VFD Approach to the Computation TE and TM Modes in Elliptic Waveguide on TM Grid

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Abstract **─** We describe here a vector finite difference approach (VFD) to the evaluation of eigenvalues and modes of elliptical waveguides. The FD is applied using a 2D elliptical grid in the waveguide section. A suitable Taylor expansion of the vector mode function allows to take exactly into account the boundary condition. To prevent the raising of spurious modes, our FD approximation results in a constrained eigenvalue problem, that we solve using a decomposition method. This approach has been evaluated comparing our results to known data for the elliptic case.

Index Terms - Elliptic waveguides, mode eigenvalues, and vector finite difference.

I. INTRODUCTION

The full-wave solution of waveguide problems can be faced both with general-purpose and specialized numerical techniques such as mode-matching (MM) [1] and methods of moments (MOM) [2]. The most effective of them is probably the mode-matching, since it exploit the modal structure of the field. However MM requires an accurate knowledge of the mode themselves to be implemented. More precisely, a quite large number of vector modes distribution and eigenvalues are needed and all the field modal functions must be known at the same set of points. The same type of information is also required in the analysis, using the method of moments (MoM), of thick-walled apertures [3-4] and slots [5]. Indeed, these apertures can be considered as stub waveguides, and the mode vectors of these guides are the natural basis functions for the MoM [6].

Apart from some simple geometries, where analytical evaluation of such mode vectors [7] is possible mode computation cannot be done in closed form, (or the closed-form solution is unsuitable for effective use), so, until now, many different numerical techniques have been proposed, and the most popular are based on FEM [8].

The most effective method to compute the field structure in a guide is the frequency-domain finite difference (FDFD) [9-10], i.e., the direct discretization of the vector eigenvalue problem [11-14]. Of course, for curved boundary, the standard rectangular grid is unfit, and a suitable curved grid should be used [15]. Moreover the vast majority of FDFD approach compute the Hertz potentials and then extract the vector mode functions using a numerical derivative. In this work we use an extension of vector generalization of FDFD approach presented in [16-17] to elliptic waveguides [18]. In order to improve both the accuracy and the computational effectiveness, a discretization grid fitting exactly the waveguide boundary is chosen. Both TE and TM modes are computed using an elliptic grid equivalent to the TM boundary condition [19] for scalar eigenvalue problem. For each grid point, a fourth-order Taylor approximations allow to replace the continuous eigenfunction problem with a discrete one. This leads to a matrix eigenvector problem, when additional conditions are added. These come out from the boundary conditions (which are included directly in the problem matrix), and the solenoidal or irrotational condition on mode vectors.

As a result, a matrix eigenvalue problem with linear constraints is obtained [20]. This is a known

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linear algebra problems, which can be quite easily reduced to a standard eigenvalue problem [21], for which effective procedure exist.

II. DESCRIPTION OF THE TECHNIQUE

Each modes vector of a metallic waveguide $\vec{e}$ is an eigenfunction of the Helmholtz equation,

$$\begin{cases} \nabla_t^2 \vec{e} + k_t^2 \vec{e} = 0 \\ \vec{e} \times \vec{i}_n \Big|_C = 0 \end{cases} \tag{1}$$

with additional condition, respectively (see Fig.1),

$$\nabla_t \cdot \vec{e} = 0 \quad \text{on C (TE modes)} \tag{2}$$

$$\nabla_t \times \vec{e} = 0 \quad \text{on C (TM modes)}, \tag{3}$$

where C is the contour of the waveguide (see Fig. 1).

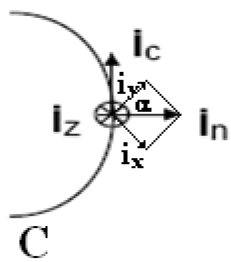

Fig. 1. Geometry of the waveguide contour.

