The Joy of Computing with Volume Integrals: Foundations for Nondestructive Evaluation of Planar Layered Media

Harold A. Sabbagh[1], R. Kim Murphy[1], Elias H. Sabbagh[1], John C. Aldrin[2], Jeremy S. Knopp[3], and Mark P. Blodgett[3]

[1] Victor Technologies LLC
Bloomington, IN 47401, USA
has@sabbagh.com, kimmurphy1@aristotle.net, ehs@sabbagh.com,

[2] Computational Tools
Gurnee, IL 60031, USA
aldrin@computationaltools.com,

[3] Air Force Research Laboratory (AFRL/RXLP)
Wright-Patterson AFB, OH 45433, USA
jeremy.knopp@wpafb.af.mil, mark.blodgett@wpafb.af.mil

Abstract **─** As an alternative to the finite-difference time-domain (FDTD), the finite-element method (FEM), and the method of moments (MoM) based on the surface integral equation (SIE), a volume-integral equation (VIE) approach using the method of moments and conjugate-gradient methods is presented to address a wide variety of complex problems in computational electromagnetics. A formulation of the volume integral method is presented to efficiently address inhomogeneous regions in multi-layered media. Since volume element discretization is limited to local inhomogeneous regions, numerical solutions for many complex problems can be achieved more efficiently than FDTD, FEM, and MoM/SIE. This is the first of a series of papers dealing with volume-integral equations; in subsequent papers of this series we will apply volume-integrals to problems in the field on nondestructive evaluation.

Index Terms **─** Aircraft structures, computational electromagnetics, electromagnetic nondestructive evaluation, volume-integral equations.

I. INTRODUCTION

The authors of a recent paper [1] claim that a surface integral equation formulation that utilizes method of moments (MoM) with higher-order basis functions may be the ‘best weapon’ for solving complex problems in electromagnetics. Following [2], only methods based upon finite-elements (FEM), finite-difference time-domain (FDTD), or MoM seem to have been considered as candidate techniques. Furthermore, certain conclusions are drawn in [2] that suggest that MoM is suited only for problems with homogeneous materials, or that finite methods are better suited to handle arbitrary bodies. By MoM, it is clear that both papers’ authors have in mind surface-integral equations. In this paper, we apply computational electromagnetics in the arena of quantitative nondestructive evaluation (NDE), and show errors with the conclusions in [1] and [2] when volume-integral equations are considered.

Even though earlier work in formulating scattering problems by integral equations [3] existed, the emergence of integral equations into the arena of contemporary computational electromagnetic was through the notion of ‘moment methods’ in 1968 [4]. Since that time, volume-integral equations (in which the unknowns are currents distributed throughout a finite volume of space) have been applied to scattering problems in free-space [5, 6], and even to biomedical studies [7]. One reason for the success of volume-integral equations in solving these scattering problems is that the very large linear systems that are obtained by discretization of these equations can be solved with very modest computer resources using conjugate-gradient search methods, coupled with fast-Fourier transforms (FFT).

In applying volume-integral equations to electromagnetic (eddy-current) nondestructive evaluation, we are once again faced with a scattering problem, but now the scatterer is generally buried within a conducting host, which may have a number of layers of different electrical properties. Furthermore, the host may be anisotropic and magnetic. Even for isotropic layered media, the Green's functions that appear in the kernels of the functionals (the integral operators) will be considerably more complex than those in free-space. Furthermore, there will be terms in the volume-integral equation that do not appear in free-space. These will be discussed briefly in Section II; but for details, the reader is invited to study [8] and [9]. By extending the ideas of previous researchers to include new kernels and functionals for layered media, we have developed VIC-3D© [10], a volume-integral code for eddy-current nondestructive evaluation (NDE), and have obtained very good results for a wide range of problems using it [8-22]. Validation of the model and code against benchmark experiments is described in [19-22].

