms, the resulting intermodulation component is given by

$$y_5\,(t; 1, 1, 0, 3) =$$

$$20a_5/32[E_2^{*}E_1^{*}E_2^{3}H^{*}(f_2)\,H^{*}(f_1)H^{3}(f_2)\,K(2f_2 - f_1)\,e^{j2\pi(2f_2 - f_1)t}]. \qquad (24)$$

In general, let the sum of identical terms corresponding to a particular frequency mix vector $\underline{m}$ of order n be denoted by $y_n(t; \underline{m})$. It follows that $y_n(t; \underline{m})$ can be expressed as

$$y_n\,(t; \underline{m}) =$$

$$\frac{(n; \underline{m})\,a_n}{2^n}\,(E_Q^{*})^{m_{-Q}}\ldots(E_1^{*})^{m_{-1}}\,(E_1)^{m_1}\ldots(E_Q)^{m_Q}$$

$$\left[H^{*}(f_Q)\right]^{m_{-Q}}\ldots\left[H^{*}(f_1)\right]^{m_{-1}}\left[H(f_1)\right]^{m_1}\left[H(f_Q)\right]^{m_Q}$$

$$K(f_{\underline{m}})e^{j2\pi f_{\underline{m}}t} \qquad (25)$$

Although $y_n(t; \underline{m})$ is complex, $y(t)$ is real. Therefore, terms in Equation (14) exist in conjugate pairs.

Characterization of a nonlinear electronic circuit by the power series model requires specification of the prefilter transfer function H(f), the postfilter transfer function K(f), and the power series coefficients $\{a_1, a_2, \ldots, a_N\}$. When accurate predictions of the nonlinear responses are desired, one might expect that accurate modeling of the power series coefficients is the most critical task. Equation (25) reveals that this is not the case. The prefilter transfer function H(f) appears as a factor n times whereas the power series coefficient a_n and the postfilter transfer function K(f) appear only once. As a result, errors in the modeling of H(f) may be much more serious than similar errors in the modeling of a_n and K(f). Because of the accuracy to which H(f) must be known in order to stay within a prescribed output error, it may be exceedingly difficult to make accurate predictions of nonlinear responses when n is large.

SUMMARY

This article discussed the characterization of weakly nonlinear electronic circuits by the power series modeling approach. This approach requires the specification of prefilter and postfilter transfer functions, and the power series coefficients. The responses of the first and second linear filters, zero-memory nonlinearities, and the total response for a particular frequency mix were mathematically described. It was shown that because of the accuracy to which the prefilter transfer function must be known in order to stay within a prescribed output error, it may be exceedingly difficult to make accurate predictions of nonlinear responses when the number of terms, n, to be considered is large. In the next article in this series, we will continue to discuss the various nonlinear modes and mechanisms that arise in practical systems. The series will conclude with a presentation on new findings of research and development to add nonlinear analysis and prediction capabilities to existing CEM tools that are used to determine detailed interference rejection requirements for large, complex systems.

REFERENCES

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