Actually, in the MoM formulation, we need the modes of the surface magnetic current $\overrightarrow{M}$ equivalent to the transverse dielectric field $\vec{e}$. Therefore, we prefer a problem description in terms of the (two-dimensional) magnetic current $\overrightarrow{M}$ equivalent to the transverse field $\vec{e} = \vec{i}_z \times \overrightarrow{M}$. We can get from equation (1), for TM modes,

$$\begin{aligned} &\nabla_t^2 \vec{e} = \nabla_t \left(\nabla_t \cdot \vec{e} \right) = \nabla_t \left(\nabla_t \cdot \left(\vec{i}_z \times \overrightarrow{M} \right) \right) = \\ &-\nabla_t \left(\vec{i}_z \left(\nabla_t \times \overrightarrow{M} \right) \right) = -\left[\vec{i}_z \times \nabla_t \times \left(\nabla_t \times \overrightarrow{M} \right) \right] = \\ &\vec{i}_z \times \left[\nabla_t^2 \overrightarrow{M} - \nabla_t \left(\nabla_t \cdot \overrightarrow{M} \right) \right] \end{aligned} \tag{4}$$

$$\begin{aligned} &\nabla_t \times \vec{e} = \\ &\nabla_t \times \left[\vec{i}_z \times \overrightarrow{M} \right] = \vec{i}_z \left(\nabla_t \cdot \overrightarrow{M} \right) - \overrightarrow{M} \left(\nabla_t \cdot \vec{i}_z \right) + \\ &\left(\overrightarrow{M} \cdot \nabla_t \right) \vec{i}_z - \left(\vec{i}_z \cdot \nabla_t \right) \overrightarrow{M} = \vec{i}_z \left(\nabla_t \cdot \overrightarrow{M} \right) = 0. \end{aligned} \tag{5}$$

By equation (5), it follows that $\nabla_t \cdot \overrightarrow{M} = 0$. When substituted in equation (1), after replacing and collecting terms we get,

$$\begin{aligned} &\nabla_t^2 \vec{e} + k_t^2 \vec{e} = \\ &\vec{i}_z \times \left[\nabla_t^2 \overrightarrow{M} - \nabla_t \left(\nabla_t \cdot \overrightarrow{M} \right) \right] + k_t^2 \left(\vec{i}_z \times \overrightarrow{M} \right) = \\ &\vec{i}_z \times \left[\nabla_t^2 \overrightarrow{M} + k_t^2 \overrightarrow{M} \right]. \end{aligned} \tag{6}$$

The TM eigenvalue problem can therefore be rewritten as,

$$\nabla_t^2 \overrightarrow{M} + k_t^2 \overrightarrow{M} = 0 \tag{7}$$

with additional conditions. sentence,

$$\overrightarrow{M} \cdot \vec{i}_n \Big|_C = 0 \tag{8}$$

$$\nabla_t \cdot \overrightarrow{M} = 0. \tag{9}$$

The dual procedure can be used to compute TE modes, and results in equations (7) and (8), while equation (9) must be replaced by,

$$\nabla_t \times \overrightarrow{M} = 0. \tag{10}$$

It is therefore clear that the only difference in computing $\vec{e}$ or $\overrightarrow{M}$ is the exchange of the additional conditions. We work, in the following, with $\overrightarrow{M}$ but the approach, using $\vec{e}$, is equivalent. It is worth noting that both, equations (9) and (10) are scalar equations (since $\overrightarrow{M}$ is transverse to the waveguide axis).

Vector FDFD approach to the solution of these problems is based on the replacement of equations (8), (9), and (10) with a discretized version. Therefore, $\overrightarrow{M}$ is evaluated only on the points of a elliptic grid (see Fig. 2) with spacing $\Delta u, \Delta v$, and the equations are replaced by difference equations. Also $\overrightarrow{M}$ is expressed in ellipitcal component so that equation (7) becomes,

$$\begin{aligned} &\left(\nabla_t^2 M \right)_u \overrightarrow{u_u} + \left(\nabla_t^2 M \right)_v \overrightarrow{u_v} = \\ &-k_t^2 \left(M_u \overrightarrow{u_u} + M_v \overrightarrow{u_v} \right). \end{aligned} \tag{11}$$

For each internal grid point (see Fig. 3), a fourth order Taylor approximation allows to evaluate the surface magnetic current in terms of the current samples at the neighboring points. The expression of the Laplace vector operator in elliptic coordinates [21] can be simplified if we let coordinates grid TE and TM $\vec{A} = h\vec{M}$, where

