AN ALTERNATIVE DESCRIPTION OF THE MAGNETIC CURRENT ANNULAR RING (FRILL) SOURCE

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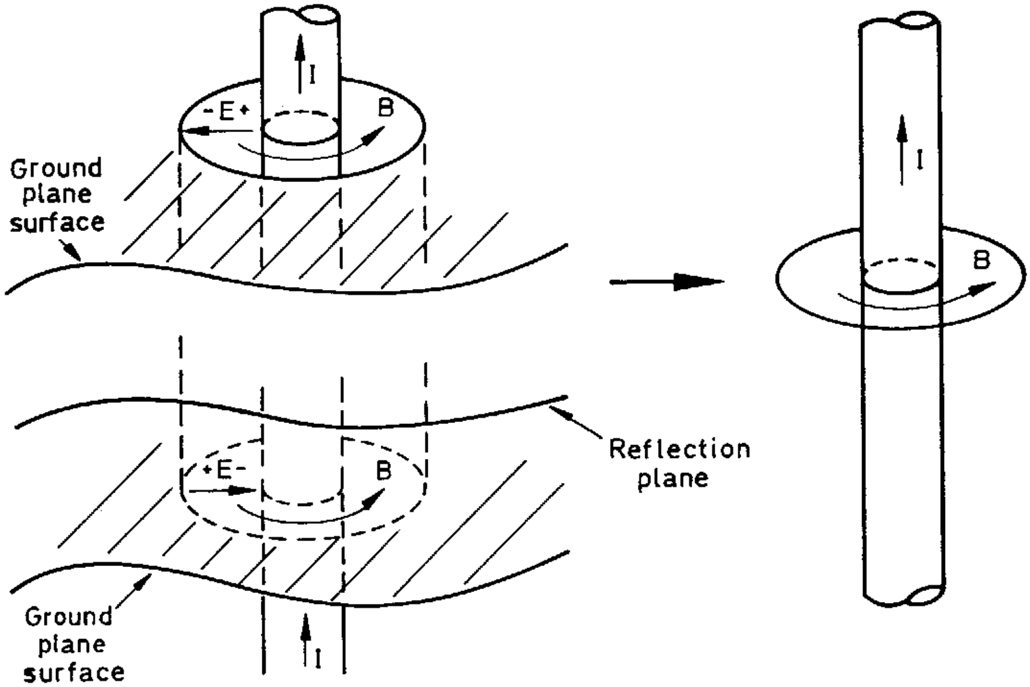
ABSTRACT

An alternative method is presented of considering, and of deriving expressions for the fields generated by, an annular ring of magnetic current (magnetic frill source). The magnetic frill appears to offer a means of numerical model excitation that is more realistic than the pulse source and, moreover, provides analytical expressions for fields in some cases. The fields for the magnetic frill source were first derived by Tsai from the electric vector potential produced by the magnetic current. The method described shows the source to resemble a toroidal transformer and the field expressions are derived from the magnetic vector potential produced by electric currents. The expressions derived are, in essence, the same as those of Tsai, but it is considered that the method yields greater physical insight into the source and so facilitates modification to suit particular applications. Tsai's expression which is of most interest for numerical calculations is derived by inspection using the method described. For calculations using point matching however, the benefits of using the frill source seem more apparent than real.

INTRODUCTION

In the literature the fields due to the magnetic current frill source are derived by calculating the electric vector potential via the magnetic current. For coaxial apertures at which a TEM wave is presented this means of calculation is mathematically convenient. This is a little unfortunate as the apparent convenience is perhaps gained at the expense of physical insight and understanding because of the 'unreal' nature of magnetic charges and currents. Possibly this accounts in part for the comparative rarity of use for the magnetic frill source, although, as indicated below, there appears to be a difficulty in its application in those moment method calculations which use point matching. However, as an assistance in understanding for the purpose of modification to specific requirements, the frill source can be shown to resemble a transformer and the fields derived from electric conduction currents.

The magnetic current frill source has been considered by several authors (Tsai[1]; Butler and Tsai[2]; Sakitani and Egashira[3]). This source appears to offer a more realistic means of driving a mathematical model of a balanced antenna than the usual pulse feed, or its limiting form, the delta gap generator. Modelling a balanced antenna is, by utilizing the method of images, equivalent to solving the problem of half the balanced antenna driven above a flat perfectly conducting ground plane of infinite extent: the conducting plane and the geometrical plane of current symmetry of the balanced antenna being coincident. Antennas mounted upon ground planes are very often driven via coaxial apertures in the ground plane where the coaxial cable feeding the aperture supports TEM propagation. Hence, the problem of a coaxially driven ground plane mounted antenna can be solved by considering two such systems (*ie* one antenna and ground plane plus the image of this combination), discarding the ground planes (so solving the equivalent balanced antenna problem) but retaining the features of coaxial driving. This is indicated diagrammatically:

