Higher-order (LT/QN) vector finite elements for waveguide analysis

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Abstract— **The finite element (FE) formulation for waveguide discontinuity analysis is reviewed and extended to multiple, arbitrarily-oriented ports. Several higher-order vector elements — specifically hierarchal linear tangential/quadratic normal (LT/QN) — are compared, and the extensions required to incorporate LT/QN elements in the formulation are presented. The improved accuracy afforded by LT/QN elements compared to constant tangential/linear normal (CT/LN) elements is investigated by considering energy conservation in an empty waveguide. Results obtained using both CT/LN and LT/QN elements are also shown for a problem of engineering interest: an E-plane bend. Results for the LT/QN elements compare especially well to approximate analytical results using quite coarse meshes. The paper concludes with a discussion of the use of iterative solvers and possible convergence problems encountered when using higher-order elements.**

Keywords— **Finite element method; higher-order vector elements; waveguide discontinuities.**

I. Introduction

The analysis of waveguide discontinuities has been a canonical problem for analytical, approximate, and now numerical approaches since the pioneering work of Marcuvitz and colleagues during the Second World War, now some sixty years back. Using variational formulations, and quasistatic approximations of the fields, Marcuvitz et al. were able to analyze an extraordinary variety of problems, documented in the classic text [1]. Subsequently, mode-matching methods were introduced for the analysis of "stepped" discontinuities — i.e. structures where the waveguide modes could be computed in a step-wise fashion, and matched at two-dimensional planes. However, for general, arbitrary discontinuities, and of course those involving non-metallic discontinuities such as dielectrics, differential equation based methods such as the finite element method (FEM) and finite difference time domain (FDTD) method are now the methods of choice.

Although an obvious application of the FEM, discontinuities in rectangular waveguide have not been widely addressed in the literature, in particular using higher-order elements. Ise [2] used "brick" elements of "first" order (constant tangential / linear normal — CT/LN) to analyze both a dielectric post and a concentric step discontinuity in rectangular waveguide ; Jin presented a detailed formulation in [3, Chapter 8], also using CT/LN elements; Webb's review paper discussed a number of related issues but did not address higher-order elements [4]; and Pekel and Lee addressed theoretical aspects of mesh refinement using an empty piece of waveguide, but again did not explicitly discuss higher-order elements [5]. Scott addressed rotationally symmetric waveguide and obtained very good results using special-purpose higher-order elements [6].

The contributions of this paper are the following. Firstly, Jin's formulation is extended to arbitrarily oriented waveguides (Jin's formulation assumes $\hat{z}$ orientation), with multiple ports (Jin assumes two port devices). Secondly, the available higher-order vector elements are reviewed, and some unifying themes underlying these are identified. Thirdly, the necessary extensions to include higher order vector elements in the formulation are outlined. Fourthly, the accuracy obtained vs. element size and number of degrees of freedom for CT/LN and linear tangential / quadratic normal (LT/QN) elements is investigated by monitoring energy conservation in a piece of empty guide. Finally, the extended formulation and implementation is validated — and the far greater accuracy obtainable with the LT/QN elements demonstrated again — by analyzing a realistic waveguide problem in X-band waveguide, namely an E-plane bend. Results for this are compared with Marcuvitz's.

Some aspects of this paper were originally presented in [7]. However, the formulation presented therein is Jin's, and does not incorporate the new extensions to be presented here, which are required to analyze general waveguide structures (such the E-plane bend analyzed here). Furthermore, the discussion of higher-order elements has been extensively revised, to highlight the connection between different published elements. Finally, some problems regarding convergence of the iterative solvers which have emerged subsequent to [7] are discussed.

II. The waveguide formulation

The formulation is a straightforward extension of Jin's approach [3]. His formulation addressed two-port, single mode analysis, with the waveguide oriented in the $\hat{z}$-direction.

Here, general waveguide orientation(s) will be considered. The formulation assumes hollow, rectangular guide at the ports (although the extension to homogeneously filled guide is straightforward). The TE_{10} mode is assumed in the following. In between the ports, in the region to be discretized using finite elements, the waveguide may contain linear, inhomogeneous, lossy, dielectric and/or magnetic material(s); and/or conductors (for instance, posts or irises); and may change orientation (eg E-plane bends) or dimension (eg E- and/or H-plane steps). The formulation to be used does, however, assume isotropic media. The generalization of the analysis to multiple ports, the inclusion of higher-order modes, and the extension to more general waveguide, will be outlined subsequently.

A. Formulation overview

The key part of the formulation is to write the electric field at port 1 (S_1) as the sum of the known incident and unknown reflected fields in terms of the (ξ,η,ζ) coordinate system local to the port, with ζ in the direction of propagation, and set to zero at each port, as follows:

$$\begin{aligned}\vec{E}(\xi,\eta,\zeta) &= \vec{E}^{\rm inc}(\xi,\eta,\zeta)+\vec{E}^{\rm ref}(\xi,\eta,\zeta)\\ &= (E_0\vec{e}_{10}(\xi,\eta)e^{-jk_{\zeta_{10}}\zeta}+\\ &\quad RE_0\vec{e}_{10}(\xi,\eta)e^{+jk_{\zeta_{10}}\zeta})\big|_{\zeta=0}\end{aligned} \quad (1)$$

$\vec{e}_{10}(\xi,\eta)$ is the relevant waveguide eigenmode (the TE_{10} eigenmode here) and $k_{\zeta_{10}}$ is the modal propagation constant. Note that it is necessary to retain the $e^{-jk_{\zeta_{10}}\zeta}$ term, even though the field is evaluated at $\zeta=0$, since the boundary condition to be discussed involves the derivative of the field, which must be evaluated *before* setting $\zeta=0$.