$h = \dfrac{1}{a_f\sqrt{\sinh^2 u + \sin^2 v}}$ is the common value of the scale factor, $2a_f$ being the inter-focal distance. The u component of $\nabla_t^2 \vec{M}$ then becomes,

$$
\begin{aligned}
&-\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial u}\cdot\frac{\partial(A_u)}{\partial u}+\frac{1}{h^3}\frac{\partial^2(A_u)}{\partial u^2}\\
&+\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial u}\cdot\frac{\partial(A_v)}{\partial v}-\frac{\partial h}{\partial v}\cdot\frac{1}{h^3}\cdot\frac{\partial(A_v)}{\partial u}\\
&+\frac{\partial h}{\partial v}\cdot\frac{1}{h^3}\cdot\frac{\partial(A_u)}{\partial v}+\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial v}\cdot\frac{\partial(A_v)}{\partial u}\\
&-\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial v}\cdot\frac{\partial(A_u)}{\partial v}+\frac{1}{h^3}\frac{\partial^2(A_u)}{\partial v^2},
\end{aligned}
\tag{12}
$$

and the v component,

$$
\begin{aligned}
&-\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial v}\cdot\frac{\partial(A_u)}{\partial u}+\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial v}\cdot\frac{\partial(A_v)}{\partial v}\\
&+\frac{1}{h^3}\cdot\frac{\partial^2(A_v)}{\partial v^2}+\frac{1}{h^3}\cdot\frac{\partial^2(A_v)}{\partial u^2}\\
&-\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial u}\cdot\frac{\partial(A_v)}{\partial u}+\frac{1}{h^5}\cdot\frac{\partial h^2}{\partial u}\cdot\frac{\partial(A_u)}{\partial v}.
\end{aligned}
\tag{13}
$$

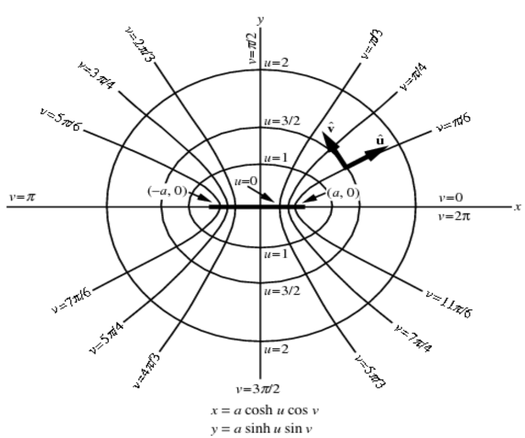

Fig. 2. Geometry of the elliptic cylindrical coordinates.

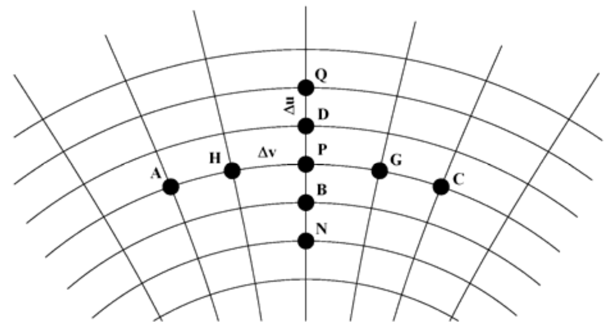

Fig. 3. Internal point of the elliptic cylindrical.

III. DISCRETIZATION OF THE EQUATIONS

For an internal point P as in Fig. 3 we can use a fourth-order Taylor expression. Letting,

$$
\begin{aligned}
A_i(u_p+\chi) &= A_{P,u}+\left.\frac{\partial A_u}{\partial u}\right|_P\cdot\chi+\frac{1}{2}\left.\frac{\partial^2 A_u}{\partial u^2}\right|_P\cdot\chi^2\\
&+\frac{1}{6}\left.\frac{\partial^3 A_u}{\partial u^3}\right|_P\cdot\chi^3+\frac{1}{24}\left.\frac{\partial^4 A_u}{\partial u^4}\right|_P\cdot\chi^4
\end{aligned}
\tag{14}
$$

where i stands for both u and v, we have

$A_{B,i} = A_i(u_p - \Delta u)$, $A_{N,i} = A_i(u_p - 2\Delta u)$

$A_{D,i} = A_i(u_p + \Delta u)$, $A_{Q,i} = A_i(u_p + 2\Delta u)$.

