A Low-Frequency Formulation of the Method of Moments via Surface Charges

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***Abstract*-In this paper a formulation of the method of moments for the analysis of low frequency problems is presented.**

In the considered frequency range, the integral solution of Maxwell equations in terms of magnetic vector potential and electric scalar potential respectively function of currents and charges is obtained imposing the Coulomb gauge.

By combining Gauss law and current continuity at the boundaries among regions with different conductivity a first set of equations is obtained. Writing Ohm's law inside the conductive regions another integral equation set that allows the determination of the conduction current and surface charges unknowns is obtained. The method of moments is then applied to this system of equations.

The use of pulse functions as subsectional bases allows a quick matrix set up especially when regular volume shapes are selected. Calculated results are compared with results obtained with other methods relating to benchmark problems.

I. Introduction

Time and frequency domain formulations of the method of moments [1], have been widely used for the analysis of the electromagnetic scattering by arbitrary shaped perfectly conducting bodies excited by electromagnetic pulses and incident waves [2], [3].

The proposed techniques, using different patches for the body modeling, have been proved to be simple and useful procedures to solve those problems handling open and closed curved structures of finite extent [4], [5].

In this paper, we investigate the extension of the method in the low frequency range. In this case, some basic features of the theoretical basis of the high frequency formulations of the method cannot be assumed:

- Considering finite conductivity, we cannot use the equation set usually given by enforcing the tangential component of the electric field equal to zero at the surface of the conductive bodies.

- Considering Coulomb gauge together with the hypothesis of div J = 0 in the conductive volume we cannot use the direct relation between vector and scalar potentials, usually given by the Lorentz gauge.

Therefore, in order to obtain a system of volume integral equations in the conduction current unknowns, we have to take into account different equations in the considered frequency range. A first set of equation in terms of scalar and vector potential is obtained considering, in presence of finite conductivity, Ohm's law and the electric field definition inside the conductive volumes. The scalar potential is given by the surface free charge distribution at the boundaries among media with different conductivity. Then, an equation set relating surface charges and volume currents has to be obtained. The relation among charges and currents is determined considering Gauss law and current density continuity at these boundaries. Time and frequency formulations can be adopted depending on the transient or steady state analysis to be performed. Of course, as in the high frequency range, the numerical efficiency of the presented procedure in terms of computational costs and accuracy of the results is influenced by the original domain approximation and by the shape function that can be adopted.

II. Formulation

We consider a massive conductor Ω with boundary Γ taking into account the presence of an external lumped parameter circuit as shown in fig. 1.

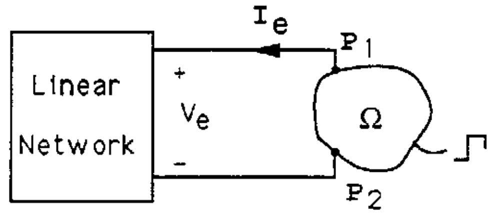

Fig. 1 Considered system

Combining the definition of the electric field $E(r,t)$ and Ohm's law we have:

$$E = -\frac{\partial A}{\partial t} - \nabla\Phi = \frac{J}{\sigma} \tag{1}$$

we consider the magnetic vector potential A expression:

$$A(r,t) = \frac{\mu_o}{4\pi}\iiint_\Omega \frac{J(r',t)}{|r-r'|}d\Omega \tag{2}$$

The scalar potential Φ is given by:

$$\Phi(r,t) = \frac{1}{4\pi\varepsilon_o}\iint_\Gamma \frac{\rho_\Gamma(r',t)}{|r-r'|}d\Gamma \tag{3}$$

where $\rho_\Gamma(r',t)$ is the surface free charge distribution at the boundary Γ. Substituting (2) and (3) in (1) we have:

$$\frac{J(r,t)}{\sigma} = -\frac{\mu_o}{4\pi}\iiint_\Omega \frac{\partial J(r',t')}{\partial t}\frac{1}{|r-r'|}d\Omega +$$
$$-\frac{1}{4\pi\varepsilon_o}\nabla\iint_\Gamma \frac{\rho_\Gamma(r',t)}{|r-r'|}d\Gamma \tag{4}$$