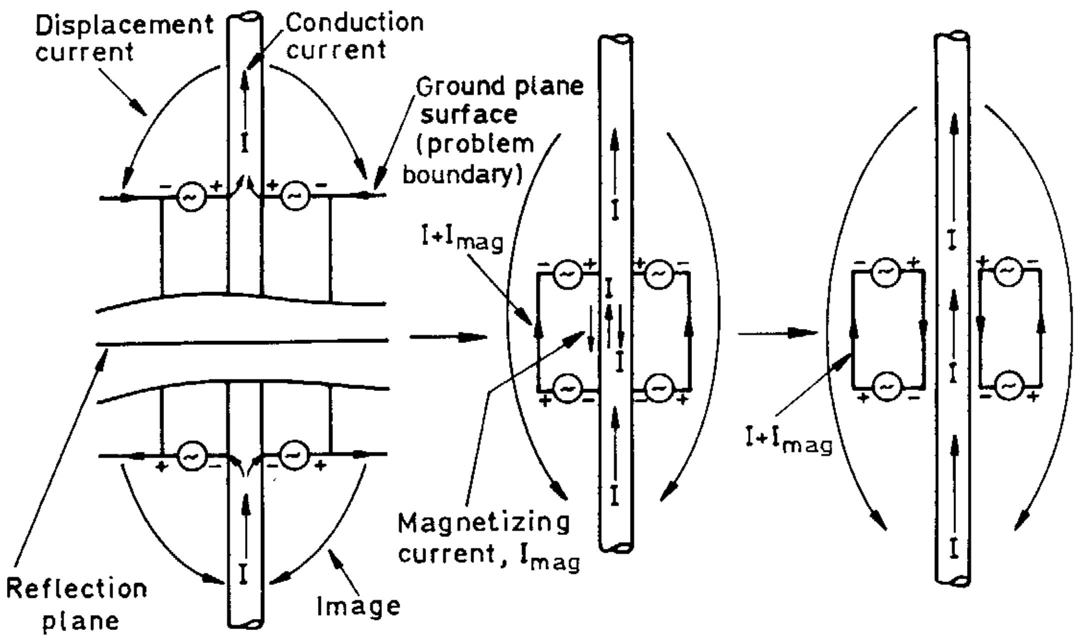


This is the conventional view of the frill but in formulation this is still essentially an aperture problem.

In aperture problems, sections of the boundary are frequently located (quite arbitrarily and for reasons of convenience) in the plane of the aperture. This requires the use of fields as sources rather than the real sources of electric conduction current and electric charge which are, of course, still the actual driving sources, but because of the arbitrary position of the boundary are exterior to the problem being solved. The particular field configuration presented at the aperture can be made to be any mode which is capable of being propagated along the cable or waveguide connected to the generator side of the aperture. The TEM mode is the simplest to treat for the coaxial cable. The aperture field (*ie* the field at the boundary) is thus assumed known: inevitable distortions caused by the discontinuity at the aperture are neglected. The arbitrary positioning of the problem boundary cutting through a particular field configuration necessitates the contrivance of fictitious charge distributions at the boundary. Such distributions would be required if there were no actual driving sources exterior to the problem. For the electric field this represents no conceptual difficulty, but for the magnetic field it requires the invention of magnetic charges (the time variation of the density of which produces magnetic current). This fictitious magnetic current can be considered as producing an electric vector potential exactly analogously to electric conduction current producing the more familiar magnetic vector potential. In this way the electric and magnetic fields existing within the problem boundary can be determined by considering the driving sources at the boundary as magnetic charges and magnetic currents (if this is convenient) rather than the more familiar electric charges and electric currents. In the case of the (electrically small) frill the convenience of doing so seems clear as there is a closed ring of magnetic current (alternating magnetic flux) at the coaxial aperture, and therefore no magnetic charge. If, on the other hand, the aperture fields were described by electric sources, these would, for this configuration, be a radial electric conduction current sheet from the inner conductor to the outer, starting and terminating in electric line charge distributions of opposite sign at the inner and outer conductors forming the aperture.

THE MAGNETIC FRILL SOURCE AS A TRANSFORMER

The apparent advantage of considering the magnetic current as the source vanishes when the generator is moved from the far end of the coaxial cable to the aperture, the ground plane mounted antenna and the image of this combination brought together, and the ground plane discarded.



This arrangement now looks like a balanced antenna driven by a 1-turn toroidal transformer. There is still an annular ring of alternating magnetic flux (*ie* magnetic current), but the origin of this is easier to appreciate. It arises from a sheet of electric current flowing as the transformer primary. However, recalling the elementary theory of transformers, it is necessary to note that the total current flowing in this sheet comprises two parts:

(i) That part which produces a magneto-motive force (mmf) which every where neutralizes exactly the mmf produced by the secondary current.

(ii) The magnetizing current: that current which alone produces the mutual flux linking the primary and secondary circuits. It thereby produces the back emf in the primary and the driving emf in the secondary.

Thus to calculate the driving fields, it is necessary to postulate a particular sheet (magnetizing) current in the primary and calculate the resultant magnetic vector potential. Clearly, as the sheet is continuous and electrically small, there are no free charges to contribute to the electric scalar potential.

That this approach does produce the results given by Tsai[1] for the magnetic frill (Tsai used the magnetic current as the source) is indicated by a few examples in the Appendices.

