Diffraction of Obliquely Incident Plane Waves by an Impedance Wedge with Surface Impedances Being Equal to the Intrinsic Impedance of the Medium

Turgut İkiz and Mustafa K. Zateroğlu

Department of Electrical and Electronics Engineering
University of Cukurova, Adana, 01330, Turkey
tikiz@cu.edu.tr, mzateroglu@cu.edu.tr

Abstract **─** Diffraction of plane waves by an impedance wedge with surface impedances equal to the intrinsic impedance of surrounding medium is investigated for oblique incidence case. In the oblique incidence case, the scattering problem cannot be solved explicitly because of the resultant coupled system of functional equations unless the system is decoupled. Therefore under the assumed condition on wedge impedance, these functional equations are decoupled and the expression for the diffraction coefficient is derived as well as the diffracted fields.

Index Terms **─** Functional equations, impedance wedge, Maliuzhinets theorem, Sommerfeld integrals.

I. INTRODUCTION

In many practical applications, scatterers are partly wedge shaped metallic structures covered by dielectric materials or metallic structures with finite conductivity which can be simulated with impedance boundary conditions. Therefore, the problem of diffraction by an impedance wedge is investigated by a number of scientists and is very important for both civil and military applications.

Diffraction by an impedance wedge was first solved by Maliuzhinets for the normal incidence case [1]. In this solution, the total field was expressed by the integral of an unknown spectral function. The unknown spectral function was determined using the boundary conditions, the edge conditions, and the radiation condition. The fundamental contribution of the Maliuzhinets method is the reduction of the integral equation into a first order functional equation. But for the oblique incidence case, the problem cannot be solved explicitly, since the resultant equations form a coupled functional equations system.

The solutions for the problem under consideration are available only for some limited wedge opening angles and only under some assumption for the surface impedance of this wedge [2-22].

In this study, applying the Leontovich boundary conditions, a coupled differential equations system is derived for the z-components of the fields. Using the similarity transformation, the relevant matrices are diagonalized assuming that the surface impedance is equal to the free space impedance.

The solution for the Helmholtz equation is sought in the form of Sommerfeld integrals. In order to solve the Maliuzhinets functional equations, the Maliuzhinets theorem is applied to the Sommerfeld integrals. Solving the functional equations, the closed form solution is derived and the uniform asymptotic solution is obtained by applying the steepest descent path method to the Sommerfeld integrals. The numerical results are obtained for different wedge opening and incidence angles and they are shown in Figs. 3 through 7.

II. FORMULATION OF THE PROBLEM

The problem under consideration is a wedge with an opening angle of 2Φ, where the edge coincides with the z-axis. The direction of propagation of the incidence wave is specified by the angles β and ϕ_0 as shown in Fig. 1. The incident field is determined by the z-components of the electromagnetic field.

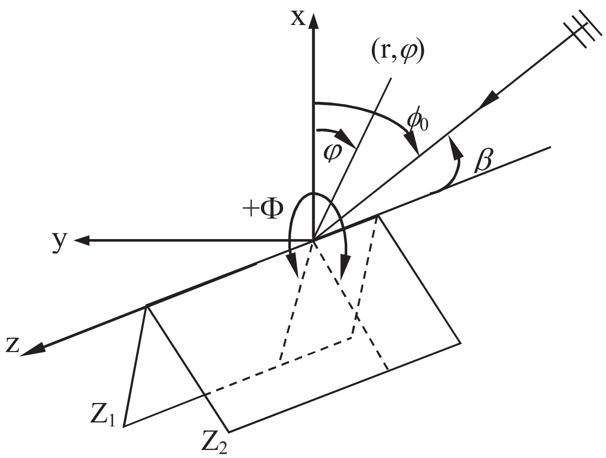

Fig. 1. The geometry of the problem.

