EM Scattering from Bodies of Revolution using the Locally Corrected Nyström Method

In memoriam: Dr. William D. Wood, Jr., 1963-2004.

A. W. Wood and J. L. Fleming

Graduate School of Engineering and Management
2950 Hobson Way, AFIT/ENC
Wright-Patterson AFB, Ohio 45433–7765

Abstract **–** The locally corrected Nyström method is applied to the magnetic field integral equation for a conducting body of revolution. A construction method is presented for the locally corrected weights for the resulting one-dimensional coupled scalar magnetic field integral equations. Special attention is paid to minimizing the cost for multi-frequency computations. Numerical results are presented for the sphere, oblate spheroid, and right circular cylinder. Good agreement with results from mature moment method codes is observed.

I. INTRODUCTION

The Locally Corrected Nyström (LCN) method brings the high-order convergence properties of the Nyström method for integral equations to those with singular kernels such as those that arise in electromagnetic boundary value problems [1–3]. Computation of the corrected quadrature weights can be efficiently accomplished.

Here we apply the LCN method to a body of revolution (BOR) under plane wave illumination. The BOR geometry allows the 2D surface integral equation to be reduced to a series of 1D integral equations through the use of a Fourier series expansion. The solution of each 1D problem is a mode function in the series expansion of the total current.

Often, we desire to solve electromagnetic scattering problems for a fixed geometry over a range of frequencies. In the context of the LCN method, the goal is to compute the corrected weights once and then reuse them for the desired frequencies and necessary mode numbers. We show that careful use of quadrature rules will allow the reuse of corrected weights over a range of frequencies and mode numbers, greatly enhancing the computational efficiency of the algorithm.

II. CONVENTIONAL AND LOCALLY CORRECTED NYSTRÖM METHOD

In the conventional Nyström method, an integral equation,

$$g(x) = \int_a^b G(x,x')u(x')dx' \tag{1}$$

is replaced by a quadrature equation,

$$g(x) \approx \sum_{p=1}^{N_s} \sum_{q=0}^{N_a-1} \omega_q G(x, x_q^p) u(x_q^p) \tag{2}$$

where x_q^p is the q^{th} abscissa on the p^{th} subinterval. Evaluating $g(x)$ at the n^{th} abscissa of the m^{th} subinterval, *i.e.*, each abscissa from the underlying quadrature rule, gives,

$$g(x_n^m) \approx \sum_{p=1}^{N_s} \sum_{q=0}^{N_a-1} \omega_q G(x_n^m, x_q^p) \tilde{u}(x_q^p) \tag{3}$$

where x_q^p is the q^{th} abscissa on the p^{th} subinterval. Solving the resultant linear system of equations yields the value of $\tilde{u}(x)$ at the quadrature points. Interpolation provides $\tilde{u}(x)$ over the integration interval. However, an obvious problem for electromagnetic integral equations is the singularity of the kernel which makes evaluation at $x_n^m = x_p^q$ impossible. In addition, quadrature convergence is slow when $||x_n^m - x_p^q|| << \lambda$.

The LCN replaces some of the quadrature weights ω_q by "locally corrected" ones, $\overline{\omega}_q$, which are used when the distance between x_n^m and x_p^q is small. The details can be found in the literature [2, 3].

III. MFIE FOR A BOR SCATTERER

The magnetic field integral equation (MFIE) over a PEC BOR geometry can be reduced to a one dimensional problem along the curve defining the BOR. Given an incident field $\vec{\boldsymbol{H}}^i$ we wish to solve the MFIE [4] for the surface current $\vec{\boldsymbol{J}}_s$,

$$\begin{aligned} \hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}^i(\vec{\boldsymbol{r}}) = \frac{1}{2}\vec{\boldsymbol{J}}_s(\vec{\boldsymbol{r}}) - \hat{\boldsymbol{n}} \times \\ \int_S \vec{\boldsymbol{J}}_s(\vec{\boldsymbol{r}}') \times \nabla' g(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}')\, ds', \qquad \vec{\boldsymbol{r}} \in S \end{aligned} \tag{4}$$

where $g(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') = \exp(ik|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|)/[4\pi|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|]$ is the free space Green function for the Helmholtz equation (assuming $e^{-i\omega t}$ time dependence) and $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector on the surface S.