II. BACKGROUND

A. Volume-integral equations for ferro-magnetic workpieces [8-9]

We start with Maxwell's equations:

$$\nabla\times\mathbf{E} = -j\omega\mathbf{B}$$
$$\nabla\times\mathbf{H} = -j\omega\mathbf{D} + \mathbf{J}^{(e)}. \quad (1)$$

Now, $\mathbf{H} = \mathbf{B}/\mu(\mathbf{r}) = \mathbf{B}/\mu_h + \mathbf{B}/\mu(\mathbf{r}) - \mathbf{B}/\mu_h = \mathbf{B}/\mu_h - \mathbf{M}_a$, where μ_h is the host permeability, and $\mathbf{M}_a$ is the anomalous magnetization vector. Thus, the second of Maxwell's equations may be written

$$\nabla\times\mathbf{B}/\mu_h = -j\omega\mathbf{D} + \mathbf{J}^{(e)} + \nabla\times\mathbf{M}_a, \quad (2)$$

which makes clear that the Amperian current, $\nabla\times\mathbf{M}_a$, is an equivalent anomalous electric current that arises because of the departures of the magnetic permeability of the workpiece from the host permeability, μ_h. $\mathbf{J}^{(e)}$, on the other hand, is an electric current that includes the anomalous current that arises due to differences in electrical conductivity; $\mathbf{J}^{(e)} = \sigma_h\mathbf{E} + (\sigma(\mathrm{r}) - \sigma_h)\mathbf{E} = \sigma_h\mathbf{E} + \mathbf{J}_a$.

Because the host conductivity and permeability are constant within each plane-parallel layer, they can be accounted for by means of Green functions. This leaves us with only the anomalous electric and magnetic sources to be determined. Even though the Amperian current is electrical (because it appears as a source term in the second Maxwell equation (Ampere's law)), we will refer to it as $\mathbf{J}^{(m)}$, to remind us that it is of magnetic origin, and to distinguish it from $\mathbf{J}^{(e)}$ (which now stands for the anomalous electric current, $\mathbf{J}_a$). The important point, however, is that because the Amperian current behaves as an electrical current, we need only use electric-electric Green functions in the formulation of the problem.

In establishing the volume-integral equations, we simply make use of the fact that the total electric field and magnetic flux density at a point is the sum of the fields due to the probe coil, which we call the incident fields, and those due to the anomalous currents, $\mathbf{J}^{(e)}$ and $\mathbf{J}^{(m)}$.

Hence, we write

$$\begin{aligned}
\mathbf{E}^{(i)}(\mathbf{r}) &= \frac{\mathbf{J}^{(e)}(\mathbf{r})}{\sigma_a(\mathbf{r})} - \mathbf{E}^{(0)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] \\
&\quad - \mathbf{E}^{(s)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] - \mathbf{E}^{(0)}(\mathbf{r})\left[\mathbf{J}^{(m)}\right] \\
&\quad - \mathbf{E}^{(s)}(\mathbf{r})\left[\mathbf{J}^{(m)}\right], \\
\mathbf{B}^{(i)}(\mathbf{r}) &= \frac{\mu(\mathbf{r})\mu_h}{\mu(\mathbf{r})-\mu_h}\mathbf{M}_a + \frac{1}{j\omega}\nabla\times\mathbf{E}^{(i)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] \\
&\quad + \frac{1}{j\omega}\nabla\times\mathbf{E}^{(s)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] \\
&\quad + \frac{1}{j\omega}\nabla\times\mathbf{E}^{(0)}(\mathbf{r})\left[\mathbf{J}^{(m)}\right] \\
&\quad + \frac{1}{j\omega}\nabla\times\mathbf{E}^{(s)}(\mathbf{r})\left[\mathbf{J}^{(m)}\right].
\end{aligned} \quad (3)$$

In arriving at the second equation, we have used the fact that $\mathbf{B} = -(1/j\omega)\nabla\times\mathbf{E}$, and $\mathbf{M}_a = ((\mu(\mathbf{r}) - \mu_h)/\mu(\mathbf{r})\mu_h)\mathbf{B}$.

The first part of the first equation in (3) is the electric-electric (ee) interaction, and the second part is the electric-magnetic (em) interaction. The two parts of the second equation are, respectively, the magnetic-electric (me) and the magnetic-magnetic (mm) interactions. We decompose the various interactions into the 'infinite-space' part, designated by the superscript, (0), and the 'layered-space' part, designated by the superscript, (s). This is done for convenience in coding and problem solving.