Equation (4) has to be evaluated for the three components x,y,z in space. In order to determine an expression that relates surface charges and conduction current we combine Gauss law, current density continuity and Ohm's law at the surface Γ:

$$div_{\Sigma}E = E_n^+ - E_n^- = \frac{\rho_\Gamma}{\varepsilon_o};$$

$$div_{\Sigma}J = -J_n^- = -\frac{\partial\rho_\Gamma}{\partial t}; \qquad J_n^- = \sigma E_n^-; \tag{5}$$

where the subscript n denotes the normal component, the superscripts + and - denote the outside and inside limit values with respect to the surface Γ and div_{Σ} is the surface divergence. Combining these equations we obtain:

$$\frac{1}{\sigma}\frac{\partial\rho_\Gamma}{\partial t} + \frac{\rho_\Gamma}{\varepsilon_o} = E_n^+ = -\frac{\mu_o}{4\pi}\iiint_\Omega \frac{\partial J(r',t')}{\partial t}\frac{1}{|r^+ - r'|}d\Omega -$$

$$-\frac{1}{4\pi\varepsilon_o}\nabla\iint_\Gamma \frac{\rho_\Gamma(r',t)}{|r^+ - r'|}d\Gamma \tag{6}$$

The current $J(r,t)$ and the surface charges $\rho_\Gamma(r,t)$ can be approximated as:

$$J(r,t) = \sum_{n=1}^{N} I_n(t) f_n(r) \tag{7}$$

$$\rho(r,t) = \sum_{n=1}^{M} \rho_n(t) g_n(r) \tag{8}$$

where $f_n(r)$ and $g_n(r)$ are the n-th component of the selected vector basis function representing the spatial variation of the two quantities, $I_n(t)$ and $\rho_n(t)$ are their time-varying unknown coefficients. Then, we obtain:

$$\frac{\sum_{n=1}^{N} I_n(t) f_n(r)}{\sigma} = -\frac{\mu_o}{4\pi}\sum_{n=1}^{N}\frac{\partial I_n(t)}{\partial t}\iiint_\Omega \frac{f_n(r')}{|r-r'|}d\Omega -$$

$$-\sum_{n=1}^{M}\frac{\rho_n(t)}{4\pi\varepsilon_o}\iint_\Gamma g_n(r')\nabla\frac{1}{|r-r'|}d\Gamma \tag{9}$$

$$\sum_{n=1}^{M} g_n(r^+)\left[\frac{\partial\rho_n(t)}{\sigma\partial t} + \frac{\rho_n(t)}{\varepsilon_o}\right] =$$

$$= -\frac{\mu_o}{4\pi}\sum_{n=1}^{N}\frac{\partial I_n(t)}{\partial t}\iiint_\Omega \frac{f_n(r')}{|r^+ - r'|}d\Omega +$$

$$-\frac{1}{4\pi\varepsilon_o}\sum_{n=1}^{M}\rho_n(t)\iint_\Gamma g_n(r')\nabla\frac{1}{|r^+ - r'|}d\Gamma \tag{10}$$

The presence of the external lumped parameter circuit is taken into account considering that at points P_1 and P_2 we have:

$$I_e - \iint_S J_n^- dS = -\iint_S \frac{\partial\rho_\Gamma}{\partial t}dS; \tag{11}$$

where S is the area of the surface elements at P_1 and P_2. Writing the nodal or mesh equations of the linear lumped parameters network we have:

$$I_e(t) = G(t) + YV_e(t) \tag{12a}$$

$$V_e(t) = \Phi(P_1,t) - \Phi(P_2,t) =$$

$$= \frac{1}{4\pi\varepsilon_o}\iint_\Gamma \frac{\rho_\Gamma(r',t)}{|P_1 - r'|} - \frac{\rho_\Gamma(r',t)}{|P_2 - r'|}d\Gamma \tag{12b}$$

where G(t) is due to the independent generators and Y is an integral-differential operator due to the impedances of the network.

