Calculations in *Mathematica* on Low-Frequency Diffraction by a Circular Disk

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Abstract

This paper is devoted to the symbolic calculation of the scattering coefficient in diffraction by a circular disk, by the use of *Mathematica*. Three diffraction problems are considered: scalar diffraction by an acoustically soft disk, scalar diffraction by an acoustically hard disk, and electromagnetic diffraction by a perfectly conducting disk. In the low-frequency approximation, the solutions of these problems are in the form of expansions in powers of ka, where a is the radius of the disk and k is the wave number. The emphasis is on the low-frequency expansion for the scattering coefficient, of which several terms are determined exactly with the help of *Mathematica*.

1. Introduction

In recent years the symbolic programming language *Mathematica* has become an important tool in the analysis of mathematical problems of which the solution involves extensive analytical calculations. In this paper we use *Mathematica* to calculate the scattering coefficient for low-frequency diffraction by a circular disk. It is appropriate to refer to Hurd [10] for a previous symbolic calculation of the scattering coefficient, as early as 1971 and therefore now of limited scope, by use of the programming language FORMAC.

More specifically, we consider the diffraction of a normally incident, plane wave by a circular disk of radius a. A harmonic time dependence of the form $\exp(-i\omega t)$, with frequency ω, is assumed and suppressed throughout. Three different diffraction problems are distinguished here and treated separately: Scalar diffraction by an acoustically soft disk in Section 2; Scalar diffraction by an acoustically hard disk in Section 3; and Electromagnetic diffraction by a perfectly conducting disk in Section 4. These diffraction problems have exact solutions in terms of spheroidal wave functions. For a survey of results and solution methods, one can refer to [8, Chapter 14].

In this paper we are especially interested in the low-frequency approximations to the exact solutions, which are valid when the disk radius a is small compared to the wavelength. In the low-frequency approximation the solution of the diffraction problem is given by a power series expansion, in powers of $\alpha := ka$, where k is the wave number. Corresponding low-frequency expansions (in powers of α) are obtainable for various field quantities such as the scattered field on the disk, the scattered far field and the scattering coefficient. Here, the scattering coefficient is defined as the ratio of the total energy scattered to the energy incident on the disk. According to the Levine and Schwinger cross-section theorem, the scattering coefficient is related to the far-field amplitude of the scattered wave in the direction of incidence; see Jones [11, §§8.19, 9.4]. The first few terms of these low-frequency expansions can easily be determined and are known from the literature. Evaluation of the higher-order terms involves a considerable amount of work and soon becomes prohibitive with increasing order. However, the calculations are completely systematic and straightforward, and are therefore well suited to be carried out by a computer algebra system. To demonstrate this by an example, we will use *Mathematica* to calculate the low–frequency expansion of the scattering coefficient for the three diffraction problems mentioned. In principle, the expansion can be evaluated up to arbitrary order; in practice, the order is limited by the available computer capacity.

Each of the three diffraction problems is solved by two independent methods, both of which are well suited for obtaining a low–frequency approximation to the solution. In the first method, due to Bazer and Brown [2], and Boersma [3], the scattered field is represented by suitable integrals which contain unknown auxiliary functions. The integral representations are designed to satisfy all conditions of the diffraction problem except for the boundary conditions on the circular disk. Imposing the latter conditions leads to Fredholm integral equations of the second kind for the auxiliary functions. The kernel of the integral equations is small (it is the same kernel) of order α, thus permitting a solution of the integral equations by Picard iteration. The solution obtained is inserted into the expression for the scattering coefficient, yielding the desired low–frequency expansion.

In the second method, which is due to Bouwkamp [5–7], the diffraction problems are formulated in terms of integral equations of the first kind or integro–differential equations for the scattered field on the disk or, in the case of electromagnetic diffraction, the currents induced in the disk. Substitution of low-frequency expansions for the scattered fields or currents, and further expansion in powers of ik, leads to a recursive system of integral equations or integro–differential equations. This system is solved by expansion in suitable Legendre polynomials, whereby the expansion coefficients are determined by a recurrence relation. These coefficients are inserted into the expression for the scattering coefficient, yielding the desired low–frequency expansion.