The BOR geometry is created by rotating a curve (ρ, z) about the z-axis. The curve is parameterized by its arc-length $\ell \in [0, L]$. The surface current has two vector components, one in the azimuthal direction and the other in the direction of the defining arc – $\vec{\boldsymbol{J}}_s = \hat{\boldsymbol{\ell}} J_\ell + \hat{\boldsymbol{\phi}} J_\phi$. In the BOR coordinate system, the surface into two variables ℓ and ϕ,

$$x = \rho(\ell)\cos\phi, \quad y = \rho(\ell)\sin\phi, \quad z = z(\ell).$$

The periodicity of the BOR geometry in the azimuthal direction allows the solution to be expanded into a Fourier series in the ϕ-direction hence reducing the integral equation to only ℓ dependence. Using the above description of the BOR geometry and some change of variables, the vector integral equation (4) can be written as the following system of two scalar second-kind integral equations [5],

$$\frac{J_\ell(\ell,\phi)}{2} = \int_0^{2\pi}\int_0^L \alpha_{12}(\ell,\ell',\phi',\phi)\, d\ell'\, d\phi' + \hat{\boldsymbol{\ell}} \cdot (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}^i) \tag{5}$$

$$\frac{J_\phi(\ell,\phi)}{2} = \int_0^{2\pi}\int_0^L \alpha_{34}(\ell,\ell',\phi',\phi)\, d\ell'\, d\phi' + \hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}^i) \tag{6}$$

where

$$\alpha_{12} = \alpha_1(\ell,\ell',\phi')J_\ell(\ell',\phi'+\phi) + \alpha_2(\ell,\ell',\phi')J_\phi(\ell',\phi'+\phi),$$

$$\alpha_{34} = \alpha_3(\ell,\ell',\phi')J_\ell(\ell',\phi'+\phi) + \alpha_4(\ell,\ell',\phi')J_\phi(\ell',\phi'+\phi).$$

The kernel functions $\alpha_{1\ldots4}$ are given in [2]. Both J_ℓ and J_ϕ are periodic in the azimuthal direction and thus both have a Fourier series expansion in the ϕ-variable,

$$J_\ell(\ell,\phi) = \sum_{n=-\infty}^{\infty} j_n^\ell(\ell) e^{in\phi}, \tag{7}$$

$$J_\phi(\ell,\phi) = \sum_{n=-\infty}^{\infty} j_n^\phi(\ell) e^{in\phi}. \tag{8}$$

Our focus now will be on an integral equation for each of the individual coefficient functions $j_n^\ell(\ell)$ and $j_n^\phi(\ell)$. The incident field can also be expressed as a Fourier series on the surface of the BOR. Using orthogonality of the exponentials we isolate one of unknown coefficient functions, and since $j_m^\ell(\ell)$ and $j_m^\phi(\ell)$ have no ϕ' dependence we can write the integral equation as,

$$\frac{j_m^\ell(\ell)}{2} = \hat{\boldsymbol{\ell}}_m + \int_0^L \left[j_m^\ell(\ell')G_m^1(\ell,\ell') + j_m^\phi(\ell')G_m^2(\ell,\ell')\right] d\ell' \tag{9}$$

$$\frac{j_m^\phi(\ell)}{2} = \hat{\boldsymbol{\phi}}_m + \int_0^L \left[j_m^\ell(\ell')G_m^3(\ell,\ell') + j_m^\phi(\ell')G_m^4(\ell,\ell')\right] d\ell', \tag{10}$$

where $\hat{\boldsymbol{\ell}}_m = [\hat{\boldsymbol{\ell}} \cdot (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}^i)]_m$, $\hat{\boldsymbol{\phi}}_m = [\hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{H}}^i)]_m$, and $G_m^i(\ell,\ell') = \int_0^{2\pi} \alpha_i(\ell,\ell',\phi') e^{im\phi'}\, d\phi'$.

Thus, the original integral equation (4) is reduced to a one dimensional problem. However, the one-dimensional problem is only solving for one component of the Fourier series solution with N_f Fourier modes; the problem must be solved N_f times. The value of k will determine the number of terms of the Fourier series that must be computed to achieve adequate accuracy.