By combining these equations we find,

$$
\begin{aligned}
\left.\frac{\partial^2 A_u}{\partial u^2}\right|_P &= \frac{1}{14\Delta u^2}\cdot\begin{pmatrix}-A_{Q,u}-A_{N,u}\\ +16A_{D,u}+16A_{B,u}\\ -30A_{P,u}\end{pmatrix}\\
\left.\frac{\partial^2 A_v}{\partial u^2}\right|_P &= \frac{1}{14\Delta u^2}\cdot\begin{pmatrix}-A_{Q,v}-A_{N,v}\\ +16A_{D,v}+16A_{B,v}\\ -30A_{P,v}\end{pmatrix}.
\end{aligned}
\tag{15}
$$

And similarly in v direction,

$$
\begin{aligned}
\left.\frac{\partial^2 A_v}{\partial v^2}\right|_P &= \frac{1}{14\Delta v^2}\cdot\begin{pmatrix}-A_{A,v}-A_{C,v}\\ +16A_{G,v}+16A_{H,v}\\ -30A_{P,v}\end{pmatrix}\\
\left.\frac{\partial^2 A_u}{\partial v^2}\right|_P &= \frac{1}{14\Delta v^2}\cdot\begin{pmatrix}-A_{A,u}-A_{C,u}\\ +16A_{G,u}+16A_{H,u}\\ -30A_{P,u}\end{pmatrix}
\end{aligned}
\tag{16}
$$

for the second-order derivatives of equations (12) and (13). Also the first-order derivatives can be evaluated much in the same way as,

$$\begin{aligned}
\frac{\partial A_u}{\partial u} &= \frac{8A_{D,u} + A_{N,u} - 8A_{B,u} - A_{Q,u}}{12\Delta u} \\
\frac{\partial A_v}{\partial u} &= \frac{8A_{D,v} + A_{N,v} - 8A_{B,v} - A_{Q,v}}{12\Delta u} \\
\frac{\partial A_u}{\partial v} &= \frac{8A_{C,u} + A_{H,u} - 8A_{A,u} - A_{G,u}}{12\Delta v} \\
\frac{\partial A_v}{\partial v} &= \frac{8A_{C,v} + A_{H,v} - 8A_{A,v} - A_{G,v}}{12\Delta v}.
\end{aligned} \tag{17}$$

Equation (17) can be used also in equations (9) and (10) to get,

$$\begin{aligned}
&\frac{1}{h^2}\cdot\left(\frac{\partial A_v}{\partial u} - \frac{\partial A_u}{\partial v}\right) = \frac{1}{12\Delta u h^2}\cdot\begin{bmatrix} 8A_{D,v} + A_{N,v} \\ -8A_{B,v} - A_{Q,v}\end{bmatrix} - \\
&\frac{1}{12\Delta v h^2}\cdot\left[8A_{C,u} + A_{H,u} - 8A_{A,u} - A_{G,u}\right] = 0.
\end{aligned} \tag{18}$$

In the same way, to discretize the condition of equation (9) (TM modes) we use equation (17) and get,

$$\begin{aligned}
&\frac{1}{h^2}\cdot\left(\frac{\partial A_v}{\partial v} + \frac{\partial A_u}{\partial u}\right) = \\
&\frac{1}{h^2}\cdot\frac{8A_{C,v} + A_{H,v} - 8A_{A,v} - A_{G,v}}{12\Delta v} + \\
&\frac{1}{h^2}\cdot\frac{8A_{D,u} + A_{N,u} - 8A_{B,u} - A_{Q,u}}{12\Delta u} = 0.
\end{aligned} \tag{19}$$

For then points close to the boundary, such as P and B in Fig. 4, an approach different must be used to evaluate the u- derivatives since less than 2 grid points (D is not a grid point) are present outside. Therefore, both the equation for P and B require the mode vector at N.K.S.

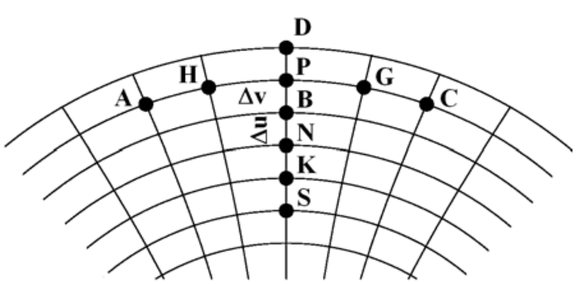

Fig. 4. Boundary point of the elliptic cylindrical.

