Adaptive Finite Element Mesh Refinement and *'a posteriori'* Error Estimation Procedures for Electromagnetic Field Computation

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Abstract - Adaptive Finite Element (FE) mesh refinement combined with a robust and functionally reliable error estimate provides nearly optimal solution accuracy. The efficiency of adaptivity depends on the effectiveness of the mesh refinement algorithm and also on the availability of a reliable and computationally inexpensive error estimation strategy. An adaptive mesh refinement algorithm utilizing a hierarchical minimal tree based data structure for 2D and 3D problems is discussed in this paper. Two different *'a posteriori'* error estimation schemes, one based on the local element by element method and the other using the gradient of field approach are also presented. The usefulness of the mesh refinement algorithm and the error estimation strategies are demonstrated by adaptively solving a set of 2D and 3D linear boundary value problems. The performance of the error estimates is also verified for adaptive modeling of a nonlinear problem involving the design of a permanent magnet synchronous machine.

I. INTRODUCTION

Due to the presence of discretization errors in any numerical modeling, the accuracy of the solution is limited. An accuracy in the range of 5% - 10% is often acceptable for most engineering applications. However certain scientific applications require solutions with a higher accuracy, in the range of 2%-3%. When the solution is plagued by the presence of domain singularities such as boundary layers, re-entrant corners, sharp bends, and multiple material discontinuities, it is necessary to selectively add more degrees of freedom where the solution varies abruptly. Under these situations, adaptivity helps to optimally improve the accuracy by selective spatial decomposition of a problem domain.

Many triangular and tetrahedral elements based adaptive mesh refinement techniques were proposed in the past [1,2]. However only a limited number of adaptive strategies are available for generating quadrilateral and hexahedral meshes [3]. The main reason is that the triangular elements match irregular boundaries better than quadrilateral and hexahedral elements. On the other hand, for the same number of unknowns, quadrilateral and hexahedral meshes require only about half that many elements as are needed in triangular or tetrahedral element meshes. Although triangular elements provide a good approximation to curved boundaries and complex geometries, they often produce elements with obtuse angles which need to be corrected using a Delaunay triangulation. Moreover visualizing higher order elements and refined meshes is easier in the case of quadrilateral/ hexahedral element meshes compared to triangular/ tetrahedral elements which produce unstructured meshes.

In an adaptive process, the critical areas of the problem domain are identified and refined based on a reliable error estimate. The equi-distribution of error in the problem signals the optimality of the adaptive mesh. An adaptive spatial decomposition technique employing first order quadrilateral elements in 2D and hexahedral elements in 3D utilizing a minimal hierarchical tree based algorithm is presented in the first part of this paper. Two different *'a posteriori'* error estimation strategies for activating the adaptive mesh refinement are discussed in the second part of the paper. In the third part of the paper, many numerical examples with experimental results are presented to demonstrate the application potential of the proposed mesh refinement algorithm and the error estimation strategies. The effectiveness of 'a posteriori' error estimates in adaptively solving a nonlinear problem for the computation of design parameters of a high field permanent magnet synchronous machine is presented in the last part of the paper.