The next key element of the formulation is to convert eqn. (1) to a boundary condition of the third type involving both the field and its normal derivative; the detail is given in [3, §8.5]:

$$\hat{n}\times(\nabla\times\vec{E})+\gamma\hat{n}\times(\hat{n}\times\vec{E})=\vec{U}^{\rm inc} \quad (2)$$

with

$$\gamma=jk_{\zeta_{10}},\;\; \vec{U}^{\rm inc}=-2jk_{\zeta_{10}}\vec{E}^{\rm inc} \quad (3)$$

It should be noted that, in obtaining eqn. (2), the transverse-only nature of the TE field is exploited. TM modes contain axial $\vec{E}$ field components, and the boundary condition cannot thus be written for an $\vec{E}$ field solver. TM mode analysis could be undertaken by using an $\vec{H}$ field solver.

The same is repeated at port 2, but at that port, there is only an unknown transmitted field:

$$\begin{aligned}\vec{E}(\xi,\eta,\zeta) &= \vec{E}^{\rm trans}(\xi,\eta,\zeta)\\ &= TE_0\vec{e}_{10}(\xi,\eta)e^{-jk_{\zeta_{10}}\zeta}\big|_{\zeta=0}\end{aligned} \quad (4)$$

Similar comments apply as regards the $e^{-jk_{\zeta_{10}}\zeta}$ term. The boundary condition at port two is

$$\hat{n}\times(\nabla\times\vec{E})+\gamma\hat{n}\times(\hat{n}\times\vec{E})=0 \quad (5)$$

In Jin's original formulation, eqn. (4) was written as

$$\begin{aligned}\vec{E}(x,y,z) &= \vec{E}^{\rm trans}(x,y,z)\\ &= TE_0\vec{e}_{10}(x,y)e^{-jk_{z_{10}}z}\end{aligned} \quad (6)$$

In this approach, $z=z_1$ at port 1, $z=z_2$ at port 2, thus the phase in Jin's formulation was referenced to each port. In the present formulation, the transmission coefficient T incorporates the "insertion" phase — i.e. for a section of empty guide length ℓ, T will have phase angle $-k_{z_{10}}\ell$, whereas in Jin's formulation, the phase was 0. The present formulation produces the same phase that would be measured using a vector network analyzer, with reference planes calibrated at the ports. (Jin's approach worked well for straight waveguide, but is inappropriate for "bent" guides or multiple ports.)

The equivalent variational functional (assuming isotropic but possibly lossy materials), subject to these boundary conditions on the ports and $\vec{E}_{tan}=0$ on the perfectly conducting walls, is well known:

$$\begin{aligned}F(\vec{E}) &= \frac{1}{2}\iiint_V\left[\frac{1}{\mu_r}(\nabla\times\vec{E})\cdot(\nabla\times\vec{E})-k_0^2\epsilon_r\vec{E}\cdot\vec{E}\right]dV\\ &\quad+\iint_{S_1}\left[\frac{\gamma}{2}(\hat{n}\times\vec{E})\cdot(\hat{n}\times\vec{E})+\vec{E}\cdot\vec{U}^{\rm inc}\right]dS\\ &\quad+\iint_{S_2}\left[\frac{\gamma}{2}(\hat{n}\times\vec{E})\cdot(\hat{n}\times\vec{E})\right]dS\end{aligned} \quad (7)$$

The FE discretization of this functional is discussed in Section IV.

B. Computation of the S-parameters

The above formulation produces R and T for port 1 (S_{11} and S_{21}). It must be repeated with an incident field at port 2 to obtain S_{12} and S_{22}. Only the excitation vector changes, so this is simply a question of repeating the matrix solve. For multiple ports, the extension is obvious: T is computed at *each* port, producing one column of the S matrix. The excitation is then repeated at each port to produce other columns. Although it will not be shown in this paper, the formulation has also been verified successfully by the author for a four-port device.

The S-parameters may be computed directly from the fields on the ports. A more accurate approach uses the orthogonality of the modes to integrate the fields computed over each port [3, §8.5]; as an example for the two-port example, the transmission coefficient is given by:

$$T=\frac{2}{abE_0}\iint_{S_2}\vec{E}(\xi,\eta,\zeta)\cdot\vec{e}_{10}(\xi,\eta)dS \quad (8)$$

As before, $\vec{e}_{10}(\xi,\eta)$ is the relevant waveguide eigenmode; a and b are the waveguide dimensions.

C. *The waveguide formulation: another perspective*

The formulation can be viewed as a finite element/boundary integral (FE/BI) formulation, using the waveguide Green's function for "exact" mesh termination. (For radiation or scattering problems, FE/BI formulations use the free space, or sometimes the half-space, Green's function, eg. [3, §9.3]). The current dominant-mode-only analysis uses only the first in the infinite series of modes comprising the waveguide Green's function. It is accurate provided that the ports are sufficiently far removed from the discontinuities (assuming, of course, that only the dominant mode is above cut-off). Higher order modes are easily included in the formulation; this does require re-computing both the LHS matrix and RHS vector, since the former has one term dependent on the propagation constant, and the latter is obviously dependent on the incident mode shape. The formulation presently assumes hollow waveguide at the ports; i.e. only TE (and TM modes, if an $\vec{H}$ field solver is also implemented) are included. More exotic modes, or numerically determined ones, could also be incorporated into the formulation. This is discussed further in Section VI.