The two methods of solution give rise to two different schemes for the calculation of the low–frequency expansion of the scattering coefficient. *Mathematica* implementations of these schemes are listed in [1, Appendices A–E]. The *Mathematica* programs were executed on a 486DX33 computer with 8 megabytes of internal memory using the *Mathematica* Enhanced Version 2.2 for Windows, and Microsoft Windows for Workgroups Version 3.11. Results of the calculations by *Mathematica* are presented at the end of Sections 2, 3, and 4. For the successive diffraction problems we have tabulated the exact values of the first ten coefficients and the numerical values (to six significant digits) of the first twenty coefficients in the low–frequency expansion of the scattering coefficient. It is found that the two different schemes do yield the same results for the scattering coefficient. This provides an excellent check on the correctness of the mathematical analysis and of the *Mathematica*–programs. It is emphasized that the *Mathematica* results to be presented are exact. The underlying calculations are in terms of only rational numbers and powers of the symbol π, and in *Mathematica* such calculations are carried out in an exact manner.

In the following sections we present only the key equations of the schemes for calculating the scattering coefficient. For the details of the underlying solutions of the diffraction problems the reader is referred to the companion report [1] which is available on request. Recalling that $\alpha = ka$, we state some further notations to be used: Im denotes that the imaginary part is to be taken; $[x]$ is the largest integer $\leq x$; and Γ denotes the gamma function.

2. Scalar diffraction by a soft circular disk

We consider the scalar diffraction of a normally incident plane wave by an acoustically soft circular disk. The diffraction problem is solved by two different methods. First, we employ the method of Bazer and Brown [2] to find that the solution may be expressed in terms of the auxiliary function $g(t)$, which satisfies the integral equation

$$g(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s}\, g(s)\, ds, \qquad -1 \le t \le 1. \tag{2.1}$$

The scattering coefficient of the soft circular disk, denoted by σ_2, is given by

$$\sigma_2 = -\frac{8}{\pi\alpha}\,\mathrm{Im}\left[\int_0^1 \cosh(\alpha t)\, g(t)\, dt\right], \tag{2.2}$$

expressed in terms of $g(t)$. The integral equation (2.1) is solved by Picard iteration, whereby one factor α is gained at each iteration step. The solution obtained is inserted into (2.2), whereupon the low–frequency expansion of σ_2 follows by a straightforward evaluation.

The second solution goes back to unpublished work of Bouwkamp, referred to in [7, p. 71]. Bouwkamp's solution of the diffraction problem, as detailed in [1, Section 2.3], is described by expansions in Legendre polynomials with expansion coefficients $a_{p,n}$, where $p = 0, 1, 2, \dots$, $n = 0, 1, \dots, [p/2]$. These coefficients are determined by the recurrence relation

$$a_{p,n} = (-1)^{n+1}\left(2n+\tfrac{1}{2}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\sum_{q=1}^{p}\Gamma^2\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{[(p-q)/2]}(-1)^m a_{p-q,m}\,\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\,\cdot$$
$$\cdot\,\frac{1}{\Gamma(\frac{1}{2}q-m-n+\frac{1}{2})\,\Gamma(\frac{1}{2}q+m-n+1)}\cdot\frac{1}{\Gamma(\frac{1}{2}q-m+n+1)\,\Gamma(\frac{1}{2}q+m+n+\frac{3}{2})}, \tag{2.3}$$

valid for $p = 1, 2, 3, \dots$, $n = 0, 1, \dots, [p/2]$, and initiated by $a_{0,0} = 1$. The low–frequency expansion of the scattering coefficient σ_2 is given by

$$\sigma_2 = -\frac{8}{\pi}\sum_{p=0}^{\infty}(-1)^p\, a_{2p+1,0}\,\alpha^{2p}, \tag{2.4}$$

which contains only the coefficients $a_{2p+1,0}$.