II. ADAPTIVE MESH REFINEMENT TECHNIQUES

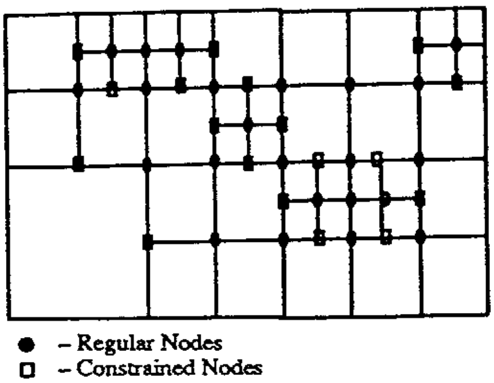
Starting with a coarse mesh, an efficient adaptive mesh refinement algorithm with the help of an error indicator and error estimator generates a nearly optimal mesh, in which the discretization error is equally distributed. It is imperative to generate a graded mesh in an adaptive process in order to produce a smooth solution. An adaptive mesh refinement algorithm should be capable of managing the computational complexity and data functions with a minimum of overhead on resources. In addition to providing an asymptotic rate of convergence, it should be able to handle different material properties and boundary conditions, while maintaining compatibility during the adaptive process. It must be flexible and robust and incorporate an efficient error estimation strategy to activate the adaptive process. Generally three different types of mesh refinement policies are available. A solution can be improved by reducing the size of an element ($h_{max} \rightarrow 0$) or increasing the order of approximating polynomial ($p \rightarrow \infty$) or combining both or by moving the mesh and relocating the nodes. Accordingly the mesh refinement procedures are classified as *h*, *p*, *h-p* and *r*-methods. In terms of number of degrees of freedom, the *p*-method is found to be nearly twice as efficient in convergence as the *h*-method [4]. Many different algorithms and the associated data structures were proposed in the past for automatic adaptive mesh refinement [1-3,5]. The proposed adaptive mesh refinement algorithm utilizes a *h*-version based mesh refinement policy.

A. *Quadrilateral & Hexahedral Mesh Refinement Strategies*

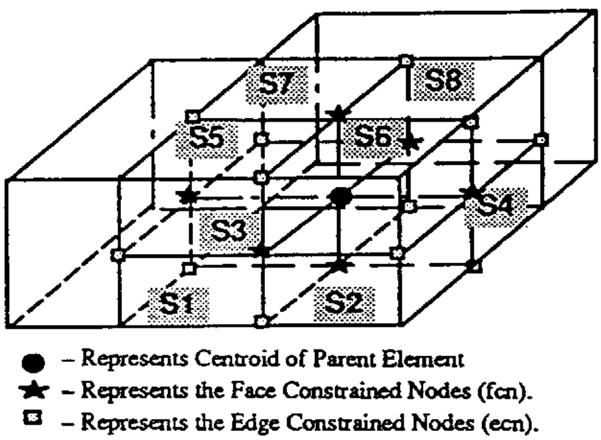
In the proposed adaptive scheme, the use of a hierarchical minimal tree data structure reduces the amount of tree travel necessary during the mesh refinement. Although most of the mesh refinement methods available in the literature employ tree based data structures they are computationally expensive. In the present approach a *one-level* rule is applied in order to generate a graded mesh with smooth mesh transition. The imposition of a *one-level* rule generates a constrained node on the boundary between two elements (common edge or face) in the quadrilateral and hexahedral meshes. Due to this, the meshes produced by this method are called 1-irregular meshes. The constraint nodes are processed in such a way that the sequence of admissible adaptive meshes produced during the course of refinement will satisfy the compatibility and continuity conditions. Utilizing the local property of quadrilateral/hexahedral elements an element is subdivided to produce a congruent element The use of the *one-level* rule imposes the following conditions on the quadrilateral/ hexahedral mesh:

- There cannot be more than one constrained node between elements sharing an edge in 2D.
- There cannot be more than one edge or face constrained node between elements sharing a common edge or face in 3D.
- The difference in the refinement level between adjacent neighbors cannot be more than one.

Before proceeding to refine an element. a check is made on the large neighbors; if a large neighbor exists, it is refined first and the actual element is refined next. Fig. 1a and fig. 1b illustrate the 2D and 3D adaptive meshes with constrained and regular nodes.



a.



b.

Fig. 1. Regular and constrained nodes. a. In quadrilateral mesh. b. In Hexahedral mesh.