III. Higher order vector elements

A. *Vector elements — a brief review*

Edge elements (also known as Nedelec elements; Whitney elements/forms; CT/LN elements; $H_0(\mathrm{curl})$ elements) were introduced during the 1980's. Nedelec [8] provided the mathematical framework for mixed order finite elements of various order. However, the polynomial spaces from which the basis functions were to be chosen were defined by him in terms of Cartesian coordinates, which is not the form vector elements are generally given in now. Cendes, Bossavit, Webb and others introduced vector elements to EM FEA analysis during the late 1980's and 1990's. (See [9] for the original references). The element shape function was then presented in terms of simplex coordinates [10], as what is now recognized as a Whitney form, dating back to much earlier work by Whitney:

$$\vec{w}_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \tag{9}$$

This element has the well-known properties of constant tangential/linear normal field (CT/LN) approximation along edges (hence, of mixed order). Since the approximation is constant in the direction tangential to the edge connecting nodes i and j, and perpendicular to *all* the other edges (two, for triangles, or five, for tetrahedrons), the degrees of freedom, defined by Nedelec as the line integrals of the finite element approximation along the respective edges, are simply the tangential fields — hence the name "edge elements". For higher order elements, additional degrees of freedom on faces must be introduced, and the name "vector elements" has now largely supplanted "edge elements". For CT/LN elements, some researchers have associated the degree of freedom with the tangential field at the centre of each edge [11].

B. *Vector vs. mixed-order elements*

It is not always appreciated that being of mixed-order is not an essential property of vector elements *per se*. Complete sets of vector elements have also been described [9], with degrees of freedom proportional to tangential field components, as for mixed-order elements. (This permits explicit enforcement of tangential field continuity only, as for mixed-order elements. As is well known, this type of field continuity is very difficult to arrange in general with nodal-based elements, which are also generally complete). However, such complete sets of vector element produce "wasted" d.o.f.'s for wave *eigenvalue* problems. See [12] for a comprehensive discussion of this. In essence, Nedelec's constraints provide mixed-order elements that model the curl-space as efficiently as possible, for a given number of degrees of freedom. Recent work by Webb [13] has indicated that some vector electromagnetic problems are more efficiently analyzed using complete-order vector elements, typically when the solution is dominated by electric fields strongly "gradient" in nature. (The specific example Webb uses is an iris in a waveguide, where the solution is strongly dominated by quasi-static field components).

C. *Higher order elements*

Although extending the "edge" elements to higher order became a topic of interest as soon as the CT/LN elements achieved widespread acceptance, it remains a topic of active research at present, a decade or more later. Development of such elements raises a number of issues, including: hierarchal vs. interpolatory behaviour; methods for the construction of the element shape functions; the interpretation of the degrees of freedom; the construction of prototype elemental matrices (analytical vs. quadrature); and the efficient iterative solution of the poorly conditioned linear algebra systems which unfortunately often result.

C.1 Hierarchal higher order LT/QN elements

For mesh refinement/enrichment purposes, hierarchal elements are required, and this paper considers only the use of such elements. Interpolatory elements have been comprehensively described in [14]. Two specific hierarchal elements have been used; the work was originally undertaken [15], [16], [7] using those proposed by Savage [17]; see Table I. Subsequently, the elements proposed by Andersen and Volakis [18], [19] have also been implemented.

As per Nedelec's definitions of suitable mixed order element, there are twenty vector based functions (v.b.f.'s) and degrees of freedom (d.o.f.'s) per tetrahedron. The lowest order v.b.f. — the Whitney form — has the usual properties, and the accompanying d.o.f. is proportional to the tangen-

CT/LN		
Edge-based	1 per edge	$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$
LT/QN		
Edge-based Face-based	1 per edge 2 per face (out of 3 possible)	$\nabla(\lambda_i \lambda_j)$ $\lambda_i(\lambda_j \nabla \lambda_k - \lambda_k \nabla \lambda_j)$

TABLE I

SAVAGE'S LT/QN HIERARCHAL ELEMENTS.

CT/LN – all		
Edge-based	1 per edge	$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$
LT/QN — Savage		
Edge-based Face-based	1 per edge 2 per face	$\nabla(\lambda_i \lambda_j)$ $\lambda_i(\lambda_j \nabla \lambda_k - \lambda_k \nabla \lambda_j)$
	(and $\{j;i;k\}$ but not $\{k;i;j\}$)	
LT/QN — Webb and Forghani		
Edge-based Face-based	1 per edge 2 per face	$\nabla(\lambda_i \lambda_j)$ $\lambda_i \lambda_k \nabla \lambda_j$
	(and $\{j;k;i\}$ but not $\{i;j;k\}$)	
LT/QN — Andersen and Volakis		
Edge-based	1 per edge	$(\lambda_i - \lambda_j)\times$ $(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)$
Face-based	2 per face (as for Savage's elements)	$\lambda_i(\lambda_j \nabla \lambda_k - \lambda_k \nabla \lambda_j)$

TABLE II

COMPARISON OF VARIOUS HIERARCHAL LT/QN ELEMENT SCHEMES.

tial electric field on the edge. Since the element shape functions are hierarchal, higher-order d.o.f.'s are not located at specific points, but defined as weighted field quantities integrated over the relevant edge or face. The latter has the interpretation as the flux through the face.