Due to the invariance of both the wedge geometry and the impedance with respect to z, the problem can be reduced to a two dimensional problem and the z-components of the electric and magnetic field vectors of the incident wave can be represented as

$$\tilde{H}_z^i = H_z^i(r,\varphi)\exp(ik\cos\beta z), \quad (1)$$

$$\tilde{E}_z^i = E_z^i(r,\varphi)\exp(ik\cos\beta z), \quad (2)$$

where

$$H_z^i = H_0 \exp\{-ikr\sin\beta\cos(\varphi-\varphi_0)\}, \quad (3)$$

and

$$E_z^i = E_0 \exp\{-ikr\sin\beta\cos(\varphi-\varphi_0)\}. \quad (4)$$

Using the Maxwell's equations, the field components can be expressed in terms of z-components as follows:

$$H_r = \frac{1}{iZ_0 k\sin^2\beta}\left(\frac{1}{r}\frac{\partial E_z}{\partial\varphi} - Z_0\cos\beta\frac{\partial H_z}{\partial r}\right), \quad (5)$$

and

$$E_r = \frac{i}{k\sin^2\beta}\left(\frac{Z_0}{r}\frac{\partial H_z}{\partial\varphi} + \cos\beta\frac{\partial E_z}{\partial r}\right), \quad (6)$$

where $k=\omega\sqrt{\varepsilon_0\mu_0}$ is the free space wave number and Z_0 is the free space impedance given by

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \quad (7)$$

On the surfaces of the wedge, the Leontovich impedance boundary condition can be represented as

$$\vec{E} - (\hat{n}.\vec{E})\hat{n} = Z_{1,2}\hat{n}\times\vec{H}. \quad (8)$$

Applying boundary conditions result in a matrix equation system defined as

$$\frac{1}{r}\begin{bmatrix}\dfrac{\partial H_z}{\partial\varphi}\\ \dfrac{\partial E_z}{\partial\varphi}\end{bmatrix}\Bigg|_{S_j,\varphi_j} = (-1)^{j+1} ik\sin^2\beta A\begin{bmatrix}H_z\\ E_z\end{bmatrix}\Bigg|_{S_j} + \cos\beta B\begin{bmatrix}\dfrac{\partial H_z}{\partial r}\\ \dfrac{\partial E_z}{\partial r}\end{bmatrix}\Bigg|_{S_j}, \quad (9)$$

where sentence

$$A = \begin{bmatrix}\dfrac{Z_j}{Z_0} & 0\\ 0 & \dfrac{Z_0}{Z_j}\end{bmatrix}, \quad (10)$$

and

$$B = \begin{bmatrix}0 & -\dfrac{1}{Z_0}\\ Z_0 & 0\end{bmatrix}, \quad (11)$$

and j=1,2. To obtain the unknown, this coupled matrix system must be diagonalized. Applying similarity transformation to matrix B can produce a diagonal matrix system. To reach this aim the transformation matrix can first be written as

$$\begin{bmatrix}H_z\\ E_z\end{bmatrix} = P\begin{bmatrix}u\\ v\end{bmatrix}, \quad (12)$$

where P is the similarity transform of matrix B, and is defined as

$$P = \begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}. \quad (13)$$

After the necessary manipulations P can be rewritten as

$$P = \begin{bmatrix}i & \dfrac{1}{Z_0}\\ Z_0 & i\end{bmatrix}. \quad (14)$$

By using this similarity transform matrix, matrices A and B are diagonalized as follows

$$P^{-1}AP=\begin{bmatrix} \frac{1}{2}\left(\frac{Z_j}{Z_0}+\frac{Z_0}{Z_j}\right) & \frac{i}{2Z_0}\left(\frac{Z_0}{Z_j}-\frac{Z_j}{Z_0}\right) \\ i\frac{Z_0}{2}\left(\frac{Z_j}{Z_0}-\frac{Z_0}{Z_j}\right) & \frac{1}{2}\left(\frac{Z_j}{Z_0}+\frac{Z_0}{Z_j}\right) \end{bmatrix}, \quad (15)$$

and

$$P^{-1}BP=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \quad (16)$$

Within equation (15) the diagonalization condition is observed as

$$\frac{Z_0}{Z_j}-\frac{Z_j}{Z_0}=0, \quad (17)$$

and finally the decoupled matrix system can be written as

$$\frac{1}{r}\begin{bmatrix} \frac{\partial u}{\partial \varphi} \\ \frac{\partial v}{\partial \varphi} \end{bmatrix}_{S_j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(-1)^{j+1} ik\sin^2\beta \begin{bmatrix} u \\ v \end{bmatrix}_{S_j} + \cos\beta \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial r} \end{bmatrix}_{S_j}. \quad (18)$$

The solutions for field components are sought in the form of Sommerfeld integrals as

$$(u,v)=\frac{1}{2\pi i}\int_{\gamma} f_j(\alpha+\varphi)e^{-ikr\sin\beta\cos\alpha}d\alpha, \quad (19)$$

where γ is the Sommerfeld double loops shown in Fig. 2, and α is the complex planes variable. The calculation of the unknown spectral functions represented by f_j is given in the section titled far field solution.