The quadrilateral and hexahedral adaptive mesh refinement algorithm is based on the assumptions that the initial meshes are structured: the subdomains and the elements generated are of the same shape and type as that of the parents; if the initial meshes are admissible then the set of meshes generated during subsequent phases of refinement are also admissible. The basic set of initial data includes the element order list, node numbers, nodal coordinates and neighbor array and node types. The refinement proceeds by connecting the mid-points of the sides of a quadrilateral and the mid-points of faces of a hexahedral to the centroid of the element. After the refinement, an element order list is generated to maintain the natural sequence in order to identify the location of elements in the domain. The dynamic data structure maintains only two levels of the tree at any point in time during the refinement. New constrained nodes are created during refinement and the existing constrained nodes may become regular nodes. The nodes after refinement are identified as regular(m), boundary(bn), edge constrained(ecn) and face constrained(fcn) nodes. In addition to the *one-level* rule the minimal tree maintains a relatively simple data structure facilitating minimum tree travel during mesh refinement and thus reducing the computational overhead.

III. *'A POSTERIORI'* ERROR ESTIMATION

Minimization of the discretization error can be achieved by incorporating an efficient error estimation procedure which computes the error indicator to mark elements with more errors and error estimator which decides on the level of mesh refinement necessary. Error estimates are computed *'a posteriori'* due to the uncertain nature of the discretization error at the beginning of the adaptation. Some of the heuristic error estimation methods are based on mathematical analysis with extensive numerical results and others are based on benchmark computations satisfying specific computational goals. This is due to the fact that the error estimates are sensitive to the complexity and structure of the problem domain, the mesh quality and the nature of singularities. Most error estimation procedures use the solution, its gradient, system energy, post-processed solution, continuity conditions of field components or the residual of the solution as the primary field components to compute the error [7,8]. The error estimates not only decide an the optimal mesh but also assess the quality of the computed solution. An error estimate with a high degree of reliability will ensure proper adaptation in all classes of problems irrespective of the nature of problems and the type of material interfaces. In order to provide an adaptive computation of electromagnetic fields, the *'a posteriori'* error estimate should be computationally inexpensive and must be able to compute errors in complex domains and singular regions. Two different types of local *'a posterior'* error estimates are briefly discussed in this paper.

A. *Error Measures*

It is important to choose a suitable error measure not only for computing error indicators and error estimators, but also to assess the quality of the computed solution. Let Φ_{ex} and Φ be the exact and approximate solutions respectively, then by using an L_2 energy norm, error measures for the local error estimate can be derived as follows,

$$\|e\|_{L_2} = \left[\int_{\Omega} e^T e \, d\Omega \right]^{1/2}, \quad e = |\Phi_{ex} - \Phi| \qquad (1)$$

The relative energy norm error in percentage is,

$$\eta = \left[\|e\| \,/\, \left(\|\Phi\|^2 + \|\Phi\|^2 \right)^{1/2} \right] * 100\% \qquad (2)$$

The admissible error is derived using the global relative error and the number of elements as

$$\|e\|_a = \frac{\eta}{100} \left[\frac{\|\Phi\|^2 + \|\Phi\|^2}{N} \right]^{1/2} \qquad (3)$$

A reliable error criterion for refinement can be derived as, $\zeta_i = \|e\|_i / \|e\|_a$ and an element gets refined whenever $\zeta_i > 1$, $\forall \ \|e\|_i > \|e\|_a$

B. *Local Error Estimates*

Local error estimates are advantageous compared to other methods since they are simple and computationally less expensive. It takes only a fraction of computational power necessary to solve a problem, since a local error estimate solves a local problem consisting of a subdomain with only a few elements. Another advantage of a local error estimate is that it accurately predicts elements to be refined in the critical regions of a problem. Of the two different types of local error estimates presented, one makes use of an improved solution to compute the error and the other utilizes the gradient of the solution to estimate the error for adaptive mesh refinement.