Since $A_{k,i} = A_i\left(u_p - 3\Delta u\right), A_{s,i} = A_i\left(u_p - 4\Delta u\right)$, $A_{K,i}$, $A_{S,i}$, $A_{B,i}$, and $A_{N,i}$ we can evaluate the derivatives by a suitable linear combination as,

$$\left.\frac{\partial^2 A_u}{\partial u^2}\right|_P = \frac{1}{12\Delta u^2}\cdot\begin{pmatrix} 6A_{B,u} + 4A_{N,u} \\ -A_{k,u} - 9A_{P,u}\end{pmatrix} \tag{20}$$

$$\left.\frac{\partial^2 A_v}{\partial u^2}\right|_P = \frac{1}{12\Delta u^2}\begin{pmatrix} -104A_{B,v} + 114A_{N,v} \\ -64A_{K,v} + 11A_{S,v} \\ -43A_{P,v}\end{pmatrix}. \tag{21}$$

In equation (20) we have also included the BC $A_{D',u} = 0$. In the same way, the condition of equations (9) and (10) becomes,

$$\begin{aligned}
&\frac{1}{h^2}\cdot\left(\frac{\partial A_v}{\partial u} - \frac{\partial A_u}{\partial v}\right) = \\
&\frac{1}{12h^2\Delta u}\begin{pmatrix} -16A_{K,v} + 36A_{N,v} \\ -48A_{B,v} + 3A_{S,v} - 25A_{P,v}\end{pmatrix} - \\
&\frac{1}{12h^2\Delta v}\cdot\left(8A_{C,u} + A_{H,u} - 8A_{A,u} - A_{G,u}\right) = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{1}{h^2}\cdot\left(\frac{\partial A_v}{\partial v} + \frac{\partial A_u}{\partial u}\right) = \\
&\frac{1}{12h^2\Delta v}\cdot\left(8A_{C,v} + A_{H,v} - 8A_{A,v} - A_{G,v}\right) + \\
&\frac{1}{12h^2\Delta u}\begin{pmatrix} -16A_{K,u} + 36A_{N,u} \\ -48A_{B,u} + 3A_{S,u} - 25A_{P,u}\end{pmatrix}.
\end{aligned} \tag{23}$$

The elliptical framework has different singular points, i.e., the foci and the points on the inter-focal segment, which require a different treatment, since the field are not regular there. For the focus of the ellipse (Fig. 5) we need the integral form of the eigenvalue equation. By integrating the first term of equation (7) on the surface S of Fig. 5,

$$\int_S \nabla_t^2 \vec{M}\cdot dS = -k_t^2 \int_S \vec{M}\cdot dS \tag{24}$$

wherein the Laplace operator is equal to,

$$\begin{aligned}
\nabla_t^2 \vec{M} &= \nabla_t\left(\nabla_t\cdot\vec{M}\right) - \nabla_t \times \nabla_t \times \vec{M} \\
&= \nabla_t\left(\nabla_t\cdot\vec{M}\right).
\end{aligned} \tag{25}$$

Substituting in equation (7) we get,

$$\int_S \nabla_t \left(\nabla_t \cdot \vec{M}\right) \cdot dS = -k_t^2 \int_S \vec{M}\, dS \qquad (26)$$

and use of the theorem of the gradient [19] results in,

$$\int_S \nabla_t \left(\nabla_t \cdot \vec{M}\right) dS = \int_C (\nabla_t \cdot \vec{M}) \vec{i}_n \, dl =$$

$$\int_C \nabla_t \vec{M} \cdot \vec{i}_n \cdot dl = \int_C \frac{1}{h^2} \cdot \frac{\partial M_u}{\partial u} \cdot \vec{i}_n dl + \int_C \frac{1}{h^2} \cdot \frac{\partial M_v}{\partial v} \cdot \vec{i}_n dl. \qquad (27)$$

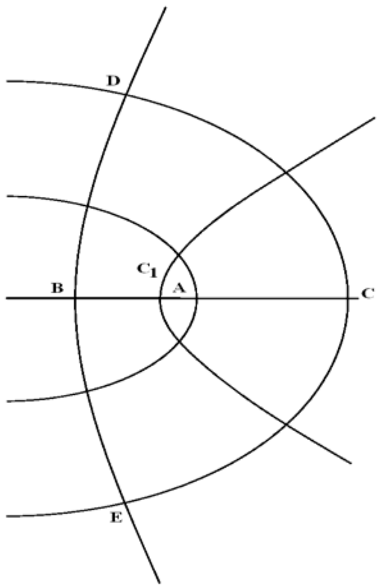

Fig. 5. Focus A of the ellipse.