The electric-electric interaction terms are given by:

$$
\begin{aligned}
\mathbf{E}^{(0)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] &= \int \mathbf{\Phi}^{(e)}(\mathbf{r}-\mathbf{r}')\mathbf{J}^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + \frac{1}{k_0^2}\nabla \int \mathbf{\Phi}^{(e)}(\mathbf{r}-\mathbf{r}')\nabla' \cdot \mathbf{J}^{(e)}(\mathbf{r}')d\mathbf{r}' \\
\mathbf{E}^{(s)}(\mathbf{r})\left[\mathbf{J}^{(e)}\right] &= \int G_{xx}^{(1)(a)}(\mathbf{r}-\mathbf{r}')\mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + a_z \int G_{xz}^{(a)}(\mathbf{r}-\mathbf{r}')\mathbf{J}_z^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + \int G_{xx}^{(1)(b)}(\mathbf{r}-\mathbf{r}')\mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + a_z \int G_{xz}^{(b)}(\mathbf{r}-\mathbf{r}')\mathbf{J}_z^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + \nabla_t \int G_{xx}^{(2)(a)'}(\mathbf{r}-\mathbf{r}')\nabla_t' \cdot \mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + \nabla_t \int G_{xx}^{(2)(b)'}(\mathbf{r}-\mathbf{r}')\nabla_t' \cdot \mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad + \frac{1}{k_0^2}\nabla \int G_{xz}^{(a)}(\mathbf{r}-\mathbf{r}')\nabla_t' \cdot \mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&\quad - \frac{1}{k_0^2}\nabla \int G_{xz}^{(b)}(\mathbf{r}-\mathbf{r}')\nabla_t' \cdot \mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}',
\end{aligned}
\tag{4}
$$

where $\nabla_t = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y}$; $\nabla_t \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y}$; $\mathbf{J}_t = a_x J_x + a_y J_y$, and

$$
\begin{aligned}
\mathbf{\Phi}^{(e)}(\mathbf{r}-\mathbf{r}') &= -j\omega\mu_0 \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{-j\omega\mu_0}{2\pi}\int_0^\infty \frac{e^{-\lambda_0|z-z'|}}{2\lambda_0} J_0(rl)l dl.
\end{aligned}
\tag{5}
$$

In the Bessel transform, $r = [(x-x')^2 + (y-y')^2]^{1/2}$. Integral expressions for the various layered-space Green functions are given in [8]. Note that the terms with the superscript (a) are convolutional (Töplitz) in all three spatial variables; whereas, those with (b) are convolutional in $(x-x')$ and $(y-y')$, but are correlational (Hankel) in $(z-z')$, where z is normal to the layers of the workpiece (see Figure 1). This decomposition is of great importance when solving large systems of equations, as we shall see later.

The Töplitz structure, as shown in the top of Fig. 1, arises when the path between the source point, z', and field point, z, includes reflections from both boundaries. The total z-directed path length between z' and z is $z - z' + 2T$ for path A, and $z' - z + 2T$ for path B. In each case, the length includes the difference between the z-coordinate of the source and field points. The Hankel structure, as shown in the bottom of Figure 1, arises when the path between source and field points includes reflections from only one of the boundaries. The total path length between z' and z is $2Z_0 - (z + z')$ for path A, and $z + z' - 2Z_{-1}$ for path B. In each case, the length includes the sum of the source and field z-coordinates.

The electric-magnetic interaction terms are given by substituting $\mathbf{J}^{(m)}$ for $\mathbf{J}^{(e)}$ in (4), and making use of the fact that $\mathbf{J}^{(m)}$ has zero divergence. We will not derive them here. Likewise, the magnetic-electric operators follow directly by taking the curl of (4) [8].

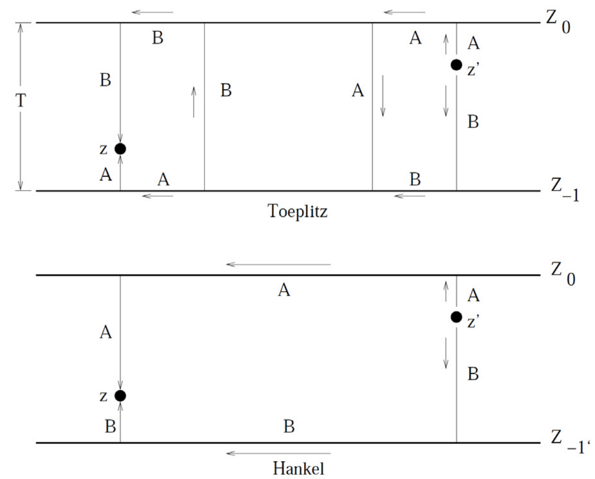


Fig. 1. Illustrating the difference between Töplitz (top) and Hankel (bottom) Green's functions.