Imposing (9) at N points in the conductive body Ω, (12a) (12b) and (10) at M-2 points on the surface Γ we obtain a system of equations in the N+M unknown coefficients $I_n(t)$ and $\rho_n(t)$ to which the testing procedure (point matching, Galerkin etc.) is applied.

Approximating the time differentiation with finite difference equations (9) (10) (12a) and (12b) directly represent the time domain formulation of the proposed procedure. The frequency domain formulation can be easily derived replacing the time differentiation with the $j\omega$ operator and assuming the $e^{j\omega t}$ dependence of the unknown quantities, obtaining the algebraic complex system:

$$\sum_{n=1}^{N} I_n\left[\frac{f_n(r)}{\sigma} + \frac{j\omega\mu_o}{4\pi}\iiint_\Omega \frac{f_n(r')}{|r-r'|}d\Omega\right] =$$

$$-\sum_{n=1}^{M}\frac{\rho_n}{4\pi\varepsilon_o}\iint_\Gamma g_n(r')\nabla\frac{1}{|r-r'|}d\Gamma \tag{13}$$

$$\sum_{n=1}^{M}\rho_n\left[g_n(r^+)\left(\frac{j\omega}{\sigma} + \frac{1}{\varepsilon_o}\right) + \frac{1}{4\pi\varepsilon_o}\iint_\Gamma g_n(r')\nabla\frac{1}{|r^+ - r'|}d\Gamma\right] =$$

$$-\frac{j\omega\mu_o}{4\pi}\sum_{n=1}^{N} I_n\iiint_\Omega \frac{f_n(r')}{|r^+ - r'|}d\Omega \tag{14}$$

III. BASIS FUNCTIONS

We define the following coefficients:

$$\iiint_\Omega \frac{f_n(r')}{|r-r'|}d\Omega = \alpha_n(r); \qquad g_n(r^+) = g_n;$$

$$\iint_\Gamma g_n(r')\nabla\frac{1}{|r-r'|}d\Gamma = -\iint_\Gamma \frac{g_n(r')}{|r-r'|^2}d\Gamma = \beta_n(r); \tag{15}$$

Then, considering for example subsectional bases [1] for N elementary volumes in the domain Ω and M elementary surfaces on the boundary Γ and writing (9) and (10) in the k-th elementary volume and surface (referring for example to the frequency domain formulation) we obtain:

$$I_k \frac{f_k(r_k)}{\sigma} + \frac{j\omega\mu_o}{4\pi}\sum_{n=1}^{N} I_n \alpha_n(r_k) = -\sum_{n=1}^{M}\frac{\rho_n}{4\pi\varepsilon_o}\beta_n(r_k) \quad (16)$$

$$\rho_k g_k\left(\frac{j\omega}{\sigma}+\frac{1}{\varepsilon_o}\right)+\frac{1}{4\pi\varepsilon_o}\sum_{n=1}^{M}\rho_n\beta_n(r^+) =$$

$$= -\frac{j\omega\mu_o}{4\pi}\sum_{n=1}^{N} I_n \alpha_n(r^+) \quad (17)$$

$$I_e = G + Y(\omega)V_e \quad (18a)$$

$$V_e = \Phi(P_1)-\Phi(P_2) = \frac{1}{4\pi\varepsilon_o}\sum_{n=1}^{M}\rho_n\left(\beta_n(P_1)-\beta_n(P_2)\right) \quad (18b)$$