C. *Element by Element Local Error Estimate*

The element by element local error method allows a solution of a small problem at the subdomain level to be solved for computing the error. Starting with a coarse mesh, a local error problem with a subdomain consisting of a patch of elements connected to a regular node is constructed. Since the local problem has only a few nodes, it is computationally less expensive. Based on the location of the regular node, Neumann and Dirichlet boundary conditions are imposed. The local problem $L\Phi=f$ thus created on each subdomain corresponding to the active node is solved using a quadratic approximation polynomial on an h-version mesh. By repeating this procedure at all the active nodes and comparing the improved solution with the original, global solution, the error on each element is computed locally. Using the error measure derived above, selected sets of elements are marked for refinement.

D. *Gradient of Field Method of Error Estimate*

In singular regions of a problem, the gradient of the field or flux will be highest, since the rate of variation of the solution is larger compared to other regions. Here the local problem is formed by creating a subdomain consisting of a patch of elements connected to an active node. With appropriate boundary conditions, the local problem is solved and an improved gradient of field is computed. The error in the gradient is computed as the difference between the gradient from the local problem and the gradient of the original solution. The derivative of the solution becomes constant in the case of a linear first-order approximation. Due to this reason the gradient g will be discontinuous across neighboring elements. In order to improve the approximation of the true gradient value, the gradient at each nodal point $\bar{g}$ is computed using the local problem formulation. Based on the fact that the nodal values are heavily influenced by the changes in the field quantities of neighboring elements, the gradient is improved by means of an averaging technique. Let g_{ex} and g be the gradient in the exact solution Φ_{ex} and in the approximate solution Φ respectively. Also let e_g be the error in the gradient. then $e_g=|g-g_{ex}|$. Since $g_{ex}=\nabla\Phi_{ex}$ and $g_{ex}=\nabla\Phi$, the true error in the gradient is $e_g=|\nabla\Phi_{ex}-\nabla\Phi|$.

IV. TEST RESULTS AND DISCUSSION

A. *Linear Problems*

To evaluate the performance of the proposed adaptive mesh refinement algorithm and the '*a posteriori*' error estimation procedures, different sets of linear and nonlinear problems in 2D and 3D are modeled using adaptive computation. The first 2D case is the classical electrostatic problem with an L-shaped domain with a corner singularity in the form $r^{2/3}sin(2\theta/3)$, where r and θ are the polar coordinates. The second is a Poisson problem on a unit square region with a charge density $\rho=1$ $Coulomb/m^2$ at the center with a permittivity of ε_0. A uniform Dirichlet boundary condition was imposed to solve the problem. For both problems, using the coarse mesh and the initial solution the proposed local error estimates are applied to initiate adaptive mesh refinement. An intermediate mesh and the final refined mesh for the L-section problem are shown in fig. 2. The corresponding equi-potential plots are shown in fig. 3. The sequence of adaptively refined meshes for the Poisson problem are shown in fig. 4. The asymetry in the meshes is due to termination of the refining process. If the process were continued, they would eventually become symmetric. The corresponding solution plots are shown in fig. 5. In the L-section problem, the singularity is present near the re-entrant corner of the problem domain. Hence the error estimate identified more elements for refinement and so the mesh is denser near the re-entrant corner. From the solution plot corresponding to the refined mesh, it can be discerned that the accuracy of solution is considerably improved.

In the Poisson problem a unit charge density exists in a small square region (0.2m x 0.2m) at the center of the domain. Due to the presence of charge density at the center, the field is stronger at the center compared to other regions and hence the refinement concentrates at the center. The sequence of solution plots verify that the accuracy of the solution is improved during each level of refinement.