B. Discretization via the method of moments (Galerkin)

Define a regular grid in three-dimensional space, with grid spacing δx, δy, δz. Relative to this grid we define $\pi(x)$ to be the unit pulse

$$
\pi(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}
\tag{6}
$$

and $\pi_{m+1}(x)$ to be the mth-order convolution of $\pi(x)$ (we define $\pi_1(x) = \pi(x)$).

Next, expand the electric current vector as:

$$
\begin{aligned}
J_x^{(e)}(\mathbf{r}) &= \sum_{KLM} J_{KLM}^{(x)} T_{KLM}^{(x)(e)}(\mathbf{r}) \\
J_y^{(e)}(\mathbf{r}) &= \sum_{KLM} J_{KLM}^{(y)} T_{KLM}^{(z)(e)}(\mathbf{r}) \\
J_z^{(e)}(\mathbf{r}) &= \sum_{KLM} J_{KLM}^{(y)} T_{KLM}^{(z)(e)}(\mathbf{r});
\end{aligned}
\tag{7}
$$

the expressions for $T_{KLM}^{(q)(e)}$ are:

$$\begin{aligned} T_{KLM}{}^{(x)(e)}(\mathbf{r}) &= \pi_{2K}(x/\delta x)\pi_{1L}(y/\delta y)\pi_{1M}(z/\delta z) \\ T_{KLM}{}^{(y)(e)}(\mathbf{r}) &= \pi_{1K}(x/\delta x)\pi_{2L}(y/\delta y)\pi_{1M}(z/\delta z) \\ T_{KLM}{}^{(z)(e)}(\mathbf{r}) &= \pi_{1K}(x/\delta x)\pi_{1L}(y/\delta y)\pi_{2M}(z/\delta z) \\ (K,L,M) &= (0,0,0),...,(N_x,N_y,N_z), \end{aligned} \tag{8}$$

where $\pi_{1M}(y/\delta y)$ is the Mth unit pulse function, and $\pi_{2K}(x/\delta x)$ is the Kth tent function, which is the convolution of $\pi_{1K}(x/\delta x)$ with itself.

The $T^{(q)(e)}(\mathbf{r})$ are called facet elements, because the qth element is constant over the qth facet of the KLMth cell. They are often referred to as 'divergence-conforming,' because the divergence of the current density is bounded. Facet elements have been called 'volumetric rooftop' functions in [5]. Volumetric rooftop functions have, also, been used in [7] and [6].

We assume that the conductivity is constant, with the value

$$\sigma_{\text{cell}} = \sigma_{\max} + V_c(\sigma_{\min} - \sigma_{\max}), \tag{9}$$

within each cell of dimension $\delta x \times \delta y \times \delta z$. $\sigma_{\max}$ and $\sigma_{\min}$ are, respectively, the maximum and minimum conductivities in the problem, and V_c is the *conductivity volume-fraction*.

Because $\mathbf{J}^{(m)}(\mathbf{r}) = \nabla \times \mathbf{M}_a(\mathbf{r})$, we expand $\mathbf{M}_a(\mathbf{r})$ in 'curl-conforming' edge-elements, which have the required differentiability of the curl operation

$$\begin{aligned} M_x(\mathbf{r}) &= \sum_{KLM} M_{KLM}^{(x)} T_{KLM}^{(x)(m)}(\mathbf{r}) \\ M_y(\mathbf{r}) &= \sum_{KLM} M_{KLM}^{(y)} T_{KLM}^{(y)(m)}(\mathbf{r}) \\ M_z(\mathbf{r}) &= \sum_{KLM} M_{KLM}^{(z)} T_{KLM}^{(z)(m)}(\mathbf{r}), \end{aligned} \tag{10}$$

where

$$\begin{aligned} T_{KLM}{}^{(x)(m)}(\mathbf{r}) &= \pi_{1K}(x/\delta x)\pi_{2L}(y/\delta y)\pi_{2M}(z/\delta z) \\ T_{KLM}{}^{(y)(m)}(\mathbf{r}) &= \pi_{2K}(x/\delta x)\pi_{1L}(y/\delta y)\pi_{2M}(z/\delta z) \\ T_{KLM}{}^{(z)(m)}(\mathbf{r}) &= \pi_{2K}(x/\delta x)\pi_{2L}(y/\delta y)\pi_{1M}(z/\delta z). \end{aligned} \tag{11}$$

These functions are called edge-elements because the expansion coefficient, $M_{KLM}^{(x)}$, is the (constant) value of M_x along the x-directed edge, ($y = (L + 1)\delta y$, $z = (M + 1)\delta z$). There are similar interpretations for $M_{KLM}^{(y)}$ and $M_{KLM}^{(z)}$.