IV. THE CORRECTED WEIGHTS

The Helmholtz kernel $\frac{e^{ikR}}{R}$ is separable into a frequency dependent factor and a frequency independent factor. The frequency dependent portion is smooth while the frequency independent portion contains the singularity. Given this situation, the corrected weights can be computed for the frequency independent portion of the kernel and the frequency dependent part can be absorbed into the solution. Hence, the locally corrected weights are computed, and any change in frequency can be accounted for by a simple multiplication of the quadrature weights. In the case of the BOR geometry the situation is not so straightforward. There are two significant differences between the standard Helmholtz kernel and the BOR kernels (G_i, $i = 1 \ldots 4$). First, not only is it necessary to be able to account for different frequencies, but the different modes must be dealt with as well. Second, the BOR kernel is not separable since the Helmholtz kernel is incorporated into an integration in the ϕ direction. The second of the two differences is the one that requires a careful approach, and when dealt with will allow a single set of weights to be used for any frequency and any mode.

In order to use the corrected weights for all frequencies and all modes in the BOR formulation, the key is to look at the integral as the original 2-D integral rather than the 1-D integral.

The first option is to perform local corrections in 2-D over the entire strip containing the singularity. Assume the singular point is in the interval (a_i, b_i). The integral of the form,

$$I(\ell) = \int_{a_i}^{b_i} u_m(\ell') G_m^i(\ell, \ell')\, d\ell' \tag{11}$$

can be written as,

$$I(\ell) = \int_{a_i}^{b_i} \int_0^{2\pi} u_m(\ell') \alpha_i(\ell, \ell', \phi') e^{im\phi'}\, d\phi'\, d\ell'. \tag{12}$$

It is clear upon inspection of the functions $\alpha_i(\ell, \ell', \phi') e^{im\phi'}$ can be written as a product $\Phi_i(\ell, \ell', \phi', m, k)\Psi_i(\ell, \ell', \phi')$ where Φ_i is smooth and contains all the frequency and mode dependent factors, while Ψ_i contains the singularity and has no dependance on frequency or mode. Therefore, the local corrections can be performed in two dimensions for the double integral using only Ψ_i as the singular kernel. The Φ_i can simply be absorbed into the solution function $u_i(\ell')$. The local corrections will produce a quadrature rule of the form,

$$\int_{a_i}^{b_i} \int_0^{2\pi} u_m(\ell') \alpha_i(\ell, \ell', \phi') e^{im\phi'}\, d\phi'\, d\ell' \tag{13}$$

$$= \int_{a_i}^{b_i} \int_0^{2\pi} u_m(\ell') \Phi_i(\ell, \ell', \phi', m, k) \Psi_i(\ell, \ell', \phi')\, d\phi'\, d\ell', \tag{14}$$

$$\approx \sum_p \sum_q u_m(\ell_p) \Phi_i(\ell, \ell_p, \phi_q, m, k) \overline{\omega}_{pq}, \tag{15}$$

where $\overline{\omega}_{pq}$ are the locally corrected weights. Now to make this quadrature rule consistent with the BOR formulation, simply factor out the $u(\ell_p)$ out of the inner sum to arrive at,

$$\int_{a_i}^{b_i} u_m(\ell') G_m^i(\ell, \ell')\, d\ell' \approx \sum_p u_m(\ell_p) \widetilde{\omega}_p^1 \tag{16}$$

where $\widetilde{\omega}_p^1 = \sum_q \Phi_i(\ell, \ell_p, \phi_q, m, k)\overline{\omega}_{pq}$. At this point, it is clear that the locally corrected weights $\overline{\omega}_{pq}$ need only be computed once, and can be updated by multiplication to account for changes in frequency or mode number. The drawback is that this yields a large local correction. Additional details in the development are somewhat tedious and can be found in [2]. The final result is,

$$\int_{a_i}^{b_i} u_m(\ell') G_m^i(\ell, \ell')\, d\ell' \approx \sum_p u_m(\ell_p) \widetilde{\omega}_p^3 \tag{17}$$

where

$$\widetilde{\omega}_p^3 = \sum_q \Phi_i(\ell, \ell_p, \phi_q, m, k)\overline{\omega}_{pq} + \sum_q \omega_p(q) \omega_q \alpha_i(\ell, \ell_p, \phi_q) e^{im\phi_q}. \tag{18}$$

Local correction need be done only once and can be easily modified for changes in frequency and mode. In this case, many more local corrections are done, but the additional corrections are small problems for one dimensional integrals. This improves accuracy without producing a very large system of local corrections.