Obviously the low–frequency expansion of σ_2 is in even powers of α, and the leading term is found to be $16/\pi^2$. Therefore we set

$$\sigma_2 = \sum_{n=0}^{\infty}\sigma_{2,2n}\,\alpha^{2n} = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\tilde\sigma_{2,2n}\,\alpha^{2n}, \tag{2.5}$$

in which $\sigma_{2,0} = 16/\pi^2$ and $\tilde\sigma_{2,0} = 1$. By use of *Mathematica*, the expansion (2.5) has been evaluated up to and including terms of order α^{18}. In Table 2.1 we present the exact values of the normalized coefficients $\tilde\sigma_{2,2n} = (\pi^2/16)\sigma_{2.2n}$ for $n = 0\,(1)\,9$. The *Mathematica* calculation has been carried out in duplicate, based on either set of key equations (2.1)–(2.2) or (2.3)–(2.4). Both approaches were found to yield the same results of Table 2.1, with calculation times of 230 and 938 seconds, respectively, for ten coefficients. It is observed that $\tilde\sigma_{2,2n}$, $n = 0\,(1)\,9$, is a polynomial in π^{-2} of degree n, with rational coefficients and leading term $(-1)^n\, 2^{2n}\, \pi^{-2n}$; these properties can be proved for general n by induction. In Table 2.2 we present the numerical values, to six significant digits, of the coefficients $\sigma_{2,2n}$ for $n = 0\,(1)\,19$. The results of Tables 2.1 and 2.2 were also obtained (and extended) by Professor D.S. Jones (Dundee) by an independent *Mathematica* calculation. It has been found recently [4] that the expansion (2.5), considered as a power series in α, has a radius of convergence 3.39879, to five decimal places.

Our expansion of the scattering coefficient σ_2 agrees with and extends the results of Bouwkamp [7, formula (8.1)] and of Bazer and Brown [2, formula (75)]. In both references the coefficients $\tilde\sigma_{2,2n}$ have been determined for $n = 0\,(1)\,3$. According to [8, formula (14.50)], the best result available so far is an expansion up to and including terms of order α^{10}, due to Hurd [9]. Later on, Hurd [10] determined two additional terms of the expansion, with coefficients $\tilde\sigma_{2,12}$ and $\tilde\sigma_{2,14}$, by use of the symbolic programming language FORMAC. The results from [9, Table III] and [10] do agree with our Table 2.1.

Table 2.1

Exact values of the normalized coefficients

$$\tilde{\sigma}_{2,2n} = (\pi^2/16)\,\sigma_{2,2n}\,, \quad n = 0\,(1)\,9,$$

in the expansion (2.5) of σ_2.

$$\tilde{\sigma}_{2,0} = 1$$

$$\tilde{\sigma}_{2,2} = -\frac{4}{\pi^2} + \frac{4}{9}$$

$$\tilde{\sigma}_{2,4} = \frac{16}{\pi^4} - \frac{8}{3\,\pi^2} + \frac{71}{675}$$

$$\tilde{\sigma}_{2,6} = -\frac{64}{\pi^6} + \frac{128}{9\,\pi^4} - \frac{1936}{2025\,\pi^2} + \frac{568}{33075}$$

$$\tilde{\sigma}_{2,8} = \frac{256}{\pi^8} - \frac{640}{9\,\pi^6} + \frac{304}{45\,\pi^4} - \frac{43168}{178605\,\pi^2} + \frac{9523}{4465125}$$

$$\tilde{\sigma}_{2,10} = -\frac{1024}{\pi^{10}} + \frac{1024}{3\,\pi^8} - \frac{28288}{675\,\pi^6} + \frac{80704}{35721\,\pi^4} - \frac{640204}{13395375\,\pi^2} + \frac{329068}{1620840375}$$

$$\tilde{\sigma}_{2,12} = \frac{4096}{\pi^{12}} - \frac{14336}{9\,\pi^{10}} + \frac{485632}{2025\,\pi^8} - \frac{2208512}{127575\,\pi^6} + \frac{17011712}{28704375\,\pi^4} - \frac{200408}{25727625\,\pi^2} + \frac{28561418}{1917454163625}$$