Considering good conductors at low frequencies we can consider $1/\varepsilon_0$ much greater than $j\omega/\sigma$, then (14) becomes:

$$\frac{\rho_k g_k}{\varepsilon_o}+\frac{1}{4\pi\varepsilon_o}\sum_{n=1}^{M}\rho_n\beta_n(r^+) = -\frac{j\omega\mu_o}{4\pi}\sum_{n=1}^{N} I_n \alpha_n(r^+) \quad (19)$$

Writing (16) in every elementary volume, (19) on the surface and equations (18) we derive the matrix forms:

$$\underline{\underline{A}}\,\underline{I} + \underline{\underline{B}}\,\underline{\rho} = 0 \qquad\qquad \underline{\underline{D}}\,\underline{I} + \underline{\underline{E}}\,\underline{\rho} = \underline{V}$$

These systems can be used in order to have an unknown set only, currents or charges.

Analogously, in the time domain equations, $\rho\sigma/\varepsilon_0$ can be considered much greater than $\partial\rho/\partial t$. Therefore, we can use (10) to express the charges as a function of currents, obtaining an expression in the current unknowns only.

The accuracy of the presented procedure and the computational cost of the matrix set up is determined by the coefficients (15). As for the high frequency problems, several kinds of basis function and patches can be proposed, and an exhaustive analysis of their numerical characteristics in several kinds of applications is not easy to be determined.

We begin implementing pulse functions as subsectional bases in order to have a quick matrix set-up. In fact, when Ω_n have particular shapes, such as parallelepipeds or cylinder sectors, the evaluation of the surface and volume integrals in (14) can be quickly and accurately obtained by means of analytical expressions [6], [7]. This choice causes the presence of fictitious surface charges at the boundary among adjacent elementary volume elements, since adjacent currents have different values as shown in fig. 2.

Nevertheless, the evaluation of integral quantities in the examined low frequency benchmark problem, has shown a good agreement between our results and results obtained with other numerical methods. Furthermore, the computational times were analogous to those obtained with Finite Elements methods.

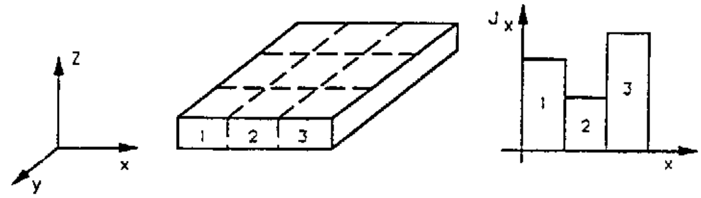

Fig. 2 Pulse function decomposition of a plate

IV. Results

In order to test the frequency domain formulation of the method, we analyse the TEAM problem 3 [8] (Bath plate with two holes) which geometry, shown in fig. 3, consists of a conducting ladder having two holes with a current coil above. The conductivity of the ladder is σ= 0.3278e8 (S/m), and the driving field originates in the coil. The coil carries a current equivalent to 1260 Amp turns at 50 and 200 Hz. and it is placed at two positions. Position 1 is directly above the center of the ladder, position 2 is directly above the center of one of the holes.