Application of the Malyuzhinets' theorem to the functions *u* and *v* gives the following equations for $\varphi=\pm\Phi$.

$$\left[\sin(\alpha-\theta)-(-1)^j\right]f_1(\alpha\pm\Phi) + \left[\sin(\alpha+\theta)+(-1)^j\right]f_1(-\alpha\pm\Phi)=C_1'\sin\alpha, \quad (20)$$

and

$$\left[\sin(\alpha+\theta)-(-1)^j\right]f_2(\alpha\pm\Phi) + \left[\sin(\alpha-\theta)+(-1)^j\right]f_2(-\alpha\pm\Phi)=C_2'\sin\alpha, \quad (21)$$

where $\theta=\cos^{-1}\left(\frac{1}{\sin\beta}\right)$.

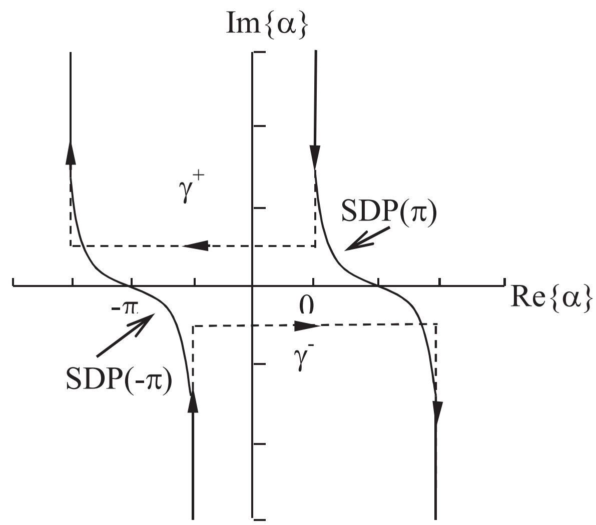

Fig. 2. The complex α plane with Sommerfeld double loops and the steepest descent paths SDP(-π) and SDP(π).

III. FAR FIELD SOLUTION

The homogeneous solutions $f_{10}(\alpha)$ and $f_{20}(\alpha)$ for the functional equations (20) and (21) can be represented in terms of χ_ϕ functions as follows

$$f_{10}(\alpha)=\frac{\chi_\phi^2\left(\alpha+\Phi+\theta+\frac{\pi}{2}\right)\chi_\phi^2\left(\alpha-\Phi-\theta+\frac{\pi}{2}\right)}{\chi_\phi^2\left(\alpha-\Phi+\theta-\frac{\pi}{2}\right)\chi_\phi^2\left(\alpha+\Phi-\theta-\frac{\pi}{2}\right)}, \quad (22)$$

$$f_{20}(\alpha)=\frac{\chi_\phi^2\left(\alpha+\Phi-\theta+\frac{\pi}{2}\right)\chi_\phi^2\left(\alpha-\Phi+\theta+\frac{\pi}{2}\right)}{\chi_\phi^2\left(\alpha-\Phi-\theta-\frac{\pi}{2}\right)\chi_\phi^2\left(\alpha+\Phi+\theta-\frac{\pi}{2}\right)}. \quad (23)$$

It is known that $\cos\left(\frac{\theta}{2}\right)=\frac{\chi_\phi(\theta+2\Phi)}{\chi_\phi(\theta-2\Phi)}$.