Many other hierarchal elements have been published, in particular of LT/QN order. Most of these (including those of Savage described above) can be seen as variants of the elements proposed by Webb and Forghani [20]. (Indeed, not only are these variants on a theme, they are also linear transforms, as will be discussed subsequently). A number are summarized in Table II. Note that all the face elements exclude (arbitrarily) one possible combination of $\{i;j;k\}$; this asymmetry has long been noted, and is required to avoid linearly dependent basis functions.

An apparent exception to this are the elements proposed by Andersen and Volakis [18], [19]; the additional six vector based functions for the edges are apparently of quadratic order. However, the Andersen and Volakis elements are linear transforms of the (non-hierarchal) elements proposed in [21] (the explicit transform for the two dimensional case was given in [19]) and the hierarchal elements proposed by Savage [17] — and used here — are in turn linear combinations of those in [21], thus the Andersen and Volakis elements are linear transforms of Savage's [17].

It might seem strange that the Andersen and Volakis elements, with apparently quadratic behaviour, can be expressed as linear combinations of elements with at most linear field dependence. This is a consequence of the mixed-order nature of the basis functions, and of course the linearly dependent nature of simplex coordinates ($\sum_{i=1}^{N} \lambda_i = 1$, with $N = 2,\ 3$ or 4 in one, two or three dimensions respectively). The $(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)$ term is of course the Whitney element, with CT/LN behaviour along edges; multiplication by the $(\lambda_i - \lambda_j)$ term (actually the Legendre polynomial P_1 redefined on the interval $[0, 1]$) yields the LT/QN behaviour.

These elements are generally constructed by "inspection", using the properties of simplex coordinates, and the gradients thereof. Nedelec required that these functions be unisolvent (that is, linearly independent) and conforming (that is, d.o.f.'s are proportional to integrals of the tangential field along edges, or over faces). The latter is easily established by inspection, but the former is less obvious. The present author has shown that all these basis functions satisfy the Nedelec constraints (restrictions on the properties of the polynomial spaces from which they are chosen) and thus (from [8, Theorem 1]) the elements are indeed both conforming and unisolvent. This proof requires expressing the basis functions in Cartesian coordinate form and then testing the Nedelec constraints explicitly; it will not be detailed here.

There is another school of thought regarding the construction of higher-order basis functions, which might be described as the *degree of freedom-centered* approach (as opposed to the above, which could be described as the *basis function-centered* approach). Salazar-Palma et al. [11] use elements from the Nedelec polynomial space and enforce Lagrangian interpolatory properties on the degrees of freedom. This produces *interpolatory* elements with well-defined degrees of freedom at *points*, but this is not possible in general with higher-order *hierarchal* elements. Yioultsis and Tsikboukis take a similar degree-of-freedom centered approach, but working with simplex instead of Cartesian coordinates [22].

IV. IMPLEMENTATION ISSUES

A. Finite element discretization

The finite element discretization of the volumetric integral term is identical to that of cavity eigenanalysis. This has been described in several references (such as [3], [23], [21], [24]) and will not be discussed further here.

Discretization of the surface integral terms, which arises due to the introduction of the ports, requires compatible surface basis functions. This is discussed in Jin in detail [3, §8.5], and need only be outlined here, since the extension to higher-order elements is obvious. Generation of the volumetric tetrahedral mesh automatically generates a triangular surface mesh. Suitable basis functions also implicitly defined, as follows:

$$\hat{n} \times \vec{E}^s = \sum_{i=1}^{n_s} \vec{S}_i^s = \{E^s\}^T \{\vec{S}^s\} \quad (10)$$

with $\{E^s\}$ are the degrees of freedom associated with the (surface) triangular element.

The surface basis functions are:

$$\vec{S}_i^s \equiv \hat{n} \times \vec{N}_i^s \quad (11)$$

$\vec{N}_i^s$ are the appropriate tetrahedral functions for the face.

Note that for CT/LN elements, this magnetic surface current ($\hat{n} \times \vec{E}^s$) discretization is identical to that produced by standard moment method RWG elements [25], providing a linear tangential/constant normal approximation of the current.

B. Elemental matrices and matrix assembly

A new elemental matrix and vector are required:

$$[B^s] = \iint_{S^s} \gamma \vec{S}^s \cdot \vec{S}^s dS \quad (12)$$

$$\{b^s\} = \iint_{S_2^s} \vec{S}^s \cdot (\vec{E}^{\mathrm{inc}} \times \hat{n}) dS \quad (13)$$

Here, γ is as in eqn. (3), and $\vec{E}^{\mathrm{inc}}$ is the incident field, as before.

$[B^s]$ can be evaluated in closed form, since it involves the integral of simplex coordinates over both ports — these integrals are known analytically. $\{b^s\}$ requires quadrature, since it involves the product of the incident mode, typically a sinusoid or product of sinusoids, with a vector based function. A four-point symmetric rule [26] was generally found to be sufficient, although a six-point rule was also implemented.

The system matrix $[A]$ is assembled from $[S]$, $[T]$ and $[B]$; the forcing vector is $\{b^s\}$, resulting in the conventional linear system $[A]\{x\} = \{b^s\}$ with $\{x\}$ the vector of degrees of freedom to be solved for. All these terms are frequency dependent, and $[B]$ and $\{b^s\}$ are additionally also dependent on the mode number and/or mode type (TE or TM), either via the propagation constant or the modal eigenfunction.