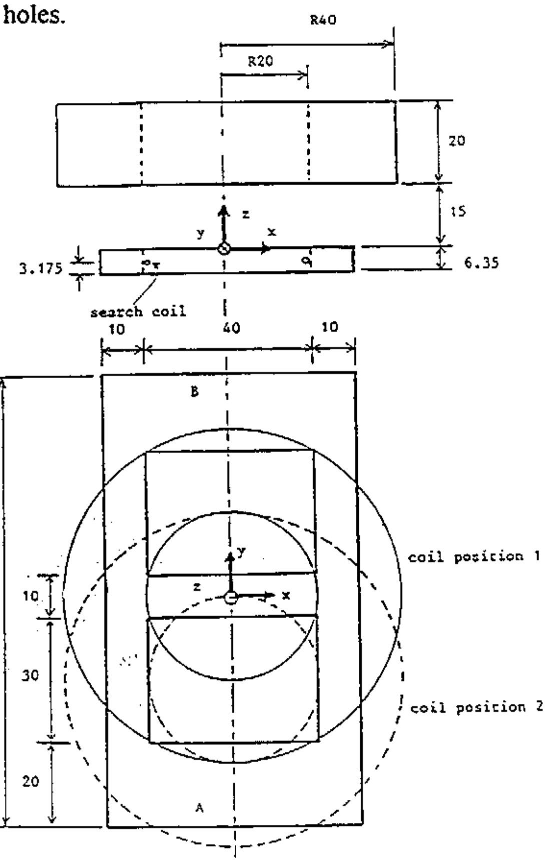

Fig. 3 Geometry of the frequency domain problem.

We calculate the magnitude and phase of the field normal to the ladder for the coil position 2, frequency of 50Hz, along a center line of the geometry 0.5 mm above the conducting ladder between +55mm and -55mm. The calculated data have been obtained considering the decomposition shown in Fig. 4. We adopted parallelepiped volumes and rectangular surfaces as subsectional bases.

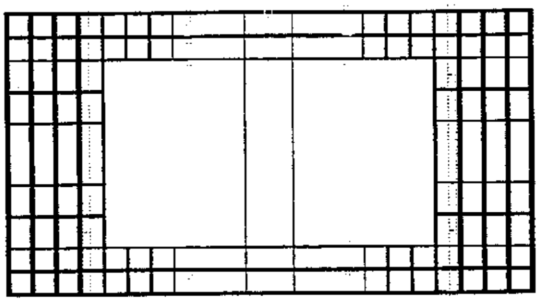

Fig. 4 Decomposition of the geometry.

Fig. 5 shows a good agreement between calculated results obtained with the proposed method and experimental data.

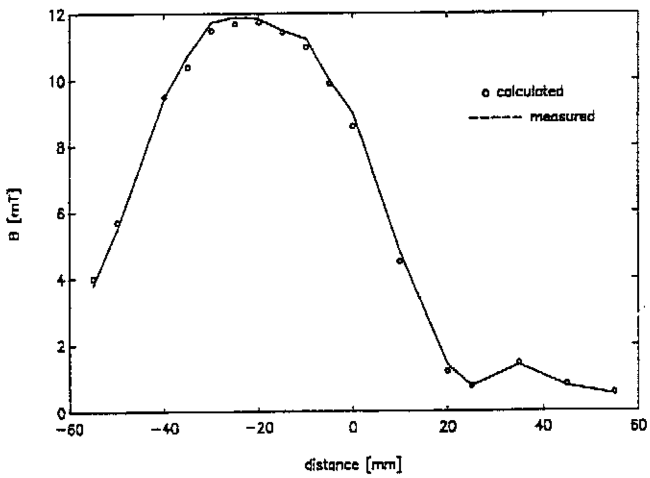

Fig. 5 Comparison between calculated and experimental data of B along the center line

The table I and II report the current in the lateral limbs for positions 1 and 2 for f=50 Hz and f=200Hz respectively.

Table I

	Position 1	Position 2	
	Ia = Ib [A]	Ia [A]	Ib [A]
Kameari [10]	68.05	81.68	38.45
Takahashi [9]	71.63	85.41	41.23
M.O.M.	69.6	82.61	40.42

Table II

	Position 1	Position 2	
	Ia = Ib [A]	Ia [A]	Ib [A]
Kameari [10]	153.89	190.48	80.14
Takahashi [9]	155.89	191.52	83.12
M.O.M.	154.91	192.55	82.95

The time domain formulation has been tested considering a problem [10] where a uniform magnetic field in the Z direction, having a constant rate B' = 1 T/sec, is applied to a square plate (20cm x 20cm) with thickness 1cm shown in fig. 6. The resistivity of the plate is 2μΩcm.

