APPLICATION OF THE *UMOM* FOR THE COMPUTATION OF THE SCATTERING BY DIELECTRICS COATED WITH WEAKLY NONLINEAR LAYERS

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ABSTRACT. The paper describes an iterative approach to the computation of the electromagnetic scattering by isotropic, dielectric objects partially made of weakly nonlinear materials. The approach is started by using a perturbative moment-method solution based on the Sherman-Morrison-Woodbury formula. The nonlinearity is assumed to be of the Kerr type, i.e., the dielectric permittivity depends on the square amplitude of the electric field. The bistatic scattering width and the field distribution are computed for some test cases, in particular, for infinite cylinders coated and filled with nonlinear materials. The convergence of the medium is numerically evaluated and the results are compared with those obtained by the iterative distorted-wave Born approximation.

1 INTRODUCTION

An interesting perturbational version of the moment method [1] was recently proposed by Yip and Tomas [2]. The approach is aimed at determining the electromagnetic scattering by a slightly perturbed scatterer, after a moment-method solution for the original unperturbed scatterer has already been obtained. The method applies the Sherman-Morrison-Woodbury (SMW) updating formula and allows one to consider changes in both the geometry and the dielectric properties of the scatterer. The above method, called by the authors the UMoM, is one of the various perturbational methods that make it possible to avoid repeating a complete computation when several scatterers, only partially different, have to be considered in the scattering evaluation. An overview of these methods was presented by Newman [3], who also described an efficient combination of the moment method with Green's function.

In this paper, the application of the UMoM, as proposed in [2], is the starting point for the development of an iterative approach to the computation of the electromagnetic scattering by nonlinear dielectric objects. The interest in evaluating the scattering by nonlinear dielectrics is generally related to the possibility of using them as coating materials, for example, in order to obtain apparatus for minimizing and maximizing scattering cross-sections in camouflage applications.

Here the nonlinearity is assumed to be of the Kerr type, i.e., the relative dielectric permittivity depends on the square amplitude of the internal electric field. As long as the nonlinearity is weak, as for most of nonlinear materials [4], the main effect of the nonlinearity is a modification to the field distribution at the frequency of the incident field, whereas the process of higher-harmonics generation may be neglected. Moreover, if the nonlinearity is weak, from a perturbation point of view, one can assume the effective dielectric permittivity to be approximated by writing it in terms of the linear field. This was done by the authors in a previous work in which they computed the bistatic scattering width for a circular nonlinear cylinder by using an iterative approach based on the distorted-wave Born approximation [5]. The main limitation of this approach is related to the fact that the effective dielectric permittivity must be weak in order that the process may converge. This is a severe limitation, in that a nonlinearity can usually be considered weak but the resulting effective dielectric permittivity is not at all weak. The authors showed that, although the nonlinearity was very weak, it affected the bistatic scattering width in a significant way.

If a nonlinear material is assumed to be only a portion of the scatterer considered, the UMoM can be successfully applied, as the weak nonlinearity can be viewed as a perturbation of the original scattering configuration. At this point, an iterative process is started by applying the SMW formula, or, in a simpler way, by using the approach described in [5], but, at the 0-*th* step, the linear field (distorted-wave Born approximation) is replaced by the field obtained by the UMoM without iterations.

In the following, the mathematical formulation of the approach is provided. Some test cases are described that involve coated cylinders of circular and irregular cross-sections. We consider infinite cylinders illuminated by transverse-magnetic waves. As $\nabla \cdot \mathbf{E} = 0$, $\mathbf{E}$ being the electric field vector, the problem reduces to a two-dimensional scalar one, for which the notation is simplified.

2 MATHEMATICAL FORMULATION

Let us consider an infinite dielectric cylinder of arbitrary cross-section, with the cylindrical axis parallel to the z axis (Figure 1). The cylinder is illuminated by a time-periodic transverse-magnetic electromagnetic field, for which $\mathbf{E}^{inc}(x,y,z,t) = E_z^{inc}(x,y,t)\mathbf{z}$ and $\mathbf{H}^{inc}(x,y,z,t) = H_x^{inc}(x,y,t)\mathbf{x} + H_y^{inc}(x,y,t)\mathbf{y}$. The propagation medium is assumed to be lossless, homogeneous, and characterized by μ_o and ε_o. Suppose the region S_2 to be made of a weakly nonlinear material (isotropic and nonmagnetic) whose dielectric permittivity is of the Kerr type [6]:

$$\varepsilon_{nl}(x,y) = \varepsilon_o[\varepsilon_2(x,y) + \xi|E_z(x,y,t)|^2] \quad (1)$$

where $\varepsilon_2(x,y)$ is the linear part and ξ is a nonlinear parameter. The medium of the region S_2 is assumed to be inhomogeneous both due to the nonlinearity and in the limit $E_z(x,y,t) \to 0$ [7]. To simplify the notation, let us assume that also the dielectric permittivity of the region S_1 is expressed by (1), with $\xi = 0$ and $\varepsilon_2(x,y)$ replaced by $\varepsilon_1(x,y)$.