$$\tilde{\sigma}_{2,14} = -\frac{16384}{\pi^{14}} + \frac{65536}{9\,\pi^{12}} - \frac{876544}{675\,\pi^{10}} + \frac{104992768}{893025\,\pi^8} - \frac{53558528}{9568125\,\pi^6} + \frac{9409312768}{72937816875\,\pi^4}$$
$$- \frac{5427789356576}{5033317179515625\,\pi^2} + \frac{24646112}{28761812454375}$$

$$\tilde{\sigma}_{2,16} = \frac{65536}{\pi^{16}} - \frac{32768}{\pi^{14}} + \frac{1519616}{225\,\pi^{12}} - \frac{219004928}{297675\,\pi^{10}} + \frac{1004324096}{22325625\,\pi^8} - \frac{248401408}{165391875\,\pi^6}$$
$$+ \frac{3275445271751792}{135899563846921875\,\pi^4} - \frac{5850372900928}{45299854615640625\,\pi^2} + \frac{2953662389}{74809474193829375}$$

$$\tilde{\sigma}_{2,18} = -\frac{262144}{\pi^{18}} + \frac{1310720}{9\,\pi^{16}} - \frac{13795328}{405\,\pi^{14}} + \frac{774815744}{178605\,\pi^{12}} - \frac{4345136128}{13395375\,\pi^{10}}$$
$$+ \frac{624041153536}{43762690125\,\pi^8} - \frac{9417147043033088}{27179912769384375\,\pi^6} + \frac{322152102668096}{81539738308153125\,\pi^4}$$
$$- \frac{7973773660981292}{589124609276406328125\,\pi^2} + \frac{200771738036}{135031100919862021875}$$

Table 2.2:

Numerical values of the coefficients

$$\sigma_{2,2n}\,, \quad n = 0\,(1)\,19,$$

in the expansion (2.5) of σ_2.

n	$\sigma_{2,2n}$	n	$\sigma_{2,2n}$
0	$1.62114 \cdot 10^{0}$	10	$-1.25015 \cdot 10^{-11}$
1	$6.34833 \cdot 10^{-2}$	11	$-8.26174 \cdot 10^{-13}$
2	$-1.21411 \cdot 10^{-3}$	12	$1.04976 \cdot 10^{-13}$
3	$-4.21848 \cdot 10^{-4}$	13	$4.75653 \cdot 10^{-15}$
4	$1.49251 \cdot 10^{-5}$	14	$-8.51679 \cdot 10^{-16}$
5	$2.99261 \cdot 10^{-6}$	15	$-2.40050 \cdot 10^{-17}$
6	$-1.53096 \cdot 10^{-7}$	16	$6.71043 \cdot 10^{-18}$
7	$-2.03550 \cdot 10^{-8}$	17	$8.81790 \cdot 10^{-20}$
8	$1.42562 \cdot 10^{-9}$	18	$-5.14921 \cdot 10^{-20}$
9	$1.33048 \cdot 10^{-10}$	19	$4.29350 \cdot 10^{-23}$

3. Scalar diffraction by a hard circular disk

This section deals with the scalar diffraction of a normally incident plane wave by an acoustically hard circular disk. Again, the diffraction problem is solved by two different methods. First, we employ the method of Bazer and Brown [2] to find that the solution may be expressed in terms of the auxiliary function $f(t)$, which satisfies the integral equation

$$f(t) = \sinh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s}\, f(s)\, ds, \qquad -1 \le t \le 1. \tag{3.1}$$

The latter equation follows by simplification of [2, formula (49)]. The simplification is possible since $f(t)$ is an odd function of t [1, Sec. 3.2.1]. The scattering coefficient of the hard circular disk, denoted by σ_1, is given by

$$\sigma_1 = \frac{8}{\pi\alpha}\,\mathrm{Im}\left[\int_0^1 \sinh(\alpha t)\, f(t)\, dt\right], \tag{3.2}$$

expressed in terms of $f(t)$. The integral equation (3.1) is solved by Picard iteration, whereby even a factor α^3 is gained at each iteration step, since $f(t)$ is an odd function of t. The solution obtained is inserted into (3.2), whereupon the low–frequency expansion of σ_1 follows in a straightforward manner.