The line integrals are divided in 4 parts (see Fig. 5). We describe here in details only the evaluation of the part over C_1. Letting $Q = \left(a, \frac{\Delta v}{2}\right)$ and $R = \left(\frac{\Delta u}{2}, 0\right)$, we have,

$$\int_{C_1} \frac{1}{h^2} \cdot \frac{\partial M_u}{\partial u} \cdot \vec{i}_n \cdot dl = \int_o^{\frac{\Delta u}{2}} \frac{1}{h^2} \cdot \frac{\partial M_u}{\partial u} \cdot \vec{i}_n \cdot h \cdot du =$$

$$\frac{1}{h} \cdot \vec{i}_n \Big|_{x_p} \cdot \int_o^{\frac{\Delta u}{2}} \frac{\partial M_u}{\partial u} \cdot du =$$

$$= \frac{-\vec{i}_x}{4h(Q)} \cdot \begin{bmatrix} M_u(A) + M_u(B) + M_u(C) \\ + M_u(D) + M_u(A)/2 \\ + M_u(B)/2 \end{bmatrix}, \qquad (28)$$

and

$$\int_{C_1} \frac{1}{h^2} \frac{\partial M_v}{\partial v} \vec{i}_n dl = \frac{1}{h} \vec{i}_n \frac{\partial M_v}{\partial v} \Big|_Q \int_0^{\frac{\Delta u}{2}} du =$$

$$\frac{1}{h(Q)} \left(-\vec{i}_x\right) \left(\frac{M_v(B) - M_v(A)}{\Delta v}\right) \frac{\Delta u}{2}. \qquad (29)$$

The same approach can be used for points on the inter-focal segment.

IV. SOLUTION OF CONSTRAINED EIGENVALUE PROBLEM

The discretized version of equation (1) for TM modes are obtained collecting equation (7) and the constraint in equations (8) and (9) to get a constrained eigenvalue problem. In the same way, equation (7) and the constraint of equations (8) and (10) are equivalent to the TE problems. Both can be written as,

$$\begin{cases} Ax = \lambda x \\ C^T x = 0 \end{cases} \qquad (30)$$

when, **A** is the discrete Laplace operator, including the boundary condition, and **C** is the discrete form of the constraint (2) or (3), **A** is a (2n, 2n) matrix, and **C** is (2n, m) with n > m and $\lambda = -k_t^2$. Following [20], we can solve equation (30) by letting $x = Q \cdot y$, where Q is the orthogonal (2n, 2n) matrix obtained by the QR factorization of the matrix C. Inserting $x = Q \cdot y$ in the first of equation (30), and pre-multiplying by Q^T we get,

$A \cdot Q \cdot y = \lambda \cdot Q \cdot y \Rightarrow Q^T \cdot A \cdot Q \cdot y =$
$= \lambda \cdot Q^T \cdot Q \cdot y = \lambda y$, which can be recast as $By = \lambda y$, where $B = Q^T \cdot A \cdot Q$ is a (2n, 2n) matrix. This matrix can then be partitioned as,

$$B \cdot y = \lambda \cdot y \qquad \Rightarrow \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \cdot \begin{vmatrix} u \\ v \end{vmatrix} = \lambda \cdot \begin{vmatrix} u \\ v \end{vmatrix}. \qquad (31)$$

Now $C = Q \cdot R$, and the constraint becomes analogously $R^T \cdot y = 0$. Since R is partitioned into an invertible T_1 and a null matrix, both $n \cdot n$ then,

$$R^T \cdot y = 0 \qquad \Rightarrow \begin{vmatrix} T_1 & 0 \end{vmatrix} \cdot \begin{vmatrix} u \\ v \end{vmatrix} = 0. \qquad (32)$$

So the constraint can be expressed as $u = 0$ [23]. Therefore, we need to extract the eigenvalues of B_{22},

$$B_{22} \cdot v = \lambda v \tag{33}$$

where B_{22} is a (n, n) matrix. Therefore, we still needs the eigenvalues of an $n \cdot n$ matrix which, at variance of the scalar case, is a full one. After the eigenvalues and eigenvectors of B_{22} are computed (by standard routines) the actual eigenvectors x can be computed $x = Q \cdot \begin{vmatrix} 0 \\ v \end{vmatrix}$.