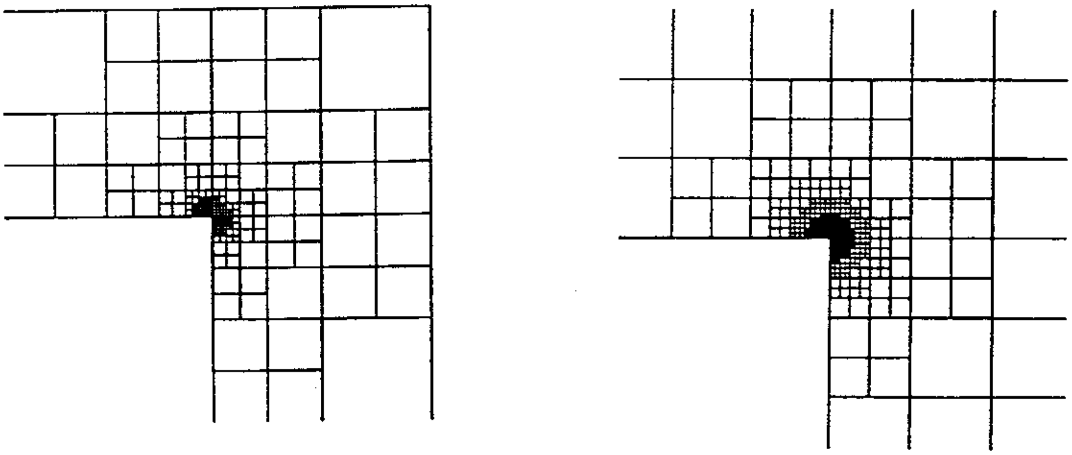


Fig. 2. Intermediate and final adaptive meshes for the L-section problem.

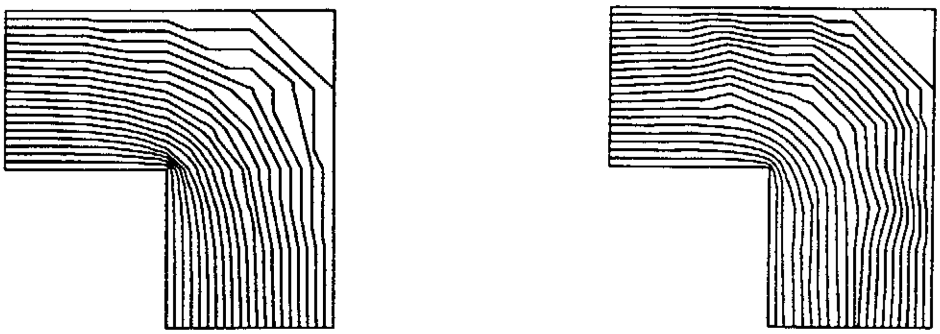


Fig. 3. Contour plots for the adaptive meshes in Fig. 2.

A magnetostatic problem in 3D is solved to compute the magnetic field and the stored magnetic energy for a given current density in a unit cube having a uniform permeability μ_0 A homogeneous boundary condition **B.n**=0 is imposed to compute the vector potential. A uniform current density $J_x=0, J_y=0, J_z=10^7\ A/m^2$ is applied along the side of the cube. The magnetic vector potential is used as the primary field variable in this problem. The initial mesh and the refined meshes are shown in fig. 6. The experimental values of stored magnetic energy and the corresponding errors are shown in table-1. The error convergence plot in fig. 7 shows a notable improvement of adaptive mesh refinement in minimizing the discretization error. From the sequence of adaptive meshes and the solution plots, the performance of adaptive the mesh refinement algorithm and the error estimation strategies are verified.

V. NONLINEAR MODEL

A carefully designed adaptive mesh refinement algorithm and error estimation method are capable of performing uniformly on linear and nonlinear problems. In order to test the performance of the proposed error estimate, a nonlinear problem involving the computation of design parameters for the design of a nonlinear high-field permanent magnet synchronous motor is modeled for adaptive FE analysis. The modeling and analysis of electrical machine design parameters is a complex task particularly due to the narrow airgap and the rotating flux due to the rotor and stator coils. To achieve optimal design and improved machine performance, accurate calculations of airgap flux density distribution and core losses are necessary.