V. NUMERICAL RESULTS

The following results show the application of the above methods to some canonical geometries. The locally corrected Nyström results are produced by AFITBOR [2] which use the methods described in the paper. Comparisons are made to CARLOS-BOR [6], a method of moments solver for the MFIE, to the three-dimensional moment method code AIM [7], or to a known analytical solution.

A. Sphere

We compare the far-zone scattered fields produced by the AFITBOR to the Mie series solution [4]. Figure 1 shows very good agreement between the radar cross-section (RCS) results obtained from the Mie series and AFITBOR for a conducting sphere with radius equal to one wavelength. The direction of the incident wave is 90° from axial incidence. The computation uses 12 modes in the Fourier series expansion. The results are very nearly identical to those obtained for axial incidence, in which only one Fourier mode is excited.

B. Oblate Spheroid

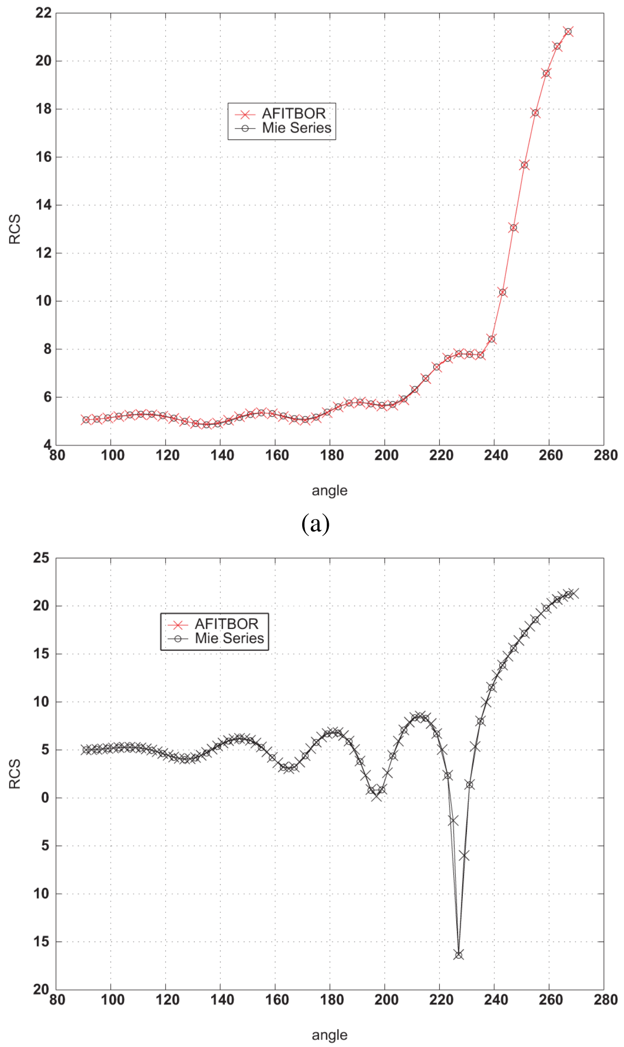
Here we apply the method to a non-spherical BOR. The only change in the code is to change the definition of the BOR defining curve. The spheroid is the BOR found by rotating half an ellipse with major axis $a = 2\,\lambda$ in the x-direction and minor axis $b = 1\,\lambda$ in the z-direction in the (x, z)-plane. The RCS for the $\theta\theta$ - polarization and $\phi\phi$ - polarization is plotted in Fig. 2.

C. Cylinder

Finally, we consider the "450" squat cylinder [8] which is a cylinder with radius 2.25 and height 2.1 inches. We let the wavelength be unity ($\lambda = 1$ inch). The incident field is 90° from axial incidence. As seen in Fig. 3 there is some disagreement in the RCS produced by AFITBOR compared to CARLOS-BOR, but comparison with a non BOR code (AIM) [7] shows AFITBOR comparing slightly better than CARLOS-BOR.