We assume that the magnetic permeability is constant, with the value

$$\mu_{\text{cell}} = \mu_{\max} + V_p(\mu_{\min} - \mu_{\max}), \tag{12}$$

within each cell of dimension $\delta x \times \delta y \times \delta z$. $\mu_{\min}$ and $\mu_{\max}$ are, respectively, the maximum and minimum permeabilities in the problem, and Vp is the *permeability volume-fraction*.

We discretize (3) by employing Galerkin's method, which uses the same vector functions for expansion and testing. The spatial derivatives that could cause problems are removed by the testing process. The procedure for discretization is to first substitute (7) and (10) into (3), and then take moments of each of the first three equations of (3) with the corresponding facet element, and of the second three equations with the corresponding edge element. The result for the electric equation is

$$\begin{aligned} \begin{bmatrix} \mathbf{E}^{(ix)} \\ \mathbf{E}^{(iy)} \\ \mathbf{E}^{(iz)} \end{bmatrix} &= \begin{bmatrix} \mathbf{Q}^{(x)} & 0 & 0 \\ 0 & \mathbf{Q}^{(y)} & 0 \\ 0 & 0 & \mathbf{Q}^{(z)} \end{bmatrix}^{(ee)} \begin{bmatrix} \mathbf{J}^{(x)} \\ \mathbf{J}^{(y)} \\ \mathbf{J}^{(z)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{G}_{(0)}^{(xx)} & \mathbf{G}_{(0)}^{(xy)} & \mathbf{G}_{(0)}^{(xz)} \\ \mathbf{G}_{(0)}^{(yx)} & \mathbf{G}_{(0)}^{(yy)} & \mathbf{G}_{(0)}^{(yz)} \\ \mathbf{G}_{(0)}^{(zx)} & \mathbf{G}_{(0)}^{(zy)} & \mathbf{G}_{(0)}^{(zz)} \end{bmatrix}^{(ee)} \begin{bmatrix} \mathbf{J}^{(x)} \\ \mathbf{J}^{(y)} \\ \mathbf{J}^{(z)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{G}_{(a)}^{(xx)} & \mathbf{G}_{(a)}^{(xy)} & \mathbf{G}_{(a)}^{(xz)} \\ \mathbf{G}_{(a)}^{(yx)} & \mathbf{G}_{(a)}^{(yy)} & \mathbf{G}_{(a)}^{(yz)} \\ \mathbf{G}_{(a)}^{(zx)} & \mathbf{G}_{(a)}^{(zy)} & \mathbf{G}_{(a)}^{(zz)} \end{bmatrix}^{(ee)} \begin{bmatrix} \mathbf{J}^{(x)} \\ \mathbf{J}^{(y)} \\ \mathbf{J}^{(z)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{G}_{(b)}^{(xx)} & \mathbf{G}_{(b)}^{(xy)} & \mathbf{G}_{(b)}^{(xz)} \\ \mathbf{G}_{(b)}^{(yx)} & \mathbf{G}_{(b)}^{(yy)} & \mathbf{G}_{(b)}^{(yz)} \\ \mathbf{G}_{(b)}^{(zx)} & \mathbf{G}_{(b)}^{(zy)} & \mathbf{G}_{(b)}^{(zz)} \end{bmatrix}^{(ee)} \begin{bmatrix} \mathbf{J}^{(x)} \\ \mathbf{J}^{(y)} \\ \mathbf{J}^{(z)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{G}_{(0)} & \mathbf{G}_{(a)} & \mathbf{G}_{(b)} \end{bmatrix} \begin{bmatrix} \mathbf{M}^{(x)} \\ \mathbf{M}^{(y)} \\ \mathbf{M}^{(z)} \end{bmatrix}, \end{aligned} \tag{13}$$