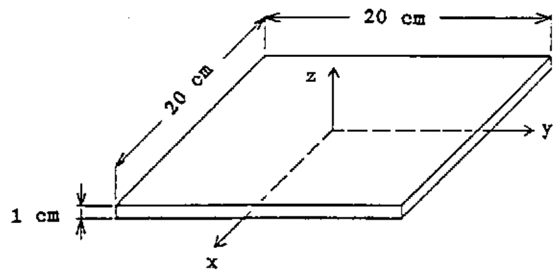

Fig. 6 Geometry of the time domain problem

The square plate has been decomposed by means of elements having different size as shown in Fig. 7.

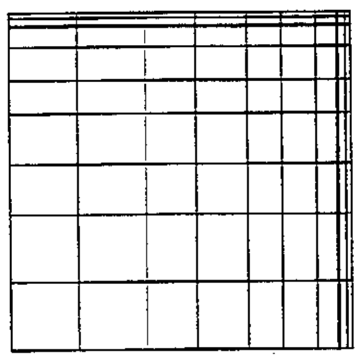

Fig. 7 Decomposition of the geometry.

The figures 8 and 9 show the current density J_y at t=10msec and, varying the x-coordinate, at y=1cm and y=x. The figures show a good agreement among the calculated data by a Finite Element method [10] and the proposed M.O.M. formulation.

The larger disagreement is obtained near the external surface of the plate, where the method [10] forces J_n=0, while the proposed formulation gives the average value in the parallelepiped nearby the corner. In the considered low frequency tests we considered point matching and Galerkin testing procedures, obtaining nearly equal numerical results.

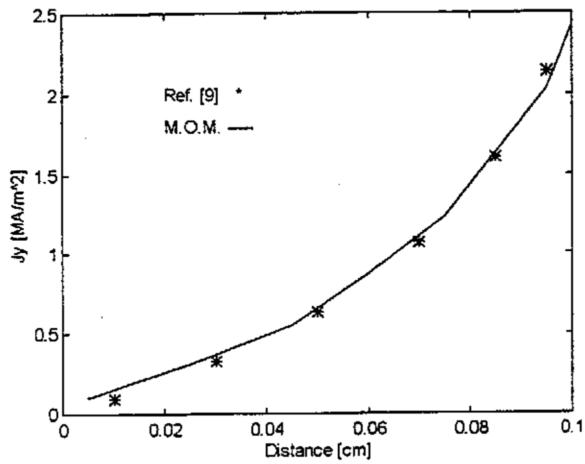

Fig. 8 Eddy current distribution at t=10msec along the line y=1cm.

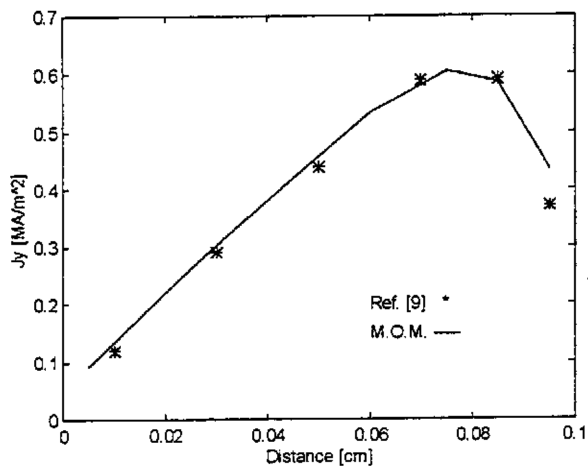

Fig. 9 Eddy current distribution at t=10msec along the line y=x.

V. Conclusions

Time and frequency domain formulations of the method of moments for the analysis of low frequency problems have been presented.

The two formulations have been tested analysing linear benchmark problems obtaining a good agreement with experimental data and results obtained with other numerical methods.

The use of pulse functions as subsectional basis has allowed a quick matrix set-up with respect to numerical integration.

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