The expressions for field components in the region that Tsai terms the 'far near zone' are derived in Appendix A. The 'far near zone' is defined to be that region where the radial distance from the source centre is much greater than the radius of the annular ring yet much smaller than the radiation wavelength. Equations (A-1), (A-2) and (A-3) (Appendix A) are to be compared with expressions 19, 20 and 21 respectively in Tsai's paper (Ref 1). It will be seen that there are minor discrepancies in the numerical coefficients of the small correction terms. It is considered by the writer that these are due to the different effects of the series approximations made in the two approaches. Tsai does not give details and refers only to making 'suitable series approximations' in the calculation of the electric vector potential integrand. It is nevertheless clear that the two sets of results are, for all practical purposes, identical.

THE MAGNETIC FRILL AS AN EXCITATION SOURCE IN POINT-MATCHING CALCULATIONS

Tsai[1] gives a numerical procedure for calculating the near zone fields. This is based upon the numerical differentiation of the electric vector potential which is itself found by numerical integration of the contribution of the incremental magnetic current sources. However, according to Tsai, this procedure explicitly excludes calculation of the electric vector potential on the frill surface. This may present a problem for some wire formulations where, to avoid the Green's function singularity occurring in the electric field integral equation, the wire current is considered to flow on the axis whilst the match points are located on the wire surface (and hence on the frill inner surface) if the axial position of the frill coincides with a match point.

As a practical means of calculating the driving electric field in the source region, perhaps one of the more useful of Tsai's expressions is that for the electric field along the wire axis (*ie* the z-axis). Using the description of the frill as a transformer, this expression is derived very simply: essentially by inspection. As indicated in Appendix B, the result (equation (B-1)) is identical to that of Tsai (Ref 1, equation 25) and produced without approximation.

An apparent difficulty with the frill when used in moment method calculations, where delta function weighting is employed, is that the expression for the axial electric field (equation (B-1)) represents a function which, for practical coaxial aperture proportions, reaches a maximum value at the frill centre and has a width of the order of the radius of the inner conductor, a . For coaxial cables, a typical value for a is $\approx 10^{-3}$ m and the radius of the outer conductor, b , is around 5-10 times this figure. Hence for frequencies below 3 GHz, the width of the function is at least 100 times smaller than the wavelength and so at least 10 times smaller than a typical distance between match points (wire segment length) for acceptable sampling densities (*ie* $\simeq \lambda/10$).

Under such circumstances, the decision as to where on the z-axis of the conductor to locate the centre of the frill is not evident: the value of the axial electric field at the driving point is dependent upon the frill axial position. Clearly then, the location of the frill with respect to the driving match point will significantly affect the solution. It is not immediately obvious how this problem can be surmounted other than by reducing wire segment lengths in the region of the drive point to the order of, and preferably rather less than, a . However, in many problems of practical interest this would require impossibly short segment lengths. To avoid this difficulty using some form of electric field averaging in the vicinity of the drive point would tend to remove any advantages of realism claimed for the frill. One means of axial electric field averaging, for example, is to perform a line integration of equation (B-1) along the z-axis and divide the result by the distance between the integration limits. Hence, taking practical values of a and b to be much smaller than the wavelength, *ie*

$$ka\ ,\quad kb \ll 1\ ,$$

arbitrarily putting $z' = 0$, and noting that the value of E_z is only significantly different from zero when $z \approx a$, allows the exponential terms in equation (B-1) to be approximated by unity. The integral between the limits $z = -d$ and $z = d$ then becomes:

$$\int_{-d}^{d} E_z d_z = \frac{i\omega\Phi}{2\ \ln(b/a)} \int_{-d}^{d} \left\{ \frac{1}{\sqrt{z^2 + b^2}} - \frac{1}{\sqrt{z^2 + a^2}} \right\} dz$$

$$= \frac{i\omega\Phi}{2\ \ln(b/a)} \left\{ \ln\left(\frac{d + \sqrt{d^2 + b^2}}{-d + \sqrt{d^2 + b^2}} \right) - \ln\left(\frac{d + \sqrt{d^2 + a^2}}{-d + \sqrt{d^2 + a^2}} \right) \right\}$$

(Ref 6, entry 200.01).

For the case where

$$d \gg a\ ,\ b$$

the above expression (using the binomial expansion) reduces to:

$$\int_{-d}^{d} E_z d_z \simeq \frac{-\ i\omega\Phi}{2\ \ln(b/a)} \ln\left(\frac{1 + 4(d^2/a^2)}{1 + 4(d^2/b^2)} \right)\ ,$$

$$= -\ i\omega\Phi\ ,$$

for d sufficiently large. This is, of course, expected when considering the frill as a transformer: the driving voltage induced in the secondary circuit is the rate of change of flux linking it. Although trivial, this result could be used to give an indication of the size of d necessary to ensure, in effect, that sufficient of the prescribed magnetizing flux is included in the calculation of the voltage driving the secondary circuit. Hence, performing the integration over a sufficiently large axial distance and dividing by 2d gives a mean driving axial electric field:

$$\bar{E}_z \simeq \frac{i\omega\Phi}{2d}\ .$$

Not surprisingly, all configurational details of the frill are lost in this process. If the axial electric field of the frill is averaged in this way around the drive point then an equivalence of the frill and pulse sources is therefore indicated.