VI. CONCLUSION

The locally corrected Nyström (LCN) method allows the high order properties of the Nyström method to be applied to electromagnetic integral equations. The primary difficulty lies in the computation of the local corrections. Even though computation of locally corrected weights is an $O(n)$ operation, it can still be a computationally



(a)

(b)

Fig. 1. Bistatic RCS of a unit sphere with $\lambda = 1$ and angle of incidence $90°$. $\phi\phi$-polarized data are shown on the top while $\theta\theta$-polarized data are shown on the bottom.

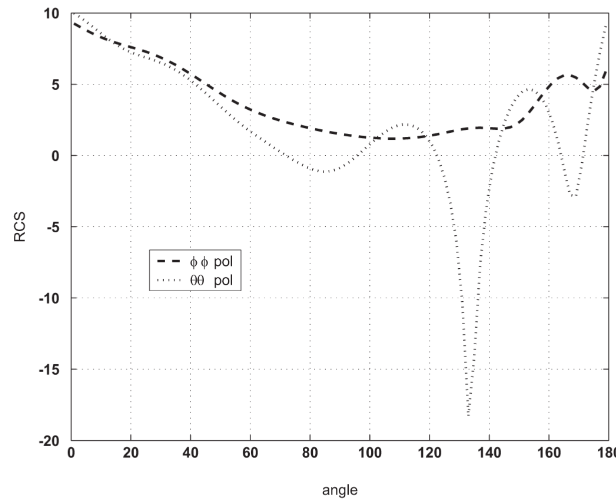


Fig. 2. Bistatic $\theta\theta$-pol and $\phi\phi$-pol RCS from AFITBOR for an oblate spheroid with $\lambda = 1$ and angle of incidence $\theta^{\text{inc}} = 30°$.

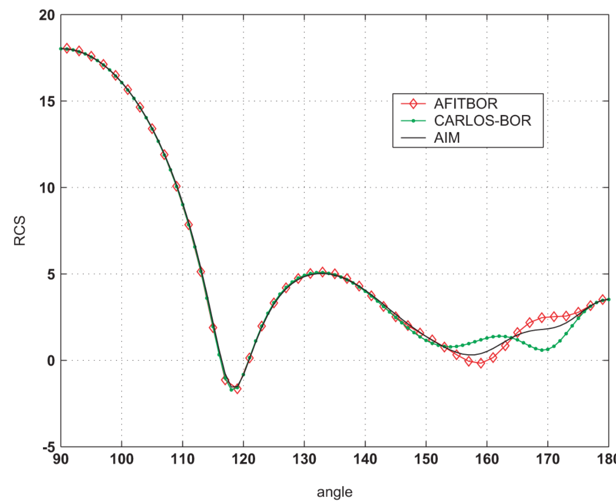


Fig. 3. Bistatic $\phi\phi$-pol RCS from AFITBOR, CARLOS-BOR and AIM for a 450 cylinder with $\lambda = 1$ and $\theta^{\text{inc}} = 90°$.

intensive procedure, so it is highly desirable to compute the corrected weights only when necessary. The paper has shown how to use a set of local corrections on a fixed geometry for a range of frequencies and a range of modes in a modal expansion for a body of revolution.

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Aihua W. Wood received her B.S. degree from Peking University, Beijing, China in 1984, and the M.S. and Ph.D. degrees from the University of Connecticut, Storrs, CT in 1988 and 1990, respectively. She was a Visiting Assistant Professor of mathematics at the Naval Postgraduate School from 1990 - 1993, and Assistant Professor at Penn State - Erie from 1993 -1994. She joined the faculty at the Air Force Institute of Technology at Wright-Patterson AFB, OH in 1994 and is currently a Professor of Mathematics. Dr. Wood’s current research interests include analytical and numerical analysis of electromagnetic scattering by cavities, wave propagation in complex media, and elliptic PDEs.

John F. Fleming received his B.A. degree from Hiram College, OH in 1995, and Ph.D. degree from Michigan State University, Est Lansing, MI in 2001. He was Research Associate at the Air Force Institute of Technology, WPAFB, OH from 2001 - 2004. He has been an Assistant Professor of mathematics at Duquesne University, Pittsburgh, PA since 2004.