The functional equations for $f_j(\alpha)$ should be supplemented by an additional condition [13] namely $f_j(\alpha)-\frac{1}{\alpha-\phi_0}$ is regular in $|\mathrm{Re}\,\alpha|\le\Phi$. Then, the solution can be represented in the following form to satisfy the additional condition

$$f_j(\alpha)=f_{j0}(\alpha)\sigma_{\phi_0}(\alpha)\xi_j(\alpha). \tag{24}$$

Here a new function is defined as

$$F_{j0}(\alpha)=f_{j0}(\alpha)\sigma_{\phi_0}(\alpha). \tag{25}$$

New unknown spectral functions $\xi_j(\alpha)$ are introduced to facilitate the solution and $\sigma_{\phi_0}(\alpha)$ is defined as

$$\sigma_{\phi_0}(\alpha)=\frac{\mu\cos(\mu\phi_0)}{\sin(\mu\alpha)-\sin(\mu\phi_0)}, \tag{26}$$

where μ is equal to $\mu=\frac{\pi}{2\Phi}$ and $\xi_j(\alpha)$ has no poles and zeros in the strip $|\mathrm{Re}\,\alpha|\le\Phi$.The function $\sigma_{\phi_0}(\alpha)$ satisfies the following relation

$$\sigma_{\phi_0}(\alpha\pm\Phi)=\sigma_{\phi_0}(-\alpha\pm\Phi). \tag{27}$$

The known functions $F_{j0}(\alpha)$ satisfy equation (20) and (21) as $f_{j0}(\alpha)$. Then, $\xi_j(\alpha)$ obey the simple functional equations where λ is wavelength sentence.

$$\xi_j(\alpha\pm\Phi)-\xi_j(-\alpha\pm\Phi)=0. \tag{28}$$

Since the residue of f_j at $\alpha=\phi_0$ must give the incident field, the following can be written

$$\mathrm{Res}\; F_{j0}(\alpha)\xi_j(\alpha)\Big|_{\alpha=\phi_0}=1, \tag{29}$$

where $\mathrm{Res}\; f(\alpha)\Big|_{\alpha=\alpha_0}$ is used for the residue of a function $f(\alpha)$ at a point α_0. It follows that

$$\xi_j(\alpha)=\frac{1}{f_{j0}(\phi_0)}. \tag{30}$$

So the solutions for the unknown spectral functions are given by

$$f_j(\alpha)=\frac{\sigma_{\phi_0}(\alpha)f_{j0}(\alpha)}{f_{j0}(\phi_0)}. \tag{31}$$

By substituting (31) into (19) and by evaluating the integral asymtoticly by the steepest descent method gives

$$U(r,\varphi)\sim\frac{e^{-i\frac{\pi}{2}}e^{i\frac{\pi}{4}}e^{-kr\sin\beta-i\frac{\pi}{4}}}{\sqrt{2\pi kr\sin\beta}}\left[f_1(\varphi-\pi)-f_1(\varphi+\pi)\right], \tag{32}$$

and

$$V(r,\varphi)\sim\frac{e^{-i\frac{\pi}{2}}e^{i\frac{\pi}{4}}e^{-kr\sin\beta-i\frac{\pi}{4}}}{\sqrt{2\pi kr\sin\beta}}\left[f_2(\varphi-\pi)-f_2(\varphi+\pi)\right]. \tag{33}$$

When inverse transformation is applied to (32) and (33), H_z and E_z are obtained as

$$\begin{bmatrix}H_z\\E_z\end{bmatrix}=\begin{bmatrix}i & \frac{1}{Z_0}\\ Z_0 & i\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}. \tag{34}$$

H_z is also written as

$$H_z=iu+\frac{1}{Z_0}v. \tag{35}$$

More specifically

$$\begin{aligned}H_z(r,\varphi)&=e^{-i\frac{\pi}{2}}e^{i\frac{\pi}{4}}\frac{e^{-kr\sin\beta-i\frac{\pi}{4}}}{\sqrt{2\pi kr\sin\beta}}\\&\times\Big\{\,i\left[f_1(\varphi-\pi)-f_1(\varphi+\pi)\right]\\&+\frac{1}{Z_0}\left[f_2(\varphi-\pi)-f_2(\varphi+\pi)\right]\Big\}=\frac{e^{-kr\sin\beta}}{\sqrt{r}}D(\varphi),\end{aligned} \tag{36}$$

where $D(\varphi)$ is the diffraction coefficient given as

$$\begin{aligned}D(\varphi)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k\sin\beta}}\Big\{\,&i\left[f_1(\varphi-\pi)-f_1(\varphi+\pi)\right]\\&+\frac{1}{Z_0}\left[f_2(\varphi-\pi)-f_2(\varphi+\pi)\right]\Big\},\end{aligned} \tag{37}$$

where $f_j(\alpha)$ is defined in (24) and the related functions $f_{10}(\alpha)$, $f_{20}(\alpha)$, $\sigma_{\phi 0}(\alpha)$, and $\xi_j(\alpha)$ are given in (22), (23), (26), and (30), respectively.