Mesh number	Number of elements	Stored magnetic energy [MJ]	Percentage of error %
1	8	1.1044	49.98
2	64	1.8983	14.03
3	288	1.9231	12.91
4	1212	2.1482	2.71

Table-1 Numerical Test Results

A. *Accuracy Improvement of Machine Design Parameters*

Permanent magnets are vital components in the design of machines. In synchronous machines, they eliminate the steady state conductor losses associated with the rotor. Since there is no need far an armature magnetizing current the stator copper losses are also reduced [9]. However most permanent magnets made of rare earth materials are very expensive. By employing different combinations of inexpensive magnets with rare earth magnets, an optimal design with improved efficiency can be obtained while maintaining the airgap field distribution. To determine the optimal design parameters, efficient modeling and computation of various design parameters is of paramount importance. The nonlinear problem modeled for adaptive accuracy improvement in this experiment involves a high-field permanent magnet synchronous machine utilizing two different types of permanent magnets for the design of the rotor. A rectangular magnet made of rare earth permanent magnets and an arc magnet made of inexpensive common materials like iron oxide are employed in the design.

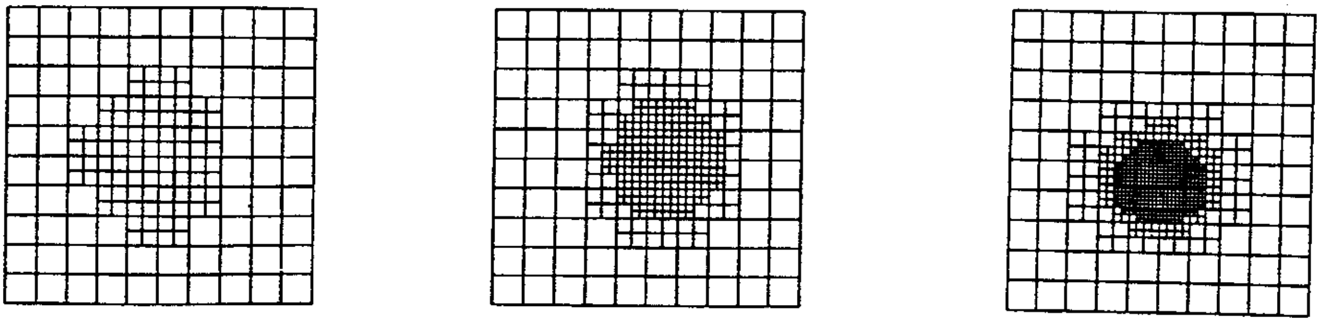


Fig. 4. Sequence of adaptive meshes for the Poisson problem.

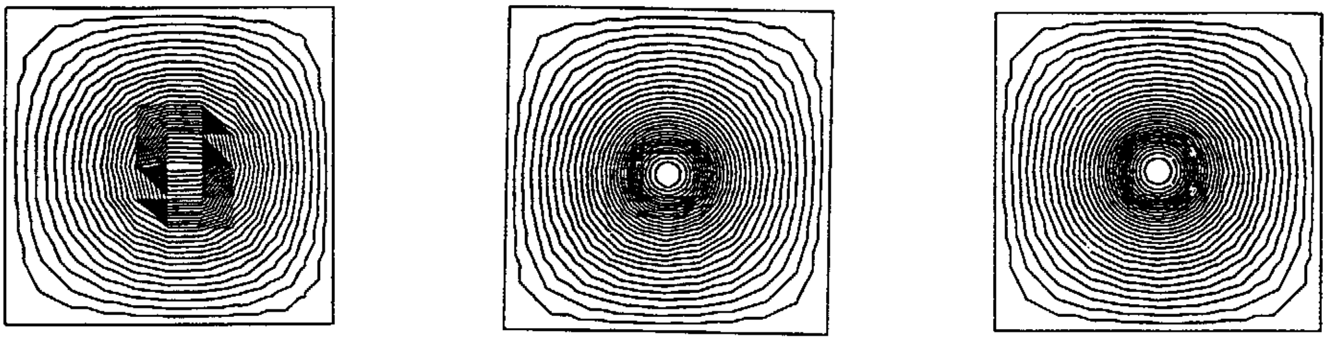


Fig. 5. Contour plots for the corresponding adaptive meshes.