A minor problem with equation (B-1) which will be manifest in some moment method formulations concerns the fact that, strictly, this equation represents the z-component of the electric field on the frill (and hence wire) axis. As remarked above, match points are frequently located on the wire surface whilst the wire current is considered to flow on the axis. Hence for simple implementation, it is either necessary to assume that the electric field has the same value at the axis as at the surface, or alter the formulation locally and arrange for the (driving) match point to lie on the wire axis rather than the surface.

Generally it seems reasonable to suppose that in order to calculate accurately quantities which depend upon the near field it is necessary to enforce the boundary conditions of the problem as precisely as possible. That this is the case is supported by the experimental studies of Brown and Woodward[4] who note that 'the exact conditions at the (antenna) terminals are extremely important in determining the impedance conditions'. The theoretical work of Albert and Synge[5] adds further credance by showing the importance of the geometry of the feed region in the determination of antenna reactance. Clearly numerical calculations must reflect this requirement and in practice increasing sampling density is usually necessary to achieve it. Thus, the procedure to make the commonly used and easily implemented pulse feed a more realistic and acceptable form of driving appears no different to that for the frill (as indicated above). It is, therefore, not easy to see any immediate advantage for the frill in point matching calculations.

CONCLUSIONS

Considering the magnetic current annular ring as a one turn toroidal transformer makes this excitation source easier to understand and, thereby, to modify to particular requirements. However, for point matching calculations at least, the source is not free from difficulties which offset the advantage of its realism.

REFERENCES

1 IEEE trans A&P, Vol AP-20 No.5, September 1972, pp 569-576. 'A numerical solution for the near and far fields of an annular ring of magnetic current'. Tsai, L.L. (See also Computer Techniques for Electromagnetics. Ed R.M. Mittra, Pergamon Press (1974), pp 72-77.)

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4 Proc IRE, April 1945, pp 257-262. 'Experimentally determined impedance characteristics of cylindrical antennas'. Brown, G.H. and Woodward, O.M.

5 Quarterly Applied Mathematics, Vol 6, 1948, pp 117-156. 'The general problem of antenna radiation and the fundamental integral equation with application to an antenna of revolution', Parts I and II. Albert, G.E. and Synge, J.L.

6 'Tables of Integrals and other mathematical data'. Dwight, H.B., Macmillan, New York. 1947.

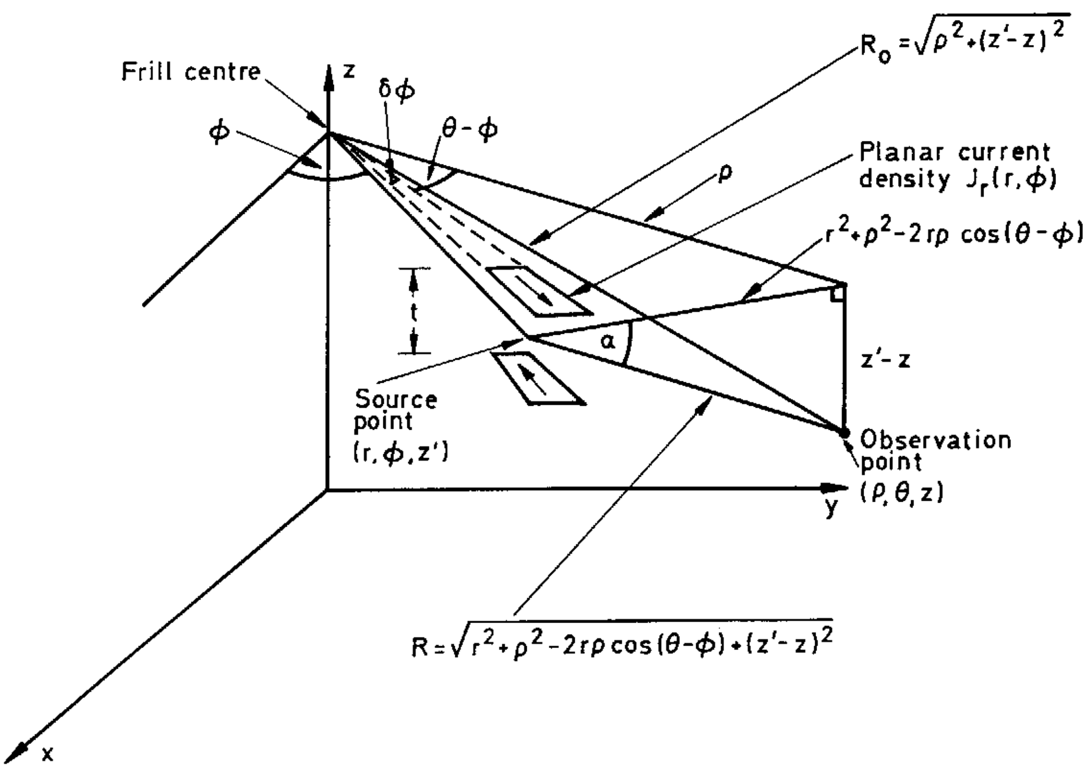
Appendix A

FIELD EXPRESSIONS FOR THE 'FAR NEAR ZONE'