where the $\mathbf{Q}$'s are tri-diagonal matrices, the $\mathbf{G}_{(0)}$'s the infinite-space matrices, the $\mathbf{G}_{(a)}$'s the convolutional layered-space matrices, and the $\mathbf{G}_{(b)}$'s the correlational layered-space matrices. The infinite-space matrices are convolutional, also. The superscript (ee) denotes electric-electric matrices, and (em) denotes electric-magnetic matrices. The $\mathbf{J}$'s are the unknown electric currents, and the $\mathbf{M}$'s are the unknown magnetic polarization vectors. The last block in (13) is simply a short-hand representation of the three blocks above it, except that it represents electric-magnetic interactions.

The magnetic equation is similar to (13), and is given by:

$$\begin{bmatrix}\mathbf{B}^{(ix)}\\ \mathbf{B}^{(iy)}\\ \mathbf{B}^{(iz)}\end{bmatrix} = \begin{bmatrix}\mathbf{Q}^{(x)} & 0 & 0\\ 0 & \mathbf{Q}^{(y)} & 0\\ 0 & 0 & \mathbf{Q}^{(z)}\end{bmatrix}^{(mm)} \begin{bmatrix}\mathbf{M}^{(x)}\\ \mathbf{M}^{(y)}\\ \mathbf{M}^{(z)}\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{G}^{(xx)}_{(0)} & \mathbf{G}^{(xy)}_{(0)} & \mathbf{G}^{(xz)}_{(0)}\\ \mathbf{G}^{(yx)}_{(0)} & \mathbf{G}^{(yy)}_{(0)} & \mathbf{G}^{(yz)}_{(0)}\\ \mathbf{G}^{(zx)}_{(0)} & \mathbf{G}^{(zy)}_{(0)} & \mathbf{G}^{(zz)}_{(0)}\end{bmatrix}^{(mm)} \begin{bmatrix}\mathbf{M}^{(x)}\\ \mathbf{M}^{(y)}\\ \mathbf{M}^{(z)}\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{G}^{(xx)}_{(a)} & \mathbf{G}^{(xy)}_{(a)} & \mathbf{G}^{(xz)}_{(a)}\\ \mathbf{G}^{(yx)}_{(a)} & \mathbf{G}^{(yy)}_{(a)} & \mathbf{G}^{(yz)}_{(a)}\\ \mathbf{G}^{(zx)}_{(a)} & \mathbf{G}^{(zy)}_{(a)} & \mathbf{G}^{(zz)}_{(a)}\end{bmatrix}^{(mm)} \begin{bmatrix}\mathbf{M}^{(x)}\\ \mathbf{M}^{(y)}\\ \mathbf{M}^{(z)}\end{bmatrix} \quad (14)$$
$$+\begin{bmatrix}\mathbf{G}^{(xx)}_{(b)} & \mathbf{G}^{(xy)}_{(b)} & \mathbf{G}^{(xz)}_{(b)}\\ \mathbf{G}^{(yx)}_{(b)} & \mathbf{G}^{(yy)}_{(b)} & \mathbf{G}^{(yz)}_{(b)}\\ \mathbf{G}^{(zx)}_{(b)} & \mathbf{G}^{(zy)}_{(b)} & \mathbf{G}^{(zz)}_{(b)}\end{bmatrix}^{(mm)} \begin{bmatrix}\mathbf{M}^{(x)}\\ \mathbf{M}^{(y)}\\ \mathbf{M}^{(z)}\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{G}_{(0)} & \mathbf{G}_{(a)} & \mathbf{G}_{(b)}\end{bmatrix}^{(me)} \begin{bmatrix}\mathbf{J}^{(x)}\\ \mathbf{J}^{(y)}\\ \mathbf{J}^{(z)}\end{bmatrix},$$

where $\mathbf{B}^{(i)}$ is the incident magnetic flux density due to the coil, the superscript (*mm*) stands for magnetic-magnetic interactions, and (*me*) stands for magnetic-electric interactions. The magnetic-magnetic **Q** matrices are a little more complicated than the electric-electric ones. The important point is that the electric-magnetic, magnetic-electric, and magnetic-magnetic matrices can be computed from the electric-electric.