IV. NUMERICAL RESULTS

In this paper, diffraction of obliquely incident plane electromagnetic waves by impedance being equal to the intrinsic impedance of surrounding medium is considered. This study is the first to investigate this case. Therefore, we reduced the problem to the normal incidence case taking $\beta=90^o$ to be able to compare our results with the known studies. In Figs. 3 through 5, it is obvious that our results and the results obtained by İkiz previously and by Büyükaksoy [17, 23] are very similar.

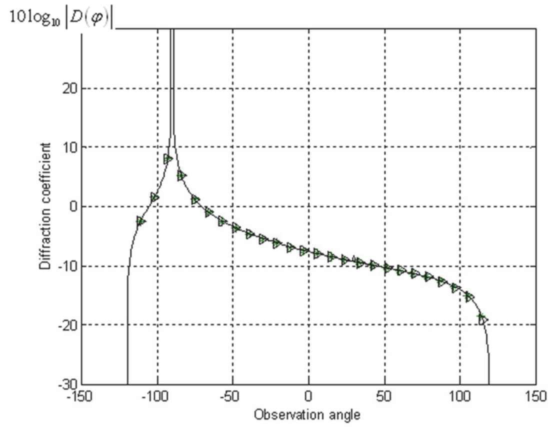

Fig. 3. Comparison of the results (Φ=120°, ϕ_o=30°). (*): results obtained previously by İkiz (▷): results obtained by İkiz in this study.

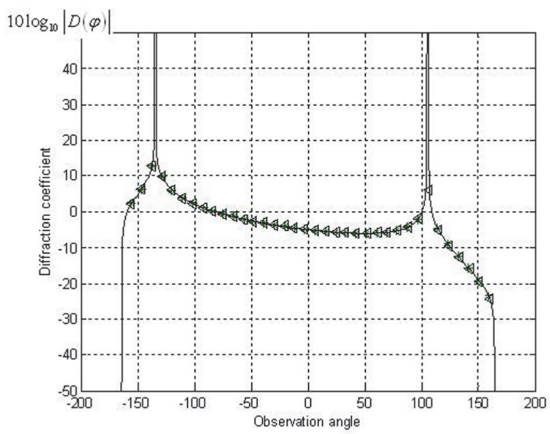

Fig. 4. Comparison of the the results (Φ=165°, ϕ_o=45°) (*): results obtained previously by İkiz (◁): results obtained by İkiz in this study.

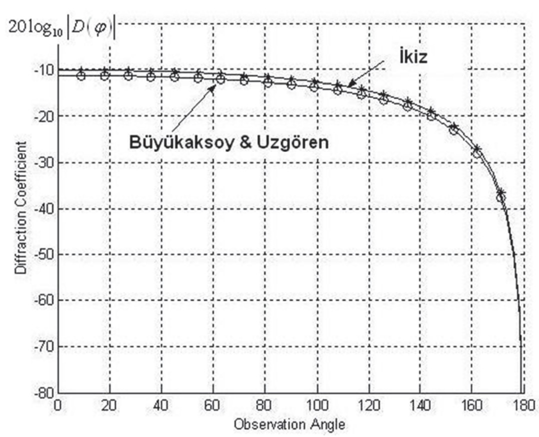

Fig. 5. Comparison of the results by Büyükaksoy & Uzgören(o) and by İkiz(*) Φ=180°.

In Figs. 6 and 7, we represent the diffraction coefficients for different values of incidence and wedge opening angles.