V. NUMERICAL RESULTS

The high-order VFD elliptic waveguide described in the previous section has been extensively validated, to assess its accuracy and effectiveness. It is well-known that an analytical solution is known for elliptic waveguide [18] but its effectiveness is very poor, so that it is unsuitable for our comparison. Therefore, we have chosen to test our data on the cut-off frequencies against the data of Zhang and Chen [24], which are very accurate but quite hard to compute, and the data of Tsogkas et.al. reported [25], which is the most recent paper on the topic. We have chosen a set of waveguide with a minor axis equal to 4 (in arbitrary units) and different eccentricities ex. The discretization step Δv has been always set to 1°, while different values of Δu has been used for each test. The resulting eigenvalue problem has been solved using standard MATLAB routines, on a PC with two Intel Xeon E5504 CPUs @ 2.00 GHz, 48 GB RAM, OS: MS Windows 7 Professional.

The main results of our validation are collected in Figs. 7 and 8. From them it appears that our VFD approach is able to give a very high accuracy, with a difference (with respect to the accurate data of [24]), which is smaller than 0.01%. On the other hand, the recent approach proposed in [25] has an accuracy around 1%. The results reported in Fig. 9, show also that the accuracy of our VFD is essentially independent from the eccentricity. The computation time of the VFD approach is the sum of the matrix filling time and the time needed to extract eigenvalue and eigenvectors of the full matrix. The latter is high since we deal with full matrices so that the total time is essentially equal to it. For example, for a grid with Du = 0.0065 and 72000 points, the filling matrix time is 6,10 sec and the time to extract eigenvalue and eigenvectors is 800 sec.

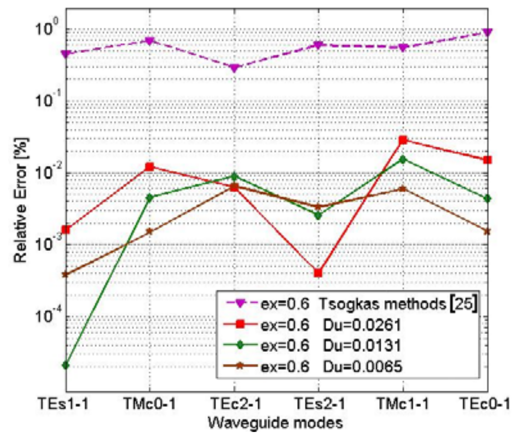

Fig. 7. Relative error on the cut-off frequency of the first modes of an elliptic waveguide ex = 0.6.

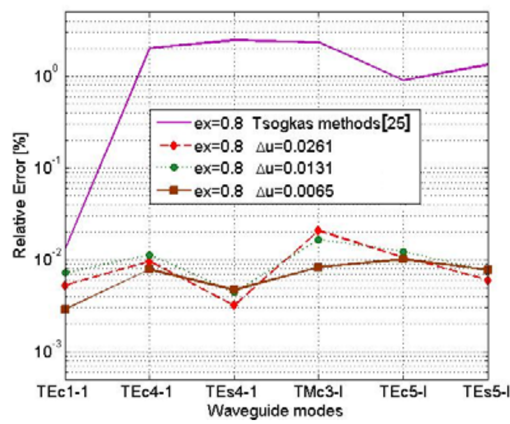

Fig. 8. Relative error on the cut-off frequency of the first modes of an elliptic waveguide ex = 0.8.

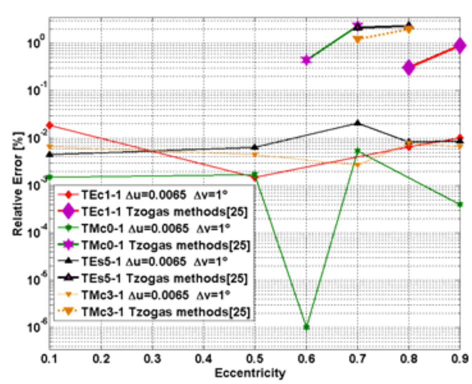

Fig. 9. Relative error on the cut-off frequency of the proposed VFD approach for different eccentricities.

VI. CONCLUSION

A new approach to the VFD computation of modes of an elliptic waveguide has been presented. We describe here a high vector finite difference frequency domain approach to the mode computation for both TE and TM modes. The main idea is the use of a discretization grid tailored to the waveguide boundary.

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