C. Solution strategies

The discretized equations that are obtained by applying the method of moments to integral equations involve dense matrices; hence, it is important to develop efficient algorithms for solving the discretized equations. For relatively small problems (~ 3000 unknowns), we use the LU factorization (direct method) of the system matrix, but if the problem is too large to accommodate the LU factorization, we employ the iterative conjugate gradient algorithm [23]. All of the examples that will be shown in subsequent papers in this series will use the conjugate gradient algorithm. One advantage that accrues to the direct method, when it is feasible, occurs when (13) and (14) have multiple left-hand sides, as occurs when the probe coil is scanned past a flaw. The solution is then computed serially for each incident field vector after the factorization step.

We take advantage of the convolutional and correlational structure of the matrices of (13) and (14) by using three-dimensional FFT's [9] to evaluate the vector-matrix products. The use of FFT's drastically alters the conclusions reached in an analysis like that in [26], reducing ~ N^3 operations to ~ $N^2\log(N)$. This is obviously important when we have multiple left-hand sides in (13) and (14).

D. Calculating the change in impedance due to flaws in ferromagnetic bodies

The development of the equations relies on the reaction and reciprocity theorems [24], as described in [8].

E. Spatial decomposition via volume-integral equations [25]

If the flaw extends over two or more layers with different electrical constitutive properties, as in Figure 2, then (13) and (14) still hold in each layer (with different matrices), but now [$\mathbf{E}^{(ix)}$, $\mathbf{E}^{(iy)}$, $\mathbf{E}^{(iz)}$] depend upon the coil current plus anomalous currents in other layers. This leads us to consider the volume-integral relation for transfer between region 0 ('source') and region q ('field')

$$\begin{aligned}
\mathbf{E}_x(\mathbf{r}) &= \int G^{(q0)(1)}(x-x',y-y';z,z')\mathbf{J}_x^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&+ \int G^{(q0)(1)}(x-x',y-y';z,z')\mathbf{J}_x^{(m)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{\partial}{\partial x}\int G^{(q0)(2)}(x-x',y-y';z,z')\nabla_t'\cdot\mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{\partial}{\partial x}\int G^{(q0)(2)}(x-x',y-y';z,z')\nabla_t'\cdot\mathbf{J}_t^{(m)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{1}{k_0^2}\frac{\partial}{\partial x}\int G^{(q0)(3)}(x-x',y-y';z,z')\nabla'\cdot\mathbf{J}^{(e)}(\mathbf{r}')d\mathbf{r}' \\
\mathbf{E}_y(\mathbf{r}) &= \int G^{(q0)(1)}(x-x',y-y';z,z')\mathbf{J}_y^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&+ \int G^{(q0)(1)}(x-x',y-y';z,z')\mathbf{J}_y^{(m)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{\partial}{\partial y}\int G^{(q0)(2)}(x-x',y-y';z,z')\nabla_t'\cdot\mathbf{J}_t^{(e)}(\mathbf{r}')d\mathbf{r}' \qquad (15)\\
&+ \frac{\partial}{\partial y}\int G^{(q0)(2)}(x-x',y-y';z,z')\nabla_t'\cdot\mathbf{J}_t^{(m)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{1}{k_0^2}\frac{\partial}{\partial y}\int G^{(q0)(3)}(x-x',y-y';z,z')\nabla'\cdot\mathbf{J}^{(e)}(\mathbf{r}')d\mathbf{r}' \\
\mathbf{E}_z(\mathbf{r}) &= \int G^{(q0)(4)}(x-x',y-y';z,z')\mathbf{J}_z^{(e)}(\mathbf{r}')d\mathbf{r}' \\
&+ \int G^{(q0)(4)}(x-x',y-y';z,z')\mathbf{J}_z^{(m)}(\mathbf{r}')d\mathbf{r}' \\
&+ \frac{1}{k_0^2}\frac{\partial}{\partial z}\int G^{(q0)(5)}(x-x',y-y';z,z')\nabla'\cdot\mathbf{J}^{(e)}(\mathbf{r}')d\mathbf{r}',
\end{aligned}$$

where $\mathbf{r} \in q$ and $\mathbf{r}' \in 0$. The superscript (*e*) denotes electrical anomalous currents and (*m*) denote magnetic anomalous currents that arise from the presence of permeable material. The numerical superscripts on the various Green's functions refer to certain properties of these functions, and are not important to the discussion.