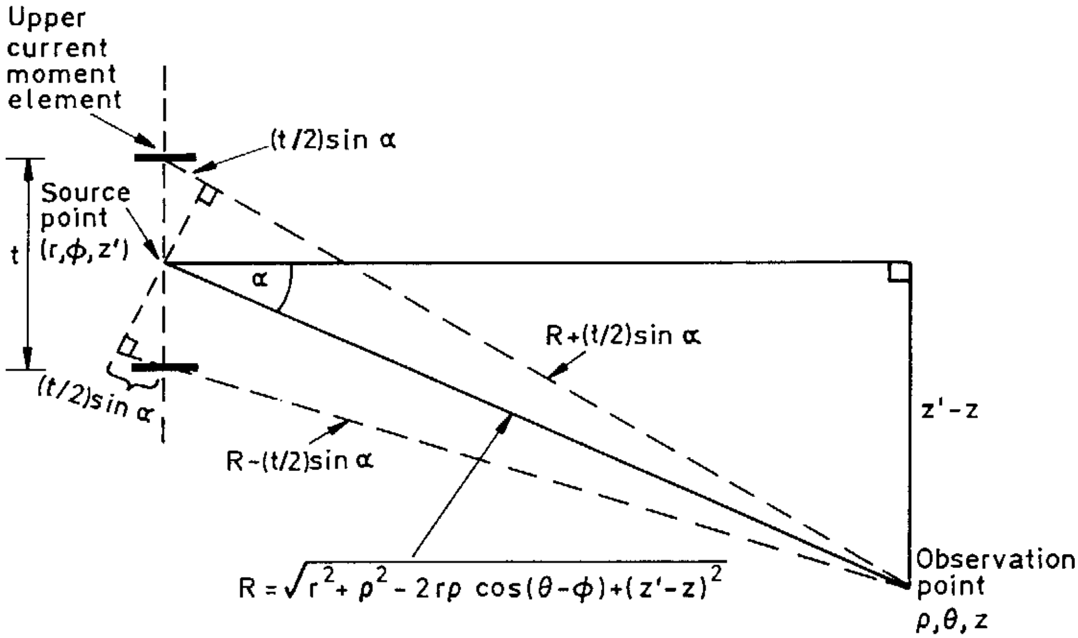
In this Appendix Tsai's expressions for the 'far near zone' are derived using the description of the frill as a one-turn toroidal transformer.

(i) The radial (ρ) component of the electric field in the 'far near zone'

Consider an elemental source consisting of two oppositely sensed current moments a distance t (the axial thickness of the frill) apart as indicated.



By the nature of the current sheets producing the frill, the direction of these currents are radially inwards and outwards respectively with respect to the frill centre. At the observation point, the contributions to the magnetic vector potential due to this elemental source are δA_ρ and δA_θ in the radial and tangential directions respectively. With reference to the following figure:



$$\sin \alpha = \frac{z' - z}{R} .$$

$$\delta A_\rho = \frac{\mu_0}{4\pi} J_r \delta\varphi \delta_r \cos(\theta - \varphi) \left\{ \frac{e^{-ik(R + t/2 \sin \alpha)}}{(R + t/2 \sin \alpha)} - \frac{e^{-ik(R - t/2 \sin \alpha)}}{(R - t/2 \sin \alpha)} \right\}$$

and

$$\delta A_\theta = \frac{\mu_0}{4\pi} J_r \delta\varphi \delta_r \sin(\theta - \varphi) \left\{ \frac{e^{-ik(R + t/2 \sin \alpha)}}{(R + t/2 \sin \alpha)} - \frac{e^{-ik(R - t/2 \sin \alpha)}}{(R - t/2 \sin \alpha)} \right\}$$

The total inward and outward radial currents are both I, so that

$$J_r = \frac{I}{2\pi r} ,$$

and the total radial and tangential magnetic vector potentials due to the equivalent frill currents in planes parallel to the x, y plane are therefore

$$A_\rho = \frac{\mu_0 I}{8\pi^2} \int_{\phi=-\pi}^{\phi=\pi} \int_{r=a}^{b} \cos(\theta - \varphi)\, e^{-ikR} \left(\frac{e^{-i(kt/2)\sin\alpha}}{(R + t/2 \sin\alpha)} - \frac{e^{+i(kt/2)\sin\alpha}}{(R - t/2 \sin\alpha)} \right) d\varphi dr$$

and

$$A_\theta = \frac{\mu_0 I}{8\pi^2} \int_{\phi=-\pi}^{\phi=\pi} \int_{r=a}^{b} \sin(\theta - \varphi)\, e^{-ikR} \left(\frac{e^{-i(kt/2)\sin\alpha}}{(R + t/2 \sin\alpha)} - \frac{e^{+ikt/2\sin\alpha}}{(R - t/2 \sin\alpha)} \right) d\varphi dr \,.$$