The second solution of the diffraction problem is taken from Bouwkamp [5]. As in Section 2, Bouwkamp's solution is described by expansions in Legendre polynomials with expansion coefficients $b_{p,n}$, where $p = 0, 1, 2, \ldots,$ and $n = 0, 1, \ldots, N_p$, with $N_p = p/2$ (p even) or $N_p = (p-3)/2$ (p odd). These coefficients are determined by the recurrence relation

$$b_{p,n} = (-1)^{n+1}\left(n+\tfrac{3}{4}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\sum_{q=2}^{p}\Gamma\left(\tfrac{1}{2}q-\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{N_{p-q}}(-1)^m\, b_{p-q,m}\,\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)}\cdot$$
$$\cdot\frac{1}{\Gamma(\frac{1}{2}q-m-n-\frac{1}{2})\,\Gamma(\frac{1}{2}q+m-n+1)}\cdot\frac{1}{\Gamma(\frac{1}{2}q-m+n+1)\,\Gamma(\frac{1}{2}q+m+n+\frac{5}{2})}, \tag{3.3}$$

valid for $p = 2, 3, 4, \ldots,$ $n = 0, 1, \ldots, N_p$, and initiated by $b_{0,0} = -2/\pi$. The low–frequency expansion of the scattering coefficient σ_1 is given by

$$\sigma_1 = \frac{4}{3}\sum_{p=0}^{\infty}(-1)^p\, b_{2p+3,0}\,\alpha^{2p+4}, \tag{3.4}$$

which contains only the coefficients $b_{2p+3,0}$.

The low–frequency expansion of σ_1 is in even powers of α, and the leading term is found to be $(16/27\pi^2)\alpha^4$. Therefore we set

$$\sigma_1 = \sum_{n=2}^{\infty}\sigma_{1,2n}\,\alpha^{2n} = \frac{16\,\alpha^4}{27\,\pi^2}\sum_{n=2}^{\infty}\tilde{\sigma}_{1,2n}\,\alpha^{2n-4}, \tag{3.5}$$

in which $\sigma_{1,4} = 16/27\pi^2$, and $\tilde{\sigma}_{1,4} = 1$. By use of *Mathematica*, the expansion (3.5) has been evaluated up to and including terms of order α^{22}. In Table 3.1 we present the exact values of the normalized coefficients $\tilde{\sigma}_{1,2n} = (27\pi^2/16)\sigma_{1,2n}$ for $n = 2\,(1)\,11$. The *Mathematica* calculation has been carried out in duplicate, based on either set of key equations (3.1)–(3.2) or (3.3)–(3.4). Both approaches were found to yield the same results of Table 3.1, with calculation times of 73 and 327 seconds, respectively, for ten coefficients. From the tabulated values and from additional calculated values of $\tilde{\sigma}_{1,2n}$, not presented here, it appears that $\tilde{\sigma}_{1,2n}$ is a polynomial in π^{-2} of degree $[(n-2)/3]$, with rational coefficients. In Table 3.2 we present the numerical values, to six significant digits, of the coefficients $\sigma_{1,2n}$ for $n = 2\,(1)\,21$. It has been found recently [4] that the expansion (3.5), considered as a power series in α, has a radius of convergence 2.12548, to five decimal places.

Our expansion of the scattering coefficient σ_1 agrees with and extends the results of Bouwkamp [5, formula (44)], [7, formula (8.2)], and of Bazer and Brown [2, formula (54)]; these results are the best available so far, according to [8, formula (14.104)]. In the references mentioned the coefficients $\tilde{\sigma}_{1,2n}$ have been determined for $n = 2\,(1)\,5$, corresponding to an expansion up to and including terms of order α^{10}.