Taking moments of (15), using the same testing functions described above (with lower-case indices), yields

$$\begin{aligned}
E^{(x)}_{KLM} &= \sum_{KLM} T^{(ee)(xx)(q0)}_{klm,KLM} J^{(x)(e)}_{KLM} + \sum_{KLM} T^{(ee)(xy)(q0)}_{klm,KLM} J^{(y)(e)}_{KLM} \\
&\quad + \sum_{KLM} T^{(ee)(xz)(q0)}_{klm,KLM} J^{(z)(e)}_{KLM} \\
E^{(y)}_{KLM} &= \sum_{KLM} T^{(ee)(yx)(q0)}_{klm,KLM} J^{(x)(e)}_{KLM} + \sum_{KLM} T^{(ee)(yy)(q0)}_{klm,KLM} J^{(y)(e)}_{KLM} \\
&\quad + \sum_{KLM} T^{(ee)(yz)(q0)}_{klm,KLM} J^{(z)(e)}_{KLM} \qquad (16) \\
E^{(z)}_{KLM} &= \sum_{KLM} T^{(ee)(zx)(q0)}_{klm,KLM} J^{(x)(e)}_{KLM} + \sum_{KLM} T^{(ee)(zy)(q0)}_{klm,KLM} J^{(y)(e)}_{KLM} \\
&\quad + \sum_{KLM} T^{(ee)(zz)(q0)}_{klm,KLM} J^{(z)(e)}_{KLM},
\end{aligned}$$

where the transfer matrices satisfy $T^{(q0)}_{klm,KLM} = T^{(q0)}_{k-K,l-L;m,M}$; i.e., they are Töplitz (2D-convolution) in (X, Y). Thus, one can still use two-dimensional FFTs to efficiently execute the computations. The superscript (ee) on the transfer matrices denotes 'electric-electric,' which means that these matrices 'transfer' an electric current in region 0 into an electric field in region q. Similar transfer matrices exist for transferring a magnetic current in region 0 into an electric field in region q, as well as transferring an electric current into a magnetic field and a magnetic current into a magnetic field. All of these other transfer matrices

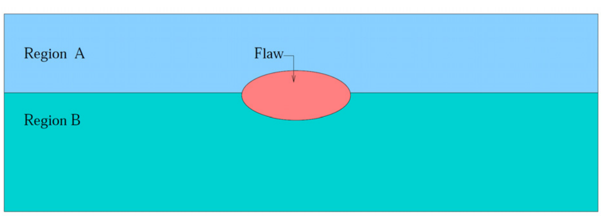


Fig. 2. A flaw in multiple layers.

can be deduced from the electric-electric ones, but the development is lengthy and will not be included here.

III. COMMENTS AND CONCLUSIONS

We have formulated the volume-integral approach in terms of the Galerkin variant of the method of moments, in which the unknown anomalous currents and the testing functions are expressed in terms of basis functions that are defined on a regular grid. This results in operators that have very special structures; they are either three-dimensional convolutions, or two-dimensional convolutions and one-dimensional correlations, which means that we can use three-dimensional FFTs to accelerate the matrix-vector operations occurring within a conjugate-gradient search algorithm. The use of a highly irregular mesh in the finite-element technique does not allow a similar advantage in the solution process. In the next paper and the remaining ones in this series, we will show how this formulation produces extremely efficient solutions of complex problems.

We have not gone into certain technical details, such as comparing operation counts for a direct matrix-vector multiply versus an FFT-assisted operation. Recent texts, such as [27-29], deal with these issues in more generality, while [30] deals with other fast algorithms for computational electromagnetics.

REFERENCES

[1] M. Yuan, T. K. Sarkar, and B. Kolundzija, "Solution of large complex problems in computational electromagnetics using higher-basis in MoM with out-of-core solvers," *IEEE Antennas and Propagation Magazine,* vol. 48, no. 2, pp. 55-62, April 2006.

[2] R. Mittra, "A look at some challenging problems in computational electromagnetic," *IEEE Antennas and Propagation Magazine,* vol. 46, no. 5, pp. 18-32, October 2004.

[3] J. H. Richmond, "Scattering by a dielectric cylinder of arbitrary cross section shape," *IEEE Trans. Antennas and Propagation,* vol. AP-13, pp. 334-341, May 1965.

[4] R. F. Harrington, *Field Computation by Moment Methods, The