Once the system matrix and right hand side vector have been assembled, the system is solved (for multiple RHS's, if the full S-matrix is required) and the S parameters are extracted as already discussed in Section II-B.

The $[S]$ and $[T]$ elemental matrices may be pre-computed, using explicit forms as given in [3], [23], [21], [24], [15]. However, a non-trivial amount of analytical work is required for new elements, and the use of cubature (three-dimensional quadrature) permits far quicker program development; new element basis functions (and their gradients, which are straightforward to compute analytically), can be added in very quickly. Since the functions being integrated are polynomials, and very efficient rules exist for integration of polynomials over simplexes, the computational overhead is modest, and of $\mathcal{O}(N)$. As an example, a symmetric rule of degree of precision four requires only eleven points, and this is sufficient for LT/QN basis functions. (The elemental matrix entries in this case require at most the integration of the product of quadratics, i.e. a polynomial of order four).

h (mm)	N	CT/LN D.o.f.'s	CT/LN $\lvert S_{21}\rvert$ (dB)	CT/LN $\angle S_{21}$ (^{o})	LT/QN D.o.f.'s	LT/QN $\lvert S_{21}\rvert$ (dB)	LT/QN $\angle S_{21}$ (^{o})
20.4	20	14	-16.18	134.4	96	-0.4113	22.11
13.8	43	29	-4.538	5.602	202	-0.3001	6.077
9.87	105	73	-1.157	-38.57	498	-0.01575	-0.2293
7.31	234	172	-0.5191	-19.72	1144	-2.78E-3	0.0448
6.47	369	273	-0.7414	-31.87	1810	-2.26E-3	-0.1619
4.99	697	600	-0.1944	-3.615	3700	-4.34E-4	0.0251
4.22	1125	962	-0.1455	-0.561	5948	-9.56E-4	0.0546

TABLE III

S_{21} FOR AN EMPTY SECTION OF WAVEGUIDE FOR THE DOMINANT TE_{10} MODE AS A FUNCTION OF THE AVERAGE EDGE LENGTH h [MM] AND NUMBER OF ELEMENTS N.

V. RESULTS

The theory discussed above has been implemented in a finite element code developed by the author, his students and industrial colleagues. The code uses the same graphical user interface as the commercial package FEKO, and is called FEMFEKO [27].

A. Empty guide

An empty section of waveguide provides a useful test of the performance of the elements, since the results are known exactly. In Table III, the transmission coefficient of a hollow piece of X-band waveguide, 40mm long, is presented. The analytical solution is trivial; the transmission coefficient is 1 (0dB), with phase angle 0^o in Jin's formulation, or $-k_{z_{10}}\ell$ for the extended formulation presented in this paper, as discussed in Section II. (The results in Table III were generated with Jin's original formulation, hence the zero phase angles). The 20 element result is of course highly inaccurate, since the problem is badly under-discretized with so few elements. (The guide wavelength was 48.630mm). The mesh refinement used in Table III was a simple h-uniform scheme.

Eigenvalue problems are appealing since one quantity (the eigenvalue) can be checked for convergence (and it is also known that the eigenvalue is variational [28]); for example, Savage and Peterson reported convergence results for LT/QN elements for eigenvalue problems in [21]. Investigating the convergence of scattering parameters is somewhat more difficult. The energy conservation term, $|S_{11}|^2 + |S_{21}|^2$, is a useful overall solution quality indicator, suggested by Jin, and will be used here. For a lossless structure, this should of course be unity. Results are presented in Figs.1 and 2. The former shows a consistently lower error (result closer to unity) for the same h — i.e. the same mesh — and thus modelling and pre-processing effort, although of course the solution using LT/QN elements uses many more degrees of freedom. The latter shows a consistently lower error for

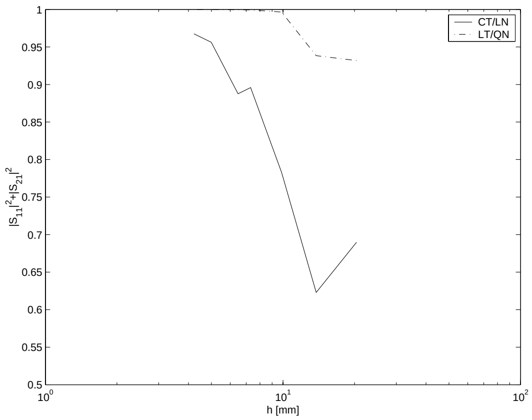

Fig. 1. $|S_{11}|^2 + |S_{21}|^2$ versus h, the average edge length in the mesh.

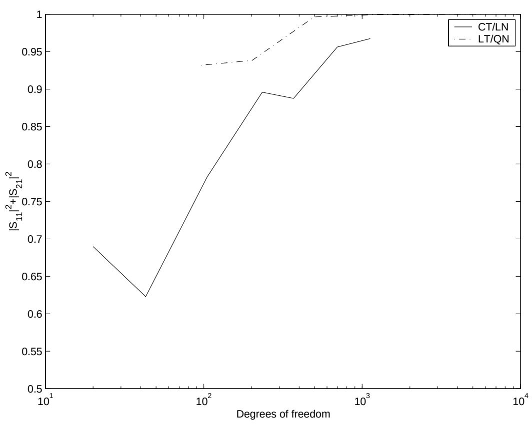

Fig. 2. $|S_{11}|^2 + |S_{21}|^2$ versus the number of degrees of freedom used.

the same number of *degrees of freedom* — and thus computational effort as well, presuming that efficient solvers are available and unaffected by the use of the higher-order elements (an assumption that will be discussed subsequently). This indeed is the major motivation for using higher-order elements.