Table 3.1

Exact values of the normalized coefficients

$$\tilde{\sigma}_{1,2n} = (27\,\pi^2/16)\,\sigma_{1,2n}\;, \quad n = 2\,(1)\,11,$$

in the expansion (3.5) of σ_1.

$$\tilde{\sigma}_{1,4} = 1$$

$$\tilde{\sigma}_{1,6} = \frac{8}{25}$$

$$\tilde{\sigma}_{1,8} = \frac{311}{6125}$$

$$\tilde{\sigma}_{1,10} = -\frac{4}{81\,\pi^2} + \frac{2612}{496125}$$

$$\tilde{\sigma}_{1,12} = -\frac{56}{2025\,\pi^2} + \frac{166918}{420217875}$$

$$\tilde{\sigma}_{1,14} = -\frac{3872}{496125\,\pi^2} + \frac{4911008}{213050462625}$$

$$\tilde{\sigma}_{1,16} = \frac{16}{6561\,\pi^4} - \frac{2466752}{1674421875\,\pi^2} + \frac{10209259}{9587270818125}$$

$$\tilde{\sigma}_{1,18} = \frac{64}{32805\,\pi^4} - \frac{10344876692}{49638236484375\,\pi^2} + \frac{555982444}{13853606332190625}$$

$$\tilde{\sigma}_{1,20} = \frac{158224}{200930625\,\pi^4} - \frac{17909145896}{762623815078125\,\pi^2} + \frac{207559549214}{16503801223538691 5625}$$

$$\tilde{\sigma}_{1,22} = -\frac{64}{531441\,\pi^6} + \frac{86777216}{406884515625\,\pi^4} - \frac{218616645875024}{998861794274875781 25\,\pi^2} + \frac{115358087888}{346579825694312522 8125}$$

Table 3.2:

Numerical values of the coefficients

$$\sigma_{1,2n}\;, \quad n = 2\,(1)\,21,$$

in the expansion (3.5) of σ_1.

n	$\sigma_{1,2n}$	n	$\sigma_{1,2n}$
2	$6.00422\cdot 10^{-2}$	12	$1.79404\cdot 10^{-8}$
3	$1.92135\cdot 10^{-2}$	13	$2.11383\cdot 10^{-10}$
4	$3.04867\cdot 10^{-3}$	14	$-8.12669\cdot 10^{-10}$
5	$1.56882\cdot 10^{-5}$	15	$-2.65462\cdot 10^{-10}$
6	$-1.44386\cdot 10^{-4}$	16	$-4.35129\cdot 10^{-11}$
7	$-4.60949\cdot 10^{-5}$	17	$-6.52346\cdot 10^{-13}$
8	$-7.39515\cdot 10^{-6}$	18	$1.92722\cdot 10^{-12}$
9	$-6.29012\cdot 10^{-8}$	19	$6.36935\cdot 10^{-13}$
10	$3.42594\cdot 10^{-7}$	20	$1.05513\cdot 10^{-13}$
11	$1.10625\cdot 10^{-7}$	21	$1.91370\cdot 10^{-15}$

4. Electromagnetic diffraction by a conducting circular disk

We consider the electromagnetic diffraction of a normally incident plane wave by a perfectly conducting circular disk. Two independent solutions of the diffraction problem are proposed, taken from Boersma [3, Sec. 3.3] and Bouwkamp [6]. As detailed in [1, Sec. 4.2], Boersma's solution may be expressed in terms of three auxiliary functions $f_0(t)$, $g_0(t)$, and $g_1(t)$, which satisfy the integral equations

$$f_0(t) = \frac{\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} f_0(s)\, ds, \qquad -1 \le t \le 1, \tag{4.1}$$

$$g_0(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_0(s)\, ds, \qquad -1 \le t \le 1, \tag{4.2}$$

$$g_1(t) = \frac{t\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_1(s) ds, \qquad -1 \le t \le 1. \tag{4.3}$$

Next, the constants C_0 and C are determined by

$$C_0 = -\frac{g_1(1)}{g_0(1)}, \qquad C = \frac{f_0(1)}{-f_0(1) + g_1'(1) + C_0\, g_0'(1)}. \tag{4.4}$$

The scattering coefficient of the conducting circular disk, denoted by σ, is found to be given by

$$\sigma = \frac{8}{\pi}\,\mathrm{Im}\left[(C+1)\int_0^1 \sinh(\alpha t)\, f_0(t)\, dt\right], \tag{4.5}$$

expressed in terms of C and $f_0(t)$. The integral equations (4.1)–(4.3) are solved by Picard iteration, whereby a factor α or α^3 is gained at each iteration step. The solutions obtained are inserted into (4.4) and (4.5), whereupon the low–frequency expansion of σ follows.