It is clear that the integrand in the expression for A_θ is an odd function of φ so that, as expected from physical considerations, integrating with respect to φ between the limits $-\pi$ to $+\pi$ makes

$$A_\theta = 0 \,.$$

It can be predicted with a fair degree of confidence that the integral for A_ρ cannot, in all cases, be evaluated in closed form. However for the near field, *ie*

where

$$a \leq r \leq b$$

and

$$b,\ \rho \ll \lambda$$

and when $\rho \gg b$, (that is, what Tsai[1] refers to as 'the far near zone'): the expression for R, *ie*

$$R^2 = r^2 + \rho^2 - 2r\rho\cos(\theta - \varphi) + (z' - z)^2 \,,$$

may be expanded:

$$R = R_0(1 + q)^{1/2} \simeq R_0\left(1 + \frac{q}{2} - \frac{q^2}{8}\right)$$

$$R^{-3} \simeq R_0^{-3} \left(1 - \frac{3q}{2} + \frac{15q^2}{8}\right) \,,$$

where R_0, the distance of the observation point from the frill centre, is clearly

$$R_0^2 = (z' - z)^2 + \rho^2$$

and

$$q = \frac{r^2 - 2r\rho\cos(\theta - \varphi)}{R_0^2} .$$

For axially thin frills ($t \to 0$) the integrand in the expression for A_ρ may be expanded:

$$A_\rho \simeq \frac{\mu_0 I}{8\pi^2} \int_{\varphi=-\pi}^{\pi} \int_{r=a}^{b} \frac{\cos(\theta - \varphi)\, e^{-ikR}}{R} \left(\left(1 - \frac{ikt\sin\alpha}{2}\right)\left(1 - \frac{t\sin\alpha}{2R}\right) - \left(1 + \frac{ikt\sin\alpha}{2}\right)\left(1 + \frac{t\sin\alpha}{2R}\right)\right) d\varphi dr,$$

ie

$$A_\rho \simeq \frac{\mu_0 I}{8\pi^2} \int_{\varphi=-\pi}^{\pi} \int_{r=a}^{b} \frac{\cos(\theta - \varphi)\, e^{-ikR}}{R} \left(-\frac{t\sin\alpha}{R} - ikt\sin\alpha \right) d\varphi dr$$

$$= -\frac{\mu_0 I t (z' - z)}{8\pi^2} \int_{\varphi=-\pi}^{\pi} \int_{r=a}^{b} \frac{\cos(\theta - \varphi)\, e^{-ikR}(1 + ikR)}{R^3}\, d\varphi dr .$$

Using the above approximation for R and expanding:

$$A_\rho \simeq -\frac{\mu_0 It(z'-z)e^{-ikR_0}}{8\pi^2 R_0^3}\int_{\varphi=-\pi}^{\pi}\int_{r=a}^{b}\cos(\theta-\varphi)\left(1-\frac{ikR_0q}{2}+i\frac{kR_0q^2}{8}\right)\left(1-\frac{3q}{2}+\frac{15q^2}{8}\right)$$
$$\left(1+ikR_0+\frac{ikR_0q}{2}-\frac{ikR_0q^2}{8}\right)d\varphi dr\ .$$

As this expression for A_ρ is an expression in R_0, in this approximation terms beyond R_0^{-6} will be neglected; that is, terms beyond R_0^{-3} in the integrand. Also, odd powers of $\cos(\theta-\varphi)$ can be ignored as they integrate to zero between the limits $-\pi$ to π. Hence

$$A_\rho \simeq \frac{\mu_0 It(z'-z)e^{-ikR_0}}{8\pi^2 R_0^3}\ \frac{\pi\rho(b^2-a^2)}{2}\ k\left[k-\frac{1}{R_0^2}\left(\frac{3}{k}+\frac{(b^2+a^2)3k}{2}\right)\right.$$
$$\left.-i\left(\frac{3}{R_0}-(b^2+a^2)\left[\frac{15}{4R_0^3}-\frac{k^2}{4R_0}\right]\right)\right].$$

But, the magnetic flux Φ is given by simple coaxial transmission line theory as

$$\Phi = \frac{\mu_0 It}{2\pi}\ln\frac{b}{a}$$

so that substituting for $It\mu_0$,

$$A_\rho \simeq \frac{\rho(z'-z)(b^2-a^2)e^{-ikR_0}(\Phi\omega)}{8\ln(b/a)R_0^3\,c}\left[k-\frac{1}{R_0^2}\left(\frac{3}{k}+\frac{(b^2+a^2)3k}{2}\right)\right.$$
$$\left.-i\left(\frac{3}{R_0}-(b^2+a^2)\left[\frac{15}{4R_0^3}-\frac{k^2}{4R_0}\right]\right)\right],$$

where c is the velocity of light in free space.