B. E plane bend

As a test of the general formulation, and also of the relative performance of the elements, an E-plane bend will be analyzed (refer to Fig. 3). The analysis will be performed for X-band (± 8-12 GHz) waveguide. It will be seen that at the lower frequency band, the bend is largely transparent, but towards the upper end of the frequency band, the effects of the bend become significant. This problem has an (approximate) analytical solution, which was first derived some sixty years ago by Marcuvitz, Schwinger and colleagues, and subsequently documented in [1].

This problem was modelled with a section of "dummy" waveguide, to ensure that only the dominant TE_{10} mode is present at the ports. A half-length of 40mm is sufficient; this translates into around 30mm of waveguide between the bend and the port (that is, $\ell - b = 30$mm). The geometry is shown in Fig. 3. It is assumed that the waveguide is air-filled, approximated as free-space.

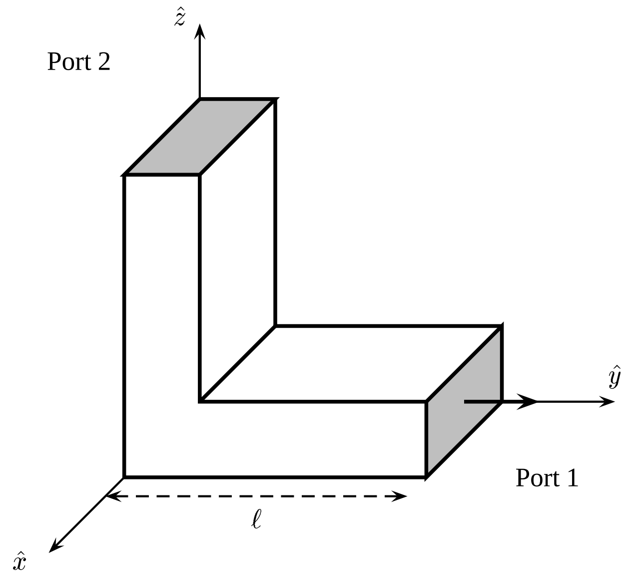

Fig. 3. The E-plane waveguide bend. Total (half) length of the bend is ℓ.

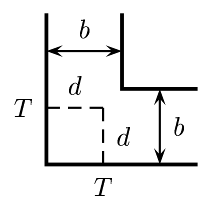

Fig. 4. Side view of the E-plane waveguide bend.

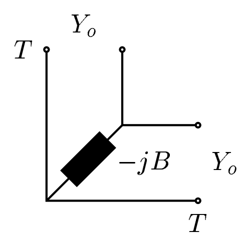

Fig. 5. Equivalent circuit of the E-plane waveguide bend.

B.1 Comparing with Marcuvitz's approximate analytical results

The equivalent circuit for the E-plane bend of Fig. 3 as derived by Marcuvitz [1, pp.312ff] is pure susceptance — a shunt inductor, $-jB$; $(B > 0)$ — at terminal planes T, located distance d from the outer corner of the bend. See Figs. 4 and 5. In the following, $Y_0 (= 1/Z_0)$ is the waveguide characteristic admittance; $Z_0 = \eta / \sqrt{1 - (\lambda/(2a))^2}$ with

λ the free-space wavelength, η the wave impedance of free space and $\lambda_g = \lambda/\sqrt{1-(\lambda/(2a))^2}$ the guide wavelength.

The formulation presented in this paper computes S parameters, whereas this model is given in terms of a shunt susceptance. There are several methods that may be used to convert Marcuvitz's model to the same format as the present formulation. A straightforward technique is to find the equivalent ABCD model for the shunt susceptance, using tables available in standard texts (for example, [29]); then convert this to S parameters; and finally, embed this within a section of guide $\ell - d$. This procedure will now be applied.

Firstly, the normalized shunt susceptance and distance d can be read off [1, Fig.5.28-3]. This is then multiplied by the normalized factor $2Y_0 b/\lambda_g$, giving B_{shunt}.

The ABCD matrix [29, Table 4.1] for this shunt load is:

$$A = 1; \;\; B = 0; \;\; C = -jB_{\mathrm{shunt}}; \;\; D = 1 \qquad (14)$$

The S-parameters are [29, Table 4.2]:

$$\begin{aligned} S_{11} &= \frac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D} \\ S_{12} &= \frac{2(AD - BC)}{A + B/Z_0 + CZ_0 + D} \\ S_{21} &= \frac{2}{A + B/Z_0 + CZ_0 + D} \\ S_{22} &= \frac{-A + B/Z_0 - CZ_0 + D}{A + B/Z_0 + CZ_0 + D} \end{aligned} \qquad (15)$$

Finally, the shunt load is embedded in a line of length $d - \ell$. This amounts to changing the phase of the individual S-parameters by $e^{-j2\theta}$, with $\theta = k_{z_{10}}(\ell - d)$ [29, p.202-204], and $k_{z_{10}} = 2\pi/\lambda_g$ the wavenumber of the TE_{10} mode. This now permits direct comparison between the results computed using the FEM and the approximate results derived by Marcuvitz.