Bouwkamp's solution [6] of the diffraction problem involves low–frequency expansions, with expansion coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, and p_n, where $n = 1, 2, 3, \ldots,$ and $\nu = 0, 1, \ldots, [(n+1)/2]$. These coefficients are determined successively by the system of equations

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=0}^{[(\tau+1)/2]} a_{\tau,\tau-2\nu}\, J(\nu, 0, n-\tau; \rho) = \sum_{\nu=0}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}(\nu!)^2}, \tag{4.6}$$

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=1}^{[(\tau+1)/2]} b_{\tau,\tau-2\nu}\, J(\nu, 1, n-\tau; \rho) = \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}(\nu-1)!(\nu+1)!}, \tag{4.7}$$

$$a_{n,n} = \sum_{\nu=1}^{[(n+1)/2]} \left((-1)^{\nu+1} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu+1)}\, a_{n,n-2\nu} + (-1)^{\nu} \frac{4\Gamma(\nu+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)}\, b_{n,n-2\nu}\right), \tag{4.8}$$

valid for $n = 1, 2, 3, \ldots,$ and initiated by $p_{-1} = 1$, $p_0 = 0$. Equation (4.8) stems from [6, formula (49)] with the second factor $\Gamma(\nu+1)$ corrected into $\Gamma(\nu)$. The J-functions in (4.6) and (4.7) are polynomials in ρ^2, generally given by

$$J(n,m,\mu;\rho) = \frac{(-1)^{n+m}\,\Gamma^2(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(n+m+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\,\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(\frac{1}{2}\mu-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-m+1)} \cdot \rho^{2m} F(-\tfrac{1}{2}\mu+n+m+\tfrac{1}{2},\ -\tfrac{1}{2}\mu-n+m;\ 2m+1;\ \rho^2), \tag{4.9}$$

where F stands for the hypergeometric function. It is easily verified that $J(n,m,\mu;\rho)$ is a polynomial in ρ^2 of degree $n+\frac{1}{2}\mu$ (μ even) or $\frac{1}{2}(\mu-1)-n$ (μ odd); in the latter case, $J(n,m,\mu;\rho)=0$ if $m+n>\frac{1}{2}(\mu-1)$. For fixed $n=1,2,3,\ldots$, equations (4.6) and (4.7) are identities for polynomials in ρ^2 of degree $[(n+1)/2]$. By equating the coefficients of $\rho^{2\nu}$ ($\nu=(0),1,2,\ldots,[(n+1)/2]$) on the left and on the right of (4.7) and (4.8), we are led to a system of linear algebraic equations for the coefficients $a_{n,n-2\nu}, b_{n,n-2\nu}, p_n$ of the nth approximation, expressed in terms of preceding coefficients for the approximations of order $n-1, n-2, \ldots, 1$. Here we also need the initial values $p_{-1}=1, p_0=0$. Together with equation (4.8), we now have a system of $2[(n+1)/2]+2$ linear equations for the same number of unknown coefficients $a_{n,n-2\nu}, b_{n,n-2\nu}$, and p_n. Thus the system (4.6)–(4.8) suffices to determine successively all coefficients a, b, and p.

Finally, the low-frequency expansion of the scattering coefficient σ is given by

$$\sigma = \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^n\, a_{2n,2n}\,\alpha^{2n}, \tag{4.10}$$

which contains only the coefficients $a_{2n,2n}$.

The low-frequency expansion of σ is in even powers of α, and the leading term is found to be $(128/27\,\pi^2)\alpha^4$. Therefore we set

$$\sigma = \sum_{n=2}^{\infty}\sigma_{2n}\,\alpha^{2n} = \frac{128\,\alpha^4}{27\,\pi^2}\sum_{n=2}^{\infty}\tilde{\sigma}_{2n}\,\alpha^{2n-4}, \tag{4.11}$$

in which $\sigma_4 = 128/27\,\pi^2$, and $\tilde{\sigma}_4=1$. By use of *Mathematica*, the expansion (4.11) has been evaluated up to and including terms of order α^{22}.