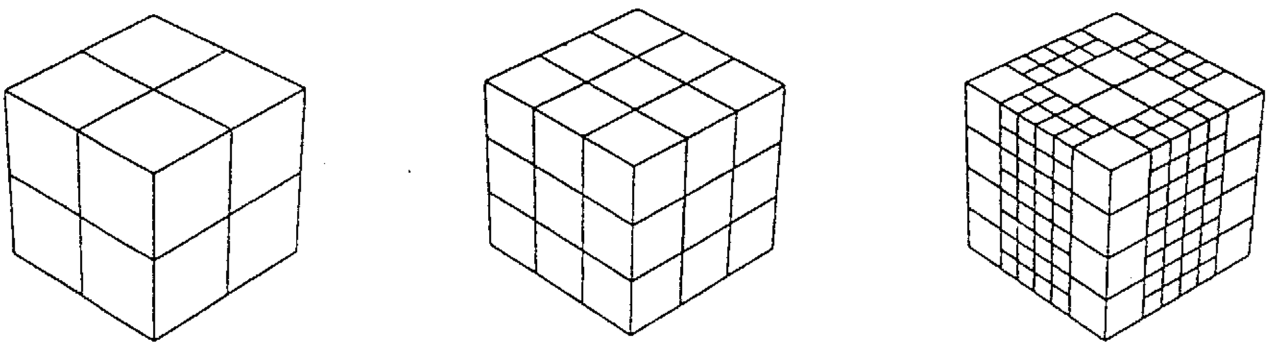


Fig. 6. Sequence of 3D adaptive meshes for the magnetostatic problem.

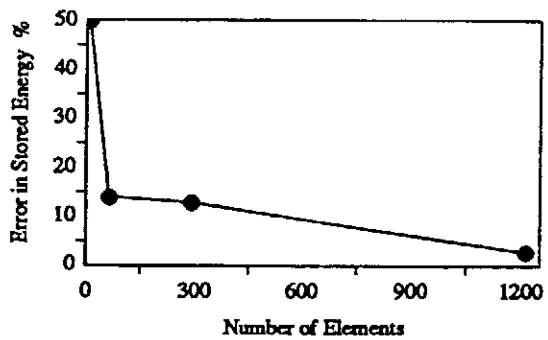


Fig. 7. Error convergence plot.

B. *Numerical Results*

Utilizing the symmetry of the domain, one fourth of the problem geometry is modeled. The modeling takes into account the steady state performance of the machine and also the sinusoidal variation of the rotor current. It is assumed that the rotor rotates at a constant synchronous speed A triangular element based *h*-version adaptive mesh refinement is employed in this model. The adaptive mesh refinement technique was initialized on a coarse mesh with 1040 triangular elements and 574 unknowns. After computing the mesh refinement parameters, the mesh refinement was allowed to progress up to 4 levels and the adaptation was terminated with 1560 elements and 838 unknowns. A sequence of adaptive meshes for the nonlinear problem and the corresponding contour plots (flux distribution) are shown in fig. 8 and fig. 9 respectively. For the sake of clarity, only an enlarged view of a section of the refined mesh is presented. From the smoothness of the flux distribution the accuracy improvement in the solution can be verified. From the numerical results the stator core loss and the airgap flux distribution can be calculated and compared with the available experimental values. The numerical test results along with the sequence of adaptive meshes and the corresponding solution plots establish the usefulness of the proposed error estimation strategy in solving a nonlinear problem for the improvement of design parameters of a permanent magnet synchronous machine.