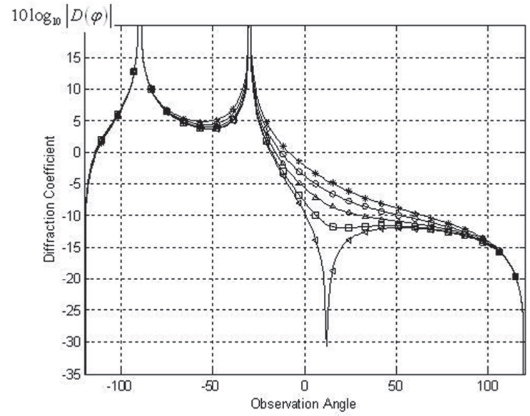

Fig. 6. Diffraction coefficient $10\log_{10}|D(\varphi)|$ versus observation angle with Φ=120°, ϕ_o=90° , β=30° (*), 45° (o), 60°.(△), 75° (□), 90° (◁).

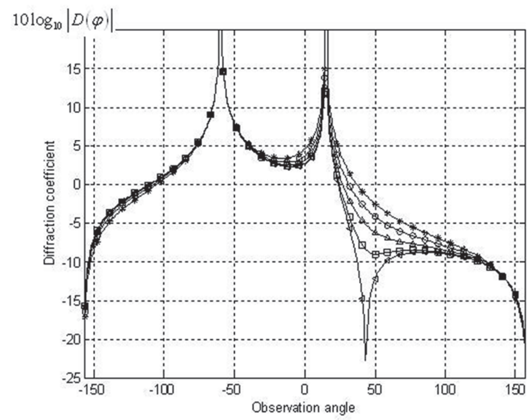

Fig. 7. Diffraction coefficient $10\log_{10}|D(\varphi)|$ versus observation angle with Φ=157,5°, ϕ_o=120° , β=30° (*), 45° (o), 60°.(△), 75° (□), 90° (◁).

V. CONCLUSION

The wedge surface impedance being equal to the intrinsic impedance of the surrounding medium, not only presents a convenient mathematical problem, but it can also correspond to a practical structure especially when it is assumed that this condition can be satisfied by choosing the appropriate ε_r and μ_r values for any composite material. From a mathematical point of view, this problem should also be considered as a first step for solving a wedge scattering problem with any surface impedance, with plane waves at any random incidence angle.

ACKNOWLEDGMENT

We would like to thank to M. A. Lyalinov for his suggestions.

REFERENCES

[1] G. D. Malyuzhinets, "Generalisation of the Reflection Method in the Theory of Diffraction of Sinusoidal Waves. Doctoral dissertation," P.N. Lebedev Phys. Inst. Acad. Sci. USSR, 1950.

[2] W. E. Williams, "Diffraction of an E-polarized Plane Wave by an Imperfectly Conducting Wedge," Proc. R. Soc. London A 252, pp. 376-393, 1959.

[3] T. B. A. Senior, "Diffraction by an Imperfectly Conducting Wedge," *Comm. Pure Appl. Math*., vol. 12, pp. 337-372, 1959.

[4] M. P. Sakharova and A. F. Filippov, "The Solution of a Nonstationary Problem of Diffraction by an Impedance Wedge in Tabulated Functions," *Zh. Vych. Matem. i Mat. Fiziki*, vol. 7, pp. 568-579, 1967.

[5] A. Mohsen, "On the Diffraction of an Arbitrary Wave by a Wedge," *Can. J. Phys*., vol. 60, pp. 1686-1690, 1982.

[6] S.-Y. Kim, RA J.-W., and S.-Y. Shin, "Edge Diffraction by Dielectric Wedge of Arbitrary Angle," *Electronic Letters*, vol. 19, no. 20, pp. 851-853, 1983.

[7] A. Ciarkowski, J. Boersma, and R. Mittra, "Plane-Wave Diffraction by a Wedge-A Spectral Domain Approach," *IEEE Trans. on Antennas and Propagation*, vol. 32, no. 1, pp. 20-29, 1984.

[8] R. Tiberio, G. Pelosi, and G. Manara, "A Uniform GTD Formulation for the Diffraction by a Wedge with Impedance Faces," *IEEE Trans. on Antennas and Propagation*, vol. 33, no. 8, pp. 867-873, 1985.