In Table 4.1 we present the exact values of the normalized coefficients $\tilde{\sigma}_{2n} = (27\,\pi^2/128)\sigma_{2n}$ for $n=2\,(1)\,11$. The *Mathematica* calculation has been carried out in duplicate, based on either set of key equations (4.1)–(4.5) or (4.6)–(4.10). Both approaches were found to yield the same results of Table 4.1, with calculation times of 2225 and 10384 seconds, respectively, for ten coefficients. From the tabulated values and from additional calculated values of $\tilde{\sigma}_{2n}$, not presented here, it appears that $\tilde{\sigma}_{2n}$ is a polynomial in π^{-2} of degree $[(n-2)/3]$, with rational coefficients. In Table 4.2 we present the numerical values, to six significant digits, of the coefficients σ_{2n} for $n=2\,(1)\,21$. It has been found recently [4] that the expansion (4.11), considered as a power series in α, has a radius of convergence 1.32335, to five decimal places.

Our expansion of the scattering coefficient σ agrees with and includes the three-term expansion due to Bouwkamp [6, formula (63)], and the five-term expansion due to Boersma [3, formula (3.54)]. Boersma's expansion up to and including terms of order α^{12} is the best result available so far, according to [8, formula (14.277)].

Table 4.1

Exact values of the normalized coefficients

$$\tilde{\sigma}_{2n} = (27\,\pi^2/128)\,\sigma_{2n}\ , \quad n = 2\,(1)\,11,$$

in the expansion (4.11) of σ.

$$\tilde{\sigma}_4 = 1$$

$$\tilde{\sigma}_6 = \frac{22}{25}$$

$$\tilde{\sigma}_8 = \frac{7312}{18375}$$

$$\tilde{\sigma}_{10} = -\frac{64}{81\,\pi^2} + \frac{60224}{496125}$$

$$\tilde{\sigma}_{12} = -\frac{2464}{2025\,\pi^2} + \frac{35048192}{1260653625}$$

$$\tilde{\sigma}_{14} = -\frac{477152}{496125\,\pi^2} + \frac{1074505984}{213050462625}$$

$$\tilde{\sigma}_{16} = \frac{4096}{6561\,\pi^4} - \frac{866102528}{1674421875\,\pi^2} + \frac{7165401088}{9587270818125}$$

$$\tilde{\sigma}_{18} = \frac{45056}{32805\,\pi^4} - \frac{10518751249408}{49638236484375\,\pi^2} + \frac{3838104543232}{41560818996571875}$$

$$\tilde{\sigma}_{20} = \frac{310123264}{200930625\,\pi^4} - \frac{1623120103424}{23109812578125\,\pi^2} + \frac{1594679901356032}{165038012235386915625}$$

$$\tilde{\sigma}_{22} = -\frac{262144}{531441\,\pi^6} + \frac{479139075584}{406884515625\,\pi^4} - \frac{1952093212715159552}{998861794274875781 25\,\pi^2} + \frac{600919849172992}{6931596513886250456 25}$$

Table 4.2:

Numerical values of the coefficients

$$\sigma_{2n}\ , \quad n = 2\,(1)\,21,$$

in the expansion (4.11) of σ.

n	σ_{2n}	n	σ_{2n}
2	$4.80337\cdot10^{-1}$	12	$2.44195\cdot10^{-3}$
3	$4.22697\cdot10^{-1}$	13	$5.15600\cdot10^{-4}$
4	$1.91142\cdot10^{-1}$	14	$-3.70006\cdot10^{-4}$
5	$1.98536\cdot10^{-2}$	15	$-4.73979\cdot10^{-4}$
6	$-4.58650\cdot10^{-2}$	16	$-2.71167\cdot10^{-4}$
7	$-4.43846\cdot10^{-2}$	17	$-6.96125\cdot10^{-5}$
8	$-2.17365\cdot10^{-2}$	18	$3.08721\cdot10^{-5}$
9	$-3.49620\cdot10^{-3}$	19	$4.82181\cdot10^{-5}$
10	$4.19730\cdot10^{-3}$	20	$2.97929\cdot10^{-5}$
11	$4.60964\cdot10^{-3}$	21	$8.90612\cdot10^{-6}$

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