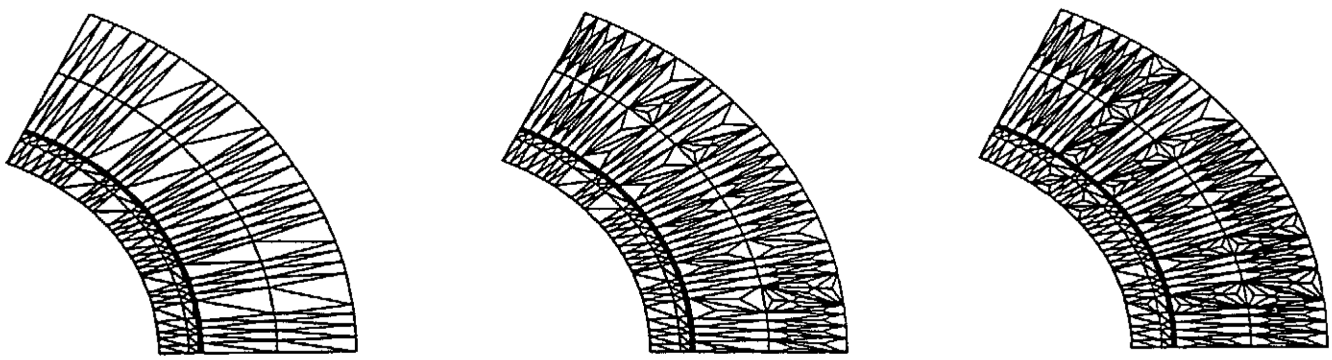


Fig. 8. Sequence of adaptive meshes for permanent magnet synchronous machine design.

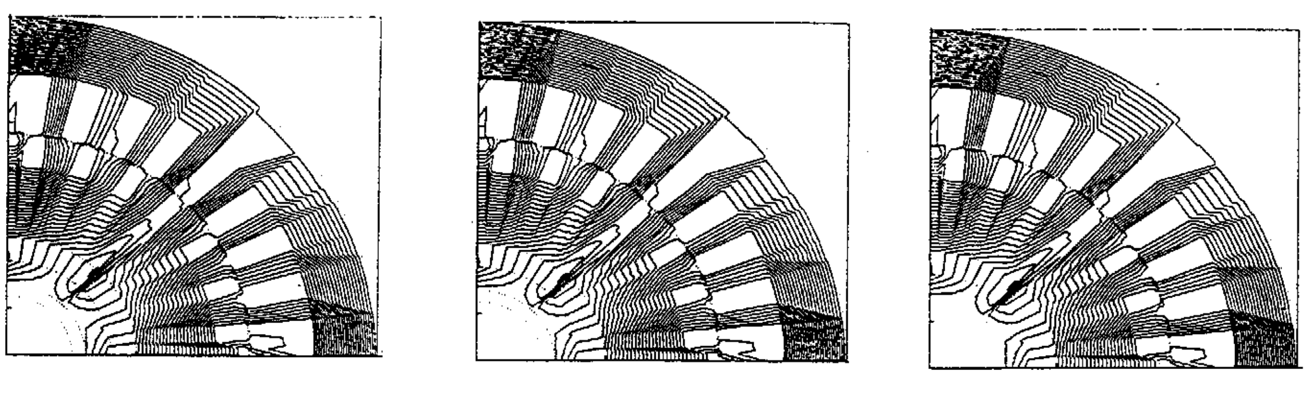


Fig. 9. Flux distribution plot for permanent magnet synchronous machine design.

VI. CONCLUSIONS

A hierarchical minimal tree based mesh refinement algorithm employing a one-level rule along with two different local *'a posteriori'* error estimation strategies are presented in this paper. The application of a minimal tree based algorithm stores only two levels of tree data structure at any step during the mesh refinement process thus reducing the tree traversal considerably, and therefore, providing a computational advantage over other tree structure based adaptive methods. The mesh refinement algorithm and the local error estimates are applied to solve linear and nonlinear elliptic boundary value problems adaptively. The numerical test results and the sequence of adaptive meshes demonstrate the application potential of the presented mesh refinement algorithm and the error estimation strategies.

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