B.2 Results

For the results to be presented, two meshes were generated; a "coarse"mesh with an average edge length of about 6mm, and a "fine" mesh, with 3.5mm average edge length. The problem was run over a frequency range of 8.25–12.25 GHz, with an accompanying guide wavelength varying from about 60–29mm. Thus, at the highest frequency, the coarse mesh was about $\lambda/5$ — too coarse for the CT/LN elements to generate reliable solutions. The fine mesh should be satisfactory.

Results are presented in Figs. 6 and 7 for the CT/LN elements for the coarse elements. The S_{11} results for the coarse mesh are indeed very inaccurate. However, the fine mesh yields acceptable results, although S_{11} is still not very accurately computed. (Note that at the low frequency end, S_{11} is small, and a very accurate solution will be required to obtain good agreement.)

Clearly, this is case where the LT/QN elements will be required to obtain really good results. These are presented in Figs. 8 and 9 for the LT/QN elements. In this case, the phase comparison is also shown, in Figs. 10 and 11. Even with the coarse mesh, the results are now acceptable; for the fine mesh, the agreement is excellent across the entire frequency band.

Both Savage's and Andersen and Volakis's LT/QN elements were used; both perform very well from a viewpoint of accuracy. There was no discernible difference between the S-parameter results computed using them. Since the basis functions are related by a linear transformation, this is to be expected. However, *both* generated ill-conditioned matrices on occasion. This remains a problem and will be discussed in the next section.

C. Iterative solver convergence and computational efficiency

A problem which has not been addressed in this paper is the issue of *computational efficiency*. Unfortunately, the higher-order elements appear to generate ill-conditioned matrices. The results presented here were obtained using iterative solvers for the linear algebra — variants on the conjugate gradient scheme (CG, Bi-CG), QMR and GMRES, with simple diagonal pre-conditioning, where relevant — but all converged erratically, some schemes converging rapidly at certain frequencies, and then converging very slowly at others, and also exhibiting different convergence behaviour for different geometries. (*All* converged at an acceptable pace for the CT/LN elements). This is a problem which is only hinted at in much of the literature on higher-order vector based elements. An exception is the work by Webb [13]; he proposes a scheme to improve the matrix conditioning by at least partially orthogonalizing the higher-order basis functions. Other recent approaches have focussed on the use of more sophisticated pre-conditioners. Incomplete LU pre-conditioning is one possibility; another is the use of a direct solution of the CT/LN solution (which can generally be computed quite cheaply) as a pre-conditioner for the LT/QN matrix.

VI. Conclusions

This paper has discussed the use of LT/QN elements for waveguide analysis. As would be expected, the LT/QN elements give *much* better solutions for the same mesh than CT/LN elements; this remains true if the number of degrees of freedom are compared. *Which* of the many published LT/QN elements are used appears insignificant in terms of solution accuracy (at least for the E-plane bend analyzed in this paper, as well as several other problems not reported here); the choice of element does impact on the convergence of the iterative solver, but unfortunately not in a consistent fashion.

Although the performance of higher-order elements is usually compared with that of lower-order elements in terms

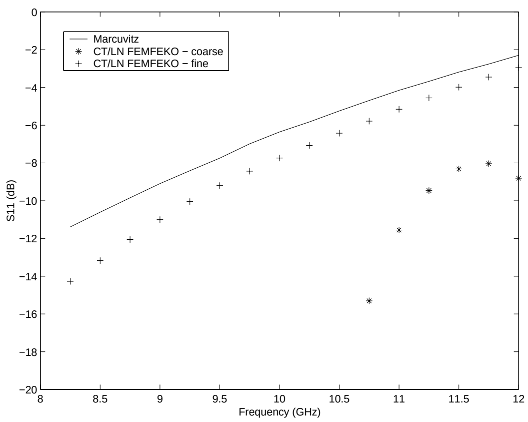

Fig. 6. Magnitude of the reflection coefficient for the E-plane bend in X-band guide; CT/LN elements.

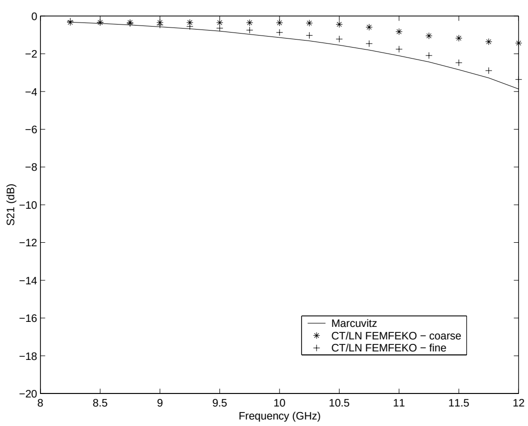

Fig. 7. Magnitude of the transmission coefficient for the E-plane bend in X-band guide; CT/LN elements.

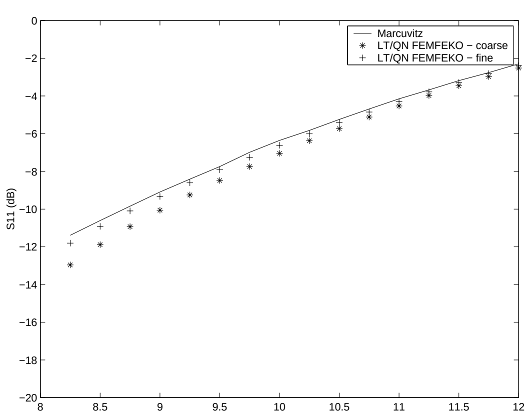

Fig. 8. Magnitude of the reflection coefficient for the E-plane bend in X-band guide; LT/QN elements.

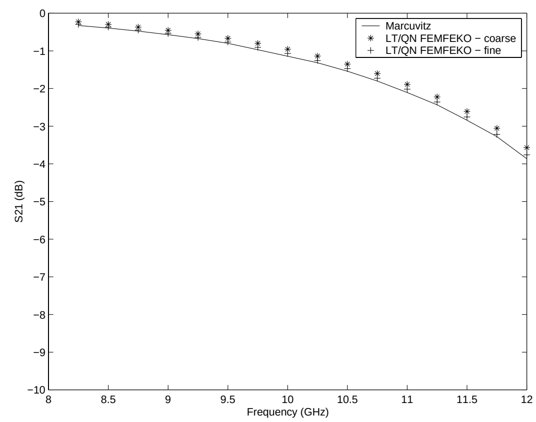

Fig. 9. Magnitude of the transmission coefficient for the E-plane bend in X-band guide; LT/QN elements.

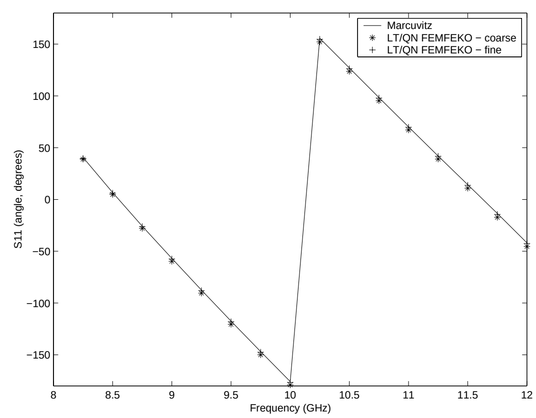

Fig. 10. Phase of the reflection coefficient for the E-plane bend in X-band guide; LT/QN elements.

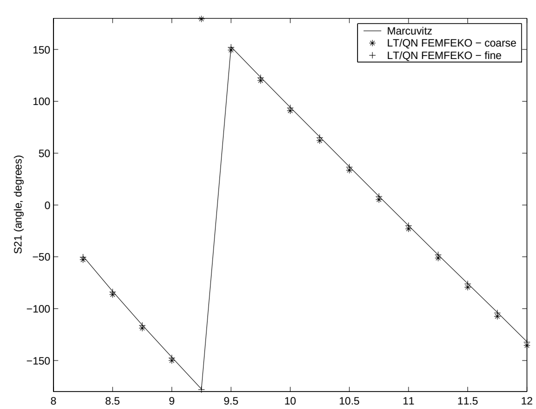

Fig. 11. Phase of the transmission coefficient for the E-plane bend in X-band guide; LT/QN elements.

of the number of degrees of freedom required for a particular accuracy, it is worth making the point that the geometrical pre-processing (and to a lesser extent, post-processing) required in a real-world FE code is largely a function of the number of elements, rather than of number of the degrees of freedom. The time required for this can become a significant fraction of the total run-time of the code. This is another practical advantage of higher-order elements not often mentioned in the literature.

Yet higher-order schemes — quadratic tangential /cubic normal and also cubic tangential / quartic normal basis functions have been published (eg [21], [17]); the extension of the present work to these is relatively straightforward theoretically, but implementing these will require meticulous work.

From the viewpoint of waveguide applications, further waveguide implementations could include TM modes; this would require an $\vec{H}$ solver (due to the boundary condition) if the same formulation is used, but this should present no major problems. Multi-mode analysis is already included in the formulation (and been implemented in the code which generated these results), but had not been tested at the time of writing.

Extending the formulation to right-circular cylindrical waveguides (or indeed any shape for which the eigenmodes are known in closed form and are transverse in nature) would be moderately straightforward. Extensions to include arbitrary, numerically-determined modes would be a desirable future addition. Inhomogeneously loaded waveguide poses new challenges, since one needs to simultaneously solve for the tangential and axial fields. Hybrid schemes, using vector elements for the former and nodal schemes for the latter have been used successfully [30], but there are a variety of approaches to this requiring exploration. These also pose some problems for the present formulation, which relies on the transverse nature of the eigenmodes to obtain the boundary condition at the ports.

Another possible extension would be to include anisotropic materials; the formulation for diagonally anisotropic materials for CT/LN elements is available [24] and this would have to be extended to fully anisotropic media and LT/QN elements. Non-linear materials are however probably best left to a time-domain approach, in particular the FDTD.

ACKNOWLEDGMENTS

The author would like to thank several colleagues and students, all of the Dept. of Electrical and Electronic Engineering, unless otherwise indicated. Riana Geschke implemented the sparse matrix routines used. Dr Frans Meyer (Electromagnetic Software and Systems Pty Ltd) implemented the graphical user interface required to run the program and extract results, and supported the work via an industrial research contract. Matthys Botha wrote a number of general I/O routines, as well as the iterative solvers, used in this work. His comments on the draft were also useful. Discussions regarding general waveguide theory with Profs. John Cloete and Petrie Meyer were helpful. The comments of two anonymous reviewers were appreciated, in particular as regards the inter-relationship of various higher-order elements, as were those of the guest editors. Finally, the (South African) National Research Foundation grants GUN 2034087 and 2046872 assisted with infrastructure and scholarships for post-graduate students working with the author.

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