[9] R. G. Rojas, "Electromagnetic Diffraction of an Obliquely Incident Plane Wave Field by a Wedge with Impedance Faces," *IEEE Trans. on Antennas and Propagation*, vol. 36, no. 7, pp. 956-970, 1988.

[10] R. Tiberio, G. Pelosi, G. Manara, and P. H. Pathak, "High-Frequency Scattering from a Wedge with Impedance Faces Illuminated by a Line Source, Part I: Diffraction," *IEEE Trans. on Antennas and Propagation*, vol. 37, no. 2, pp. 212-218, 1989.

[11] A. D. Rawlins, "Diffraction of an *E*-or *H*-polarized Plane Wave by a Right-Angle Wedge with Imperfectly Conducting Faces," *Q.JI Mech. Appl. Math*., 43, pt. 2, pp. 161-172, 1990.

[12] Y. Liu and R. Ciric, "Improved Formulas for the Diffraction by a Wedge," *Radio Science*, vol. 28, no. 5, pp. 859-863, 1993.

[13] A. V. Osipov and A. N. Norris, "The Malyuzhinets Theory for Scattering from Wedge Boundaries: A Review," *Wave Motion*, vol. 29, pp. 313-340, 1999.

[14] A. N. Norris and A. V. Osipov, "Far-Field Analysis of the Malyuzhinets Solution for Plane and Surface Waves Diffraction by an Impedance Wedge," *Wave Motion*, vol. 30, pp. 69-89, 1999.

[15] M. A. Lyalinov and N. Y. Zhu, "Diffraction of Skewly Incident Plane Wave by an Anisotropic Impedance Wedge- A Class of Exactly Solvable Cases," *Wave Motion*, vol. 30, pp. 275-288, 1999.

[16] A. D. Andeev, "On the Special Function of the Problem of Diffraction by a Wedge in an Anisotropic-Plasma," *Radio Tekhnika I Elektronika*, vol. 39, pp. 885–892, 1994.

[17] T. İkiz and F. Karaömerlioğlu, "Diffraction of Plane Waves by a Two-Impedance Wedge in Cold Plasma," *Journal of Electromagnetic Waves and Applications*, vol. 18, no.10, pp. 1361-1372, 2004.

[18] T. B. A. Senior, "Diffraction Tensros for Imperfectly Conducting Edges," *Radio Sci*., vol. 10, pp. 911-919, Oct. 1975.

[19] O. M. Bucci and G. Franceschetti, "Electromagnetic Scattering by a Half-Plane with Two Impedances," *Radio Sci*., vol. 11, pp. 49-59, Jan. 1976.

[20] V. G. Vaccaro, "Electromagnetic Diffraction from a Right-Angled Wedge with Soft Conditions on One Face," *Opt. Acta*, vol. 28, pp. 293-311, Mar. 1981.

[21] T. B. A. Senior and J. L. Volakis, "Scattering by an Imperfect Right-Angled Wedge," *IEEE Trans. Antennas and Propagation*, vol. 34, pp. 681-689, May 1986.

[22] J. L. Volakis, "A Uniform Geometrical Theory of Diffraction for an Imperfectly

Conducting Half-Plane," *IEEE Trans. Antennas and Propagation*, vol. AP-34, pp. 172-180, Feb. 1986.

[23] A. Büyükaksoy and G. Uzgören, "Kırınım Problemleri (Diffraction Problems)," *Gebze High Technology Istitute Publications*, ISBN 975-8316-03-6, 1999.

Turgut İkiz was born in Ereğli, Turkey on August 1, 1966. He received the B.S. degree in Electrical and Electronics Engineering from Middle East Technical University in 1989. He received the M.S. and Ph.D. degrees from Çukurova University in 1992 and 1998, respectively. His research interests are propagation and diffraction of electromagnetic waves. He is currently an Associate professor in Electromagnetics at Çukurova University.

Mustafa Kemal Zateroğlu was born in İskenderun, Turkey on August 15, 1977. He received the B.S. degree in Electronics Engineering from Erciyes University in 1999. He received the M.S. degree in Electrical and Electronics Engineering from Çukurova University in 2002. He is expected to complete his Ph.D. study in 2011 at Çukurova University. His research interests are diffraction problems and numerical analysis of electromangetic waves, full wave solvers.