But as there are no electric charges (the current sheets are electrically small and continuous)

$$E_\rho = -i\omega A_\rho$$

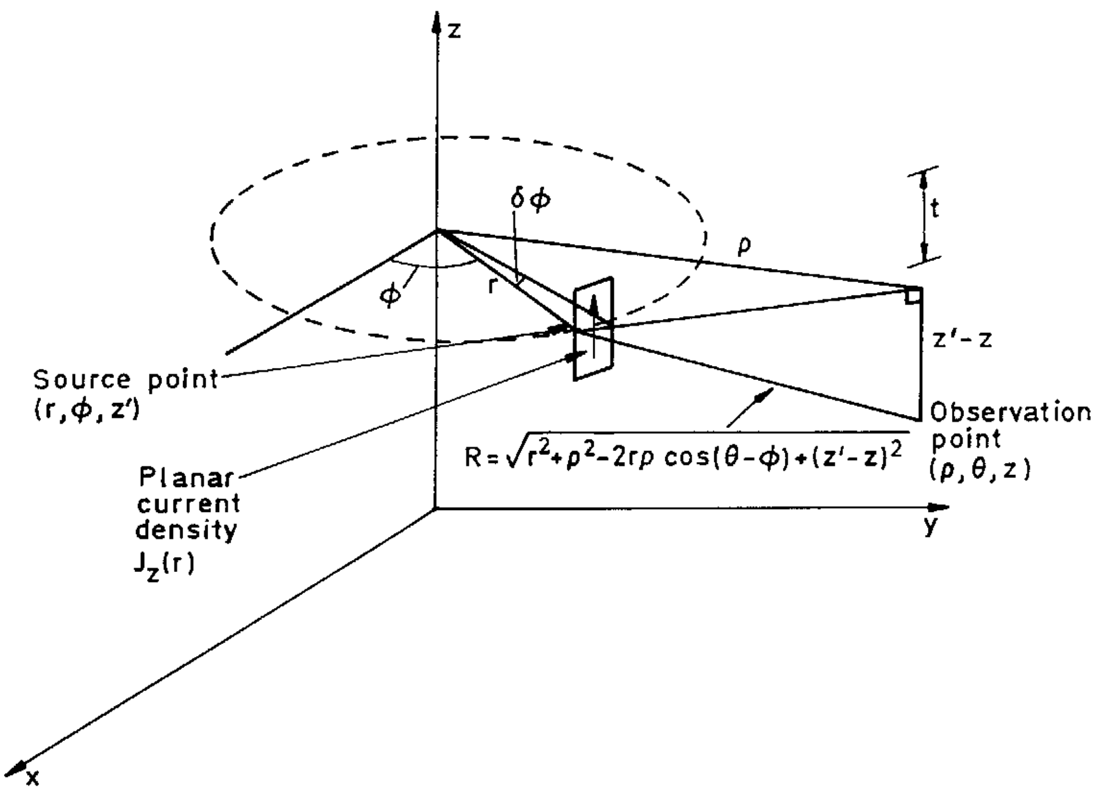
or

$$\frac{E_\rho \Sigma}{k} = -icA_\rho = -\frac{(i\omega\Phi)(z'-z)\,\rho(b^2-a^2)e^{-ikR_0}}{8\ln(b/a)R_0^3}\left[k - \frac{1}{R_0^2}\left(\frac{3}{k} + \frac{(b^2+a^2)3k}{2}\right) + i\left(-\frac{3}{R_0} + (b^2+a^2)\left[\frac{15}{4R_0^3} - \frac{k^2}{4R_0}\right]\right)\right]. \quad \text{...... (A-1)}$$

The term $i\omega\Phi$ is recognised as the magnetic current, usually made equal to 1 volt for convenience.

(ii) The axial (z) component of electric field in the 'far near zone'

The z component of the magnetic vector potential off the z-axis is evaluated with reference to the following figure for a circle of axially directed current at radius r .



The planar current density (as taken to be uniform) J_z is given by

$$J_z = \frac{I}{2\pi r},$$

and hence the z component of the magnetic vector potential is given by

$$A_z(r) = \frac{\mu_0}{4\pi} \int_{\varphi=-\pi}^{\pi} \frac{J_z e^{-ikR} t r \delta\varphi}{R} .$$

Where the axial thickness of the source, t is assumed very small, *ie*

$$t << R .$$

Making previous approximations for R and using the same notation,

$$A_z = -\frac{\mu_0 I t e^{-ikR_0}}{8\pi^2 R_0^3} \int_{-\pi}^{\pi} \left(1 - \frac{q}{2} + \frac{3q^2}{8} - \frac{5q^3}{16}\right) \left(1 - \frac{ikR_0 q}{2} + \frac{ikR_0 q^2}{8} - \frac{k^2 R_0^2 q^2}{8}\right) d\varphi .$$

Integrating, re-arranging, and taking account of the sign convention for the currents indicated above, this becomes

$$A_z = -\frac{\pi\mu_0 I t e^{-ikR_0} k(b^2 - a^2)}{8\pi^2 R_0^2} \left[- i - \frac{1}{kR_0} + \frac{(b^2 + a^2)(3 + 3ikR_0 - k^2 R_0^2)}{4kR_0^3} + \frac{\rho^2}{kR_0^3} \left(\frac{3}{2} + \frac{3ikR_0}{2} - \frac{k^2 R_0^2}{2} + \frac{6k^2(b^2 + a^2)}{8}\right)\right] .$$