In order to devise an iterative approach to the computation of the electromagnetic field distribution, let us compute the scattering by an inhomogeneous linear scatterer of section $S = S_1 \cup S_2$, obtained by setting $\xi = 0.0$ everywhere. The electric field integral equation (EFIE) for this problem can be expressed as [8]:

$$\Phi^t(x,y) = \Phi^i(x,y) - j(k_o^2/4)\int_S[\varepsilon_{lin}(x,y) - 1]\Phi^t(x',y') \times H_o^{(2)}(k_o\rho)dx'dy' \quad (2)$$

where $\Phi^i(x,y)$ and $\Phi^t(x,y)$ are the space-dependent parts of the incident and the total electric fields (the time-dependence $exp\{j\omega t\}$ is assumed and suppressed); $H_o^{(2)}(k_o\rho)$ is the Hankel function of the second kind and the zero-th order, ρ is given by: $\rho = [(x - x')^2 + (y - y')^2]^{1/2}$, and $\varepsilon_{lin}(x,y) = \varepsilon_1(x,y)$ if $(x,y) \in S_1$, $\varepsilon_{lin}(x,y) = \varepsilon_2(x,y)$ if $(x,y) \in S_2$. By applying the Richmond formulation [9] to (2), the problem solution is reduced to solving the following algebraic system of linear equations:

$$[G]\underline{\Phi}^t = \underline{\Phi}^i \quad (3)$$

where:

$\underline{\Phi}^t$: unknown array of dimensions P × 1, P being the number of subdomains. The pth element of $\underline{\Phi}^t$ is given by: $\phi_p^t = \Phi^t(x_p,y_p)$, where (x_p,y_p) is the center of the pth subdomain;

$\underline{\Phi}^i$: excitation array of dimensions P × 1 whose elements are given by: $\phi_p^i = \Phi^{i(1)}(x_p,y_p)$, $p = 1,...,P$;

[G]: Green's matrix of dimensions P × P whose generic elements are:

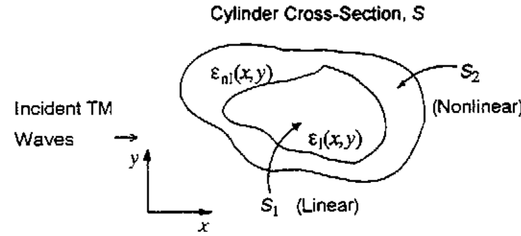

Fig. 1 Problem geometry.

$g_{pq} = (j/2)[\varepsilon_{lin}(x_q,y_q) - 1][\pi k_o a_q H_1^{(2)}(k_o a_q) - 2j]$ if $p = q$

$g_{pq} = (j/2)[\varepsilon_{lin}(x_q,y_q)-1]\pi k_o a_q J_1(k_o a_q)H_0^{(2)}(k_o\rho_{pq})$ if $p \neq q$

where $\rho_{pq} = [(x_p - x_q)^2 + (y_p - y_q)^2]^{1/2}$ and $a_q = (S_q/\pi)^{1/2}$, S_q being the area of the q-th subdomain.

Let us now apply the UMoM, considering the perturbed configuration obtained by computing the dielectric permittivity in (1) in terms of the linear field distribution. If we use the same scheme as for the Richmond formulation, the problem turns out to be expressed by:

$$[G']\underline{\Phi}^{nl} = \underline{\Phi}^i \quad (4)$$

where:

$\underline{\Phi}^{nl}$: unknown array of dimensions P × 1;

[G']: Green's matrix whose generic elements are:

$g_{pq} = (j/2)[\varepsilon_{nl}(x_q,y_q) - 1][\pi k_o a_q H_1^{(2)}(k_o a_q) - 2j]$ if $p = q$

$g_{pq} = (j/2)[\varepsilon_{nl}(x_q,y_q) - 1]\pi k_o a_q J_1(k_o a_q)H_o^{(2)}(k_o\rho_{pq})$ if $p \neq q$,

As the geometrical properties are kept unperturbed and only the electric properties are made to change, the matrix

$$[\Delta G] = [G]-[G'] \quad (5)$$

has only P_2 non-zero columns, corresponding to the subdomains with perturbed characteristics. For the solution of (4), the UMoM uses the SMW updating formula [10]:

$$[G']^{-1} = [G]^{-1} + [G]^{-1}[U]([I] - [V]^T[G]^{-1}[U])^{-1}[V]^T[G]^{-1} \quad (6)$$

where [U] and [V] are matrices defined in the following and [I] is a $P_2 \times P_2$ identity matrix. The problem is the same as for case (a) in [2], so the identification of matrices [U] and [V] is quite immediate. In particular, U is a $P \times P_2$ matrix whose columns are the P_2 non-zero columns of [ΔG], and [V] is a $P_2 \times P$ matrix whose elements are given by: $v_{ij} = \delta_{ij}$, where δ_{ij}

denotes the Kronecker symbol. It follows that:

$$[\Delta G] = [U][V]^T \quad (7)$$

where $[V]^T$ is the transposed matrix of [V]. The use of the SMW is discussed in several papers and books (see the exhaustive list given in [2]; in the Appendix of that paper, the formula is derived for completeness). A discussion of the convergence of the series on which the SMW formula is based, in terms of the matrix eigenvalues, can be found, for example, in [10].

At this point, the nonlinear problem can be solved in two ways. Once the first-order approximation has been obtained, one can start the iterative process by applying the UMoM recursively, according to the following scheme:

- At step $k = 0$, set:
  $\underline{\Phi}^{k=0} = \underline{\Phi}^t$; $\varepsilon_{nl(k=0)}(x_p,y_p) = \varepsilon_o[\varepsilon_{lin}(x_p,y_p) + \xi|\phi^t_p|^2]$;
  $[G^{k=0}] = [G]$
- At step k, assume:
  $\underline{\Phi}^k = [G^k]^{-1}\underline{\Phi}^i = \{[G^{k-1}]^{-1} + [G^{k-1}]^{-1}[U]([I]-[V]^T[G^{k-1}]^{-1}[U])^{-1}[V]^T[G^{k-1}]^{-1}\}$;
  $\varepsilon_{nl(k)}(x_p,y_p) = \varepsilon_o[\varepsilon_{lin}(x_p,y_p) + \xi|\phi^k_p|^2]$

In a simplified version of the approach, the nonlinear solution obtained by approximating the solution for $\underline{\Phi}^{nl}$ in (4) by the SMW formula is used to start an iterative process expressed by ($k \geq 1$):

$$\Phi^t_k(x,y) = \Phi^i(x,y) - j(k_o^2/4)\int_S[(\varepsilon_{lin}(x',y')-1) + \xi|\Phi^t_{k-1}(x',y')|^2]\Phi^t_{k-1}(x',y')H_0^{(2)}(k_o\rho)dx'dy' \quad (8)$$

where $\Phi^t_{k=0}(x,y)$ is computed, in an approximate way, by (4) and (6). This simplified version constitutes an improvement over the distorted-wave Born-approximation iterative approach proposed in [5]. In a discretized form, the above iterative process can be written as:

$$\phi^k_q \approx \phi^k_q - j(k_o^2/4)\sum_{p=1}^{P}[\varepsilon_{lin}(x_p,y_p) + \xi|\phi^k_p|^2]\phi^k_p \times H_0^{(2)}(k_o\rho_p)\Delta s_p \quad (9)$$

where $\rho_p = [(x - x_p)^2 + (y - y_p)^2]^{1/2}$ and q = 1, .., P.

In a linear case, the possibility of applying the UMoM is related to the convergence of the series for the SMW formula, which in turn is related to the matrix eigenvalues. This sets rigid limits on the validity of the approach for slightly perturbed geometries. When the scatterer's geometry is unperturbed, the above condition can be satisfied for a weak "excess" of permittivity. Analogously, in the case of nonlinear scatterers, we can expect the process to converge for weak nonlinearities only. Unfortunately, in the present case, convergence depends on various factors: the linear part of the dielectric permittivity, the nonlinear coefficient, and the incident electric field. Unlike linear scattering, for a monochromatic plane-wave TM illumination, the amplitude, phase and frequency values contribute to the process convergence or divergence. At present, this makes it impossible for the authors to define a criterion that establishes whether convergence can be reached or not, for given perturbed and unperturbed configurations. In the Results section, however, this aspect will be discussed by way of several numerical examples.

To this end, let us define the following residual error:

$$\Re\{k+1\}[dB] = 10log_{10}(S^{-1}\int_S\{\Phi^t_{k+1}(x',y') - \Phi^i(x',y') + j(k_o^2/4)\int_S[\varepsilon_{nl}(u,v)-1]\Phi^t_{k+1}(u,v)H_0^{(2)}(k_o\xi)\,du\,dv\}) \quad (10)$$

The approach is assumed to be convergent if $\Re\{k\} \to 0$, as $k \to \infty$.

3 NUMERICAL RESULTS

Some test cases are now described. In the first example, a homogeneous dielectric cylinder ($\varepsilon_1 = 1.8$), coated with a nonlinear layer ($\varepsilon_2 = 1.1$), was illuminated by a unit uniform plane TM-wave propagating along the x axis (Figure 2(*a*)). The radii of the two cylinders were such that $k_oa_1 = 0.49\pi$ and $k_oa_2 = 0.6\pi$. Figure 2 gives the values of the bistatic scattering width (BSW), defined as [11]:

$$W(\phi)\,[dB] = 10log_{10}\left[\lim_{\rho\to\infty}2\pi\rho\frac{|\Phi^t(x,y) - \Phi^i(x,y)|^2}{|\Phi^i(x,y)|^2}\right] \quad (11)$$

The values of the nonlinear parameter were assumed to be (*a*) $\xi = 0.01$, (*b*) $\xi = 0.1$, and (*c*) $\xi = 0.8$. The linear values ($\xi = 0.0$) are also provided. They were analytically computed by using the recursive Richmond formula [12] (slightly corrected in [13]). For comparison, the figure also gives the values obtained by applying the iterative distorted-wave Born-approximation (DWBA) approach [5]. It can be noticed that the behaviours for $\xi = 0.01$, corresponding to a very weak nonlinearity, and for $\xi = 0.1$ are similar. For $\xi = 0.8$, the iterative DWBA solution did not converge, so only the first two iterations are shown. As expected, the iterative UMoM always converged very fast (it should be stressed that the relative high permittivity of the internal cylinder made the convergence of the DWBA problematic, even for very weak nonlinearities). It is worth noting that, for the considered values of the field intensity and of the scatterers' dimensions most of the chosen values for ξ correspond to weak

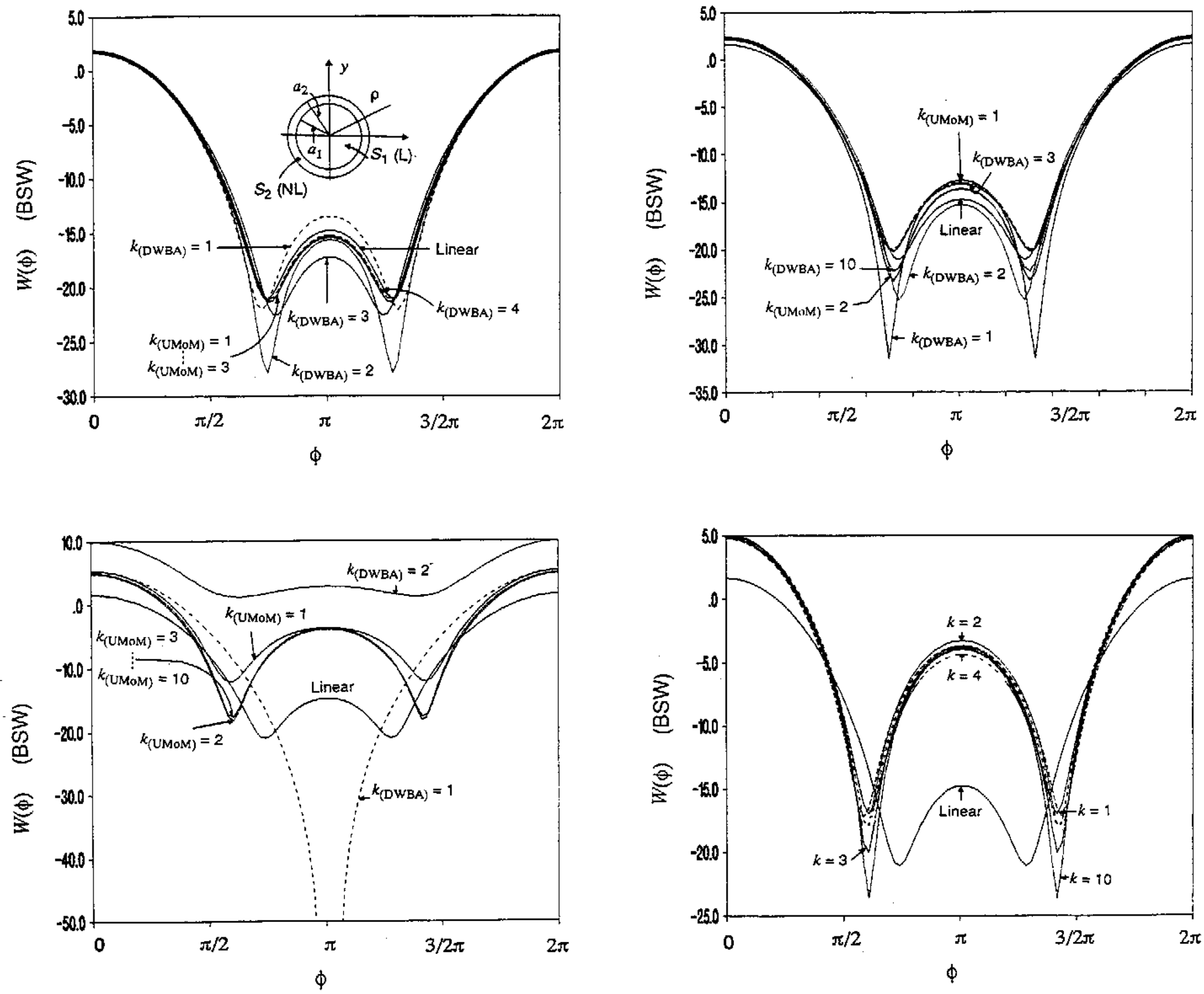

Fig. 2 Bistatic scattering width of a circular cylinder (ε_1 = 1.8) coated with a nonlinear layer (ε_2 = 1.1, k_0a_1 = 0.49π; k_0a_2 = 0.6π, P = 121, P_2 = 40). Comparison between the iterative approaches using the distorted-wave Born approximation (DWBA) and the UMoM. *(a)* ξ = 0.01; *(b)* ξ = 0.1; *(c)* ξ = 0.8; *(d)* ξ = 0.8, simplified iterative version (relation (8)).

nonlinearities, in the sense the resulting scatterers are such that the obtained scattering distributions (predicted by the assumed nonlinear electromagnetic model) can be regarded as slight perturbations of those of the corresponding linear cases.

The simplified version of the proposed approach (relation (8)) was also applied. As an example, Figure 2(*d*) gives the BSW value for ξ = 0.8. The solution converged, even though rather slowly, whereas, for the other two values of ξ, the behaviours were similar to that of the iterative DWBA approach.

Figure 3 gives, for the same values of the nonlinear parameter, the amplitude of the total electric field along the x axis [y = 0]. Finally, Figure 4 shows the plots of the residual errors (relation (10)) for different numbers of iterations. As long as the nonlinearity was weak (and hence the configuration was slightly perturbed), the iterative UMoM approach converged independently of the linear permittivity. This is confirmed by Figure 5, which gives the BSW values for a multilayer cylinder equal to that in the previous example, but with an internal dielectric permittivity equal to 5.0. In this example, the nonlinearity was partially blinded by the high value of the nonlinear permittivity. The simplified version of the approach (which exhibited obvious limitations similar to those of the DWBA approach) did not converge

($\Re(1)$ = -36.2, $\Re(2)$ = -31.3, $\Re(3)$ = -24.7, $\Re(5)$ = -2.0, $\Re(8)$ = 40.5).

In another example, we considered the effects of the ratio between the wavelength and the scatterer's dimensions by considering a multilayer cylinder with the cross-section shown

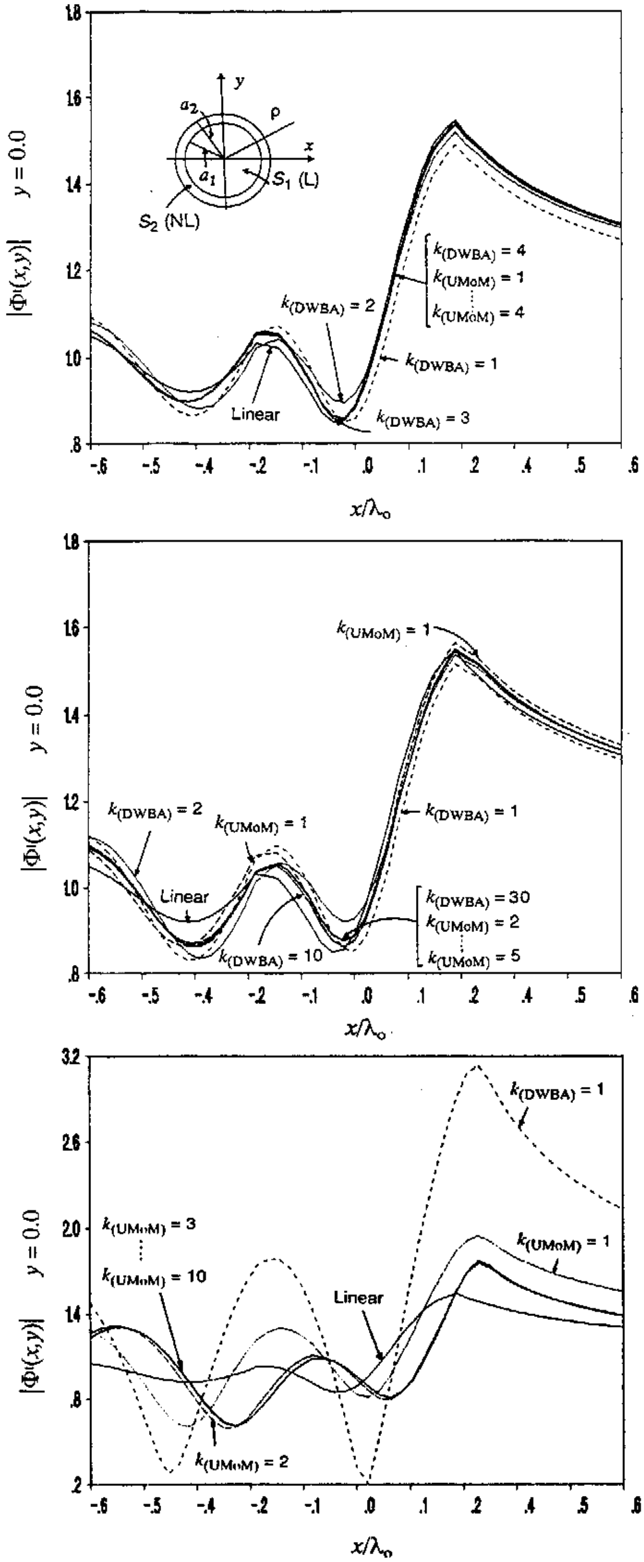

Fig. 3 Scattering by a circular cylinder (ε_1 = 1.8) coated with a nonlinear layer (ε_2 = 1.1, $k_o a_1$ = 0.49π; $k_o a_2$ = 0.6π, P = 121, P_2 = 40). Amplitude of $\Phi^t(x,y)$. Comparison between the iterative approaches using the distorted-wave Born approximation (DWBA) and the UMoM. (*a*) ξ = 0.01; (*b*) ξ = 0.1; (*c*) ξ = 0.8.

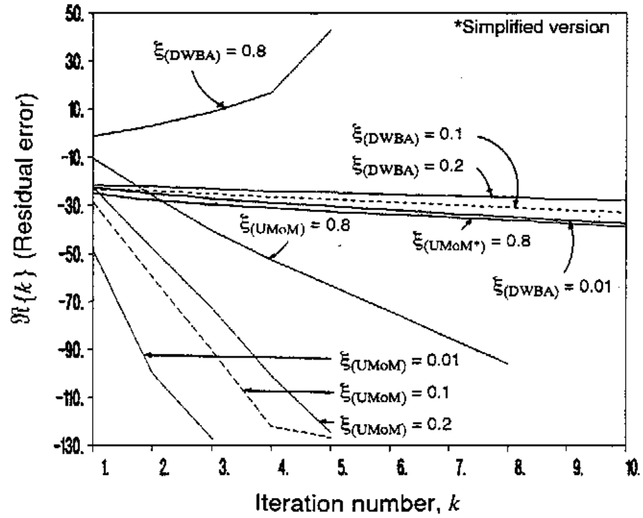

Fig. 4 Residual errors $\Re\{k\}$ (dB) for different numbers of iterations. Simulations in Figs. 2 and 3.

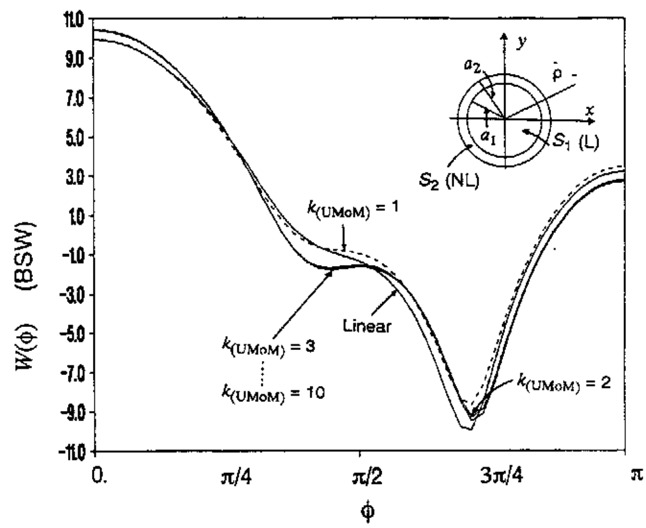

Fig. 5 Bistatic scattering width of a circular cylinder (ε_1 = 5.0) coated with a nonlinear layer (ε_2 = 1.1, ξ = 0.1, $k_o a_1$ = 0.49π; $k_o a_2$ = 0.6π, P = 121, P_2 = 40). Iterative UMoM.

in Figure 6(*a*). The internal layer was linear (ε_1 = 1.5), whereas the external was nonlinear (ε_2 = 1.5, ξ = 0.2). The illumination conditions were the same as in the previous examples. The BSW was computed by using the iterative UMoM for (*a*) $k_o l$ = 0.48π, (*b*) $k_o l$ = 0.8π, and (*c*) $k_o l$ = 1.2π. The linear values (ξ = 0.) are also given in Figure 6, and the residual errors are given in Table I.

A linear circular cylinder (ε_1 = 3.0) with a nonlinear nucleus (ε_2 = 3.0, ξ = 0.2) was then considered. The radii of the two cylinders were such that $k_o a_1$ = 1.27π and $k_o a_2$ = 1.63π.

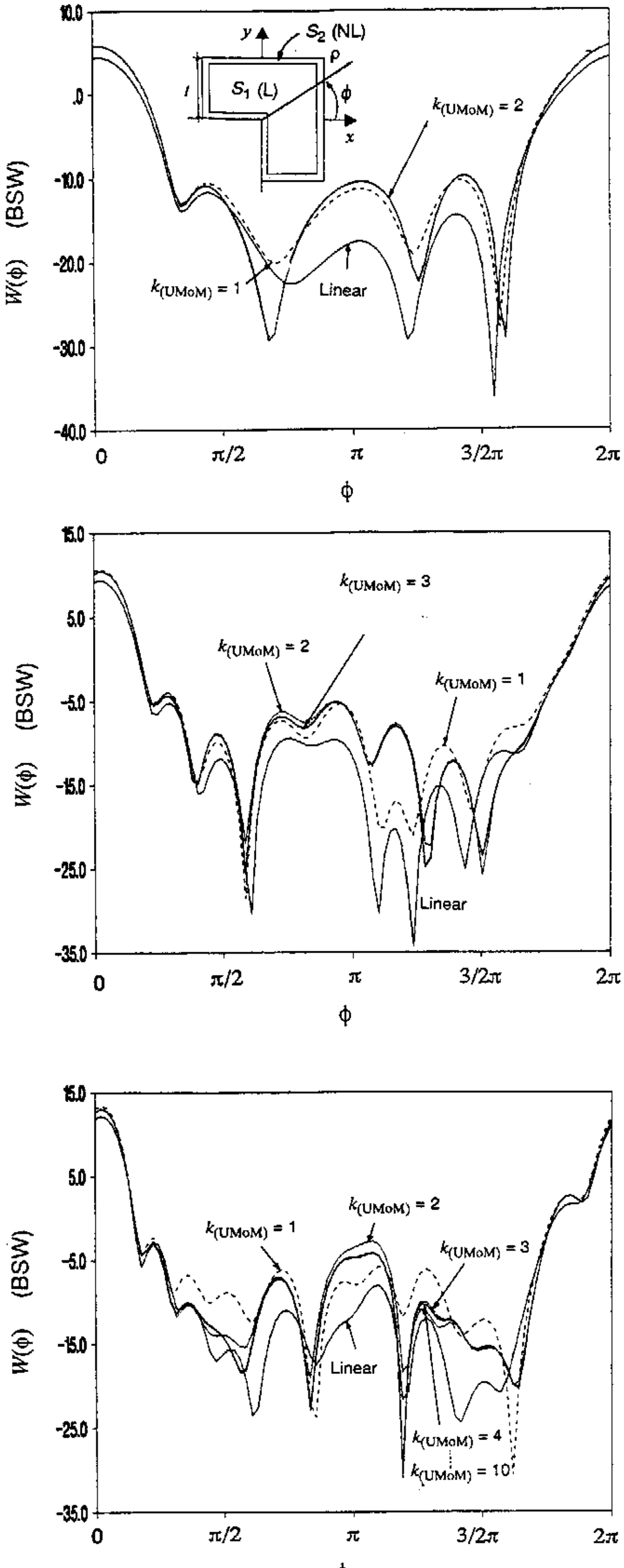

Fig. 6 Bistatic scattering width of a cylinder of irregular cross section ($\varepsilon_1 = \varepsilon_2 = 1.5$, $\xi = 0.2$, $P = 132$, $P_2 = 52$). (*a*) $k_0 l = 0.48\pi$; (*b*) $k_0 l = 0.8\pi$;(*a*) $k_0 l = 1.2\pi$.

$\downarrow k_0 l$	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
0.48π	-20.4	-44.6	-66.2	-86.5	-103.2	-121.1	-125.5	-126.0
0.8π	-15.9	-36.1	-51.1	-67.7	-81.8	-95.0	-109.2	-121.7
1.2π	-13.3	-28.6	-45.1	-58.9	-75.6	-88.5	-101.0	-111.9

Table I. Residual errors $\Re\{k\}$ (dB) for different numbers of iterations. Simulations in Fig. 6.

Figure 7 gives the BSW values computed at various iteration steps, and Table II provides the values of the residual errors.

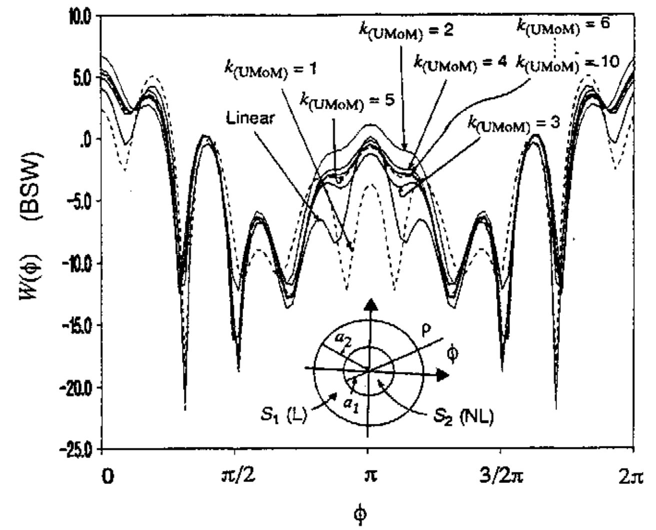

Fig. 7 Bistatic scattering width of a circular cylinder ($\varepsilon_1 = 3.0$) with a nonlinear nucleus ($\varepsilon_2 = 3.0$, $\xi = 0.2$, $k_0 a_1 = 1.27\pi$; $k_0 a_2 = 1.63\pi$, $P = 81$, $P_2 = 49$). Iterative UMoM.

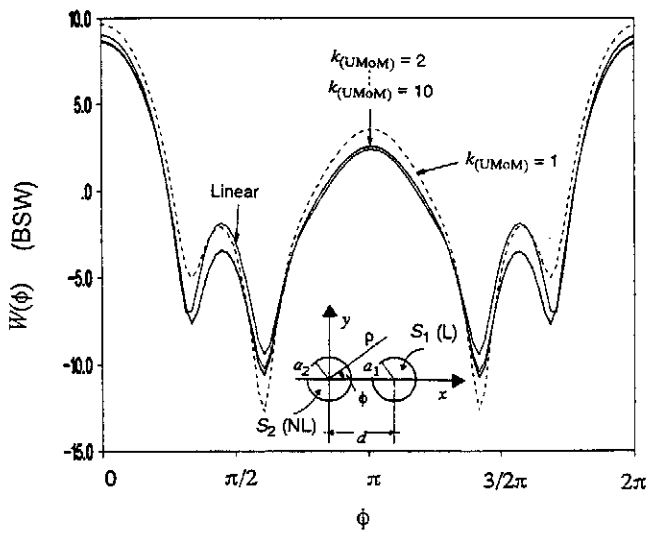

Fig. 8 Bistatic scattering width of two separate circular cylinders ($\varepsilon_1 = \varepsilon_2 = 4.0$, $\xi = 0.2$, $k_0 a_1 = k_0 a_2 = 0.82\pi$; $P = 162$, $P_2 = 81$). Iterative UMoM.

k	1	2	3	4	5	10	15	20
-	-11.1	-19.7	-25.3	-33.0	-39.1	-72.9	-107.1	-119.0

Table II. Residual errors $\Re\{k\}$ (dB) for different numbers of iterations. Simulations in Fig. 7.

Finally, the plane-wave scattering by two equal, separate, homogeneous, circular cylinders was considered ($\varepsilon_1 = \varepsilon_2 = 4.0$). The cylinders' radii were such that $k_o a_1 = k_o a_2 = 0.82\pi$, and the distance between the two centers was such that $k_o d = 1.2\pi$. As a perturbed configuration, one of the two cylinders was assumed to be nonlinear, with $\xi = 0.2$. Figure 8 gives the BSW values computed for this configuration at various steps. The linear values (numerically computed) are also given for comparison.

4 DISCUSSION AND CONCLUSIONS

In this paper, the perturbational UMoM has been applied to develop an iterative approach to the numerical computation of the electromagnetic scattering by dielectric cylinders of arbitrary shapes, coated with layers made of weakly nonlinear materials (of the Kerr type). Some test cases have been described, including multilayer circular cylinders coated or filled with nonlinear dielectrics, under TM illumination conditions. A comparison with data obtained by the iterative DWBA has been made.

The approach converged very fast as long as the nonlinearity was weak, corresponding to a slightly perturbed configuration. For example, in the example shown in Figure 7, the approach did not converge for $\xi = 2.0$. But this nonlinearity seems too high for the considered simplified electromagnetic model of the nonlinear process (neglecting the harmonics generation) to be realistic [14].

Future work will be devoted to applying the proposed iterative approach to perturbed geometries for which the nonlinearities, although weak, are such that the harmonics generation cannot be neglected. As shown in [14], each field component can then be expressed in integral form by coupling coefficients that take into account the harmonics mixing. From a perturbation point of view, the nonlinear field provided by the UMoM could be used to start an iterative process by which the higher-order harmonics (initially, the third-order harmonic, if a Kerr-like nonlinearity is assumed to be under monochromatic illumination) are first computed in terms of the field of the fundamental frequency, and the effects of the higher-order harmonics on the effective dielectric permittivity are then recursively evaluated. Convergence will of course remain an issue.

The approach has so far been applied only to dielectric infinite cylinders under TM illumination. For these configurations, the Richmond formulation is quite effective. Extensions to the TE-wave case and to the three-dimensional case are conceptually feasible, even though more accurate testing and weighting functions should be used for the MoM implementation.

The UMoM considered here is not restricted to dielectric configurations (the test case described in [2] actually concerns perfectly conducting scatterers). Therefore, the approach could in principle be applied to conductive objects coated with nonlinear materials. To this end, MoM solutions suitable for heterogeneous structures made of dielectric and conductive materials should be used.

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