Substituting for $It\mu_0$ as before and noting that

$$E_z = - i\omega A_z ,$$

$$\frac{E_z}{k} = -\frac{(i\omega\Phi)e^{-ikR_0}(b^2-a^2)}{8\ln(b/a)\,R_0^2}\left[2\left[\frac{1}{kR_0} + i - \frac{(b^2+a^2)(3+3ikR_0-k^2R_0^2)}{4kR_0^3}\right] + \frac{\rho^2}{R_0}\left(-\frac{3}{kR_0^2} - \frac{3i}{R_0} + k - \frac{3k(b^2+a^2)}{2R_0^2}\right)\right].$$

...... (A-2)

(iii) The tangential (ϕ) component of the magnetic field in the 'far near zone'

For currents of the orientation indicated, the φ-component of flux density B_φ is the only component which is non-zero in this (cylindrical) coordinate system. By Maxwell's third equation:

$$\mathrm{curl}E = -\frac{\partial B}{\partial t},$$

so that

$$B_\varphi = -\frac{1}{i\omega}\left(\frac{dE_\rho}{dz} - \frac{dE_z}{d\rho}\right) = -\frac{1}{i\omega}\left(\frac{\partial E_\rho}{\partial z} + \frac{\partial E_\rho}{\partial R_0}\frac{dR_0}{dz} - \frac{\partial E_z}{\partial \rho} - \frac{\partial E_z}{\partial R_0}\frac{dR_0}{d\rho}\right).$$

Hence, after a lengthy process of differentiation and re-arranging, it can be shown that, in terms of the magnetic field H_φ,

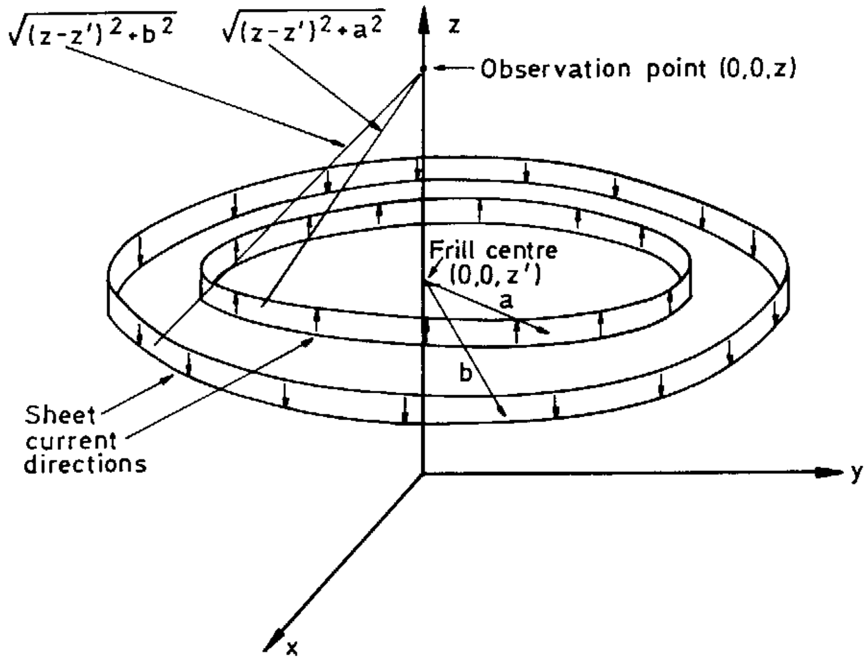
$$\frac{B_\varphi}{\mu_0 k} = \frac{H_\varphi}{k} = -\frac{i(i\omega\Phi)k(b^2-a^2)\rho\, e^{-ikR_0}}{120\pi\, 8\ln(b/a)R_0^2}\left[\frac{1}{kR_0} + i - \frac{i(b^2+a^2)}{R_0^2}\right].$$

...... (A-3)

Appendix B

ELECTRIC FIELD ALONG THE WIRE AXIS

The electric field along the z-axis is found very simply and without approximation when considering the frill as a one-turn toroidal transformer:



$$A_z(0,0,z) = \frac{\mu_0}{4\pi}\left\{\frac{It\, e^{-ik\sqrt{(z'-z)^2+a^2}}}{\sqrt{(z'-z)^2+a^2}} - \frac{It\, e^{-ik\sqrt{(z'-z)^2+b^2}}}{\sqrt{(z'-z)^2+b^2}}\right\}$$

$$= \frac{\Phi}{2\ln(b/a)}\left\{\frac{e^{-ik\sqrt{(z'-z)^2+a^2}}}{\sqrt{(z'-z)^2+a^2}} - \frac{e^{-ik\sqrt{(z'-z)^2+b^2}}}{\sqrt{(z'-z)^2+b^2}}\right\},$$

(substituting for $\mu_0 It$ as before),

so that

$$E_z(0,0,z) = -i\omega A_z(0,0,z) = \frac{(i\omega\Phi)}{2\ln(b/a)}\left\{\frac{e^{-ik\sqrt{(z'-z)^2+b^2}}}{\sqrt{(z'-z)^2+b^2}} - \frac{e^{-ik\sqrt{(z'-z)^2+a^2}}}{\sqrt{(z'-z)^2+a^2}}\right\}.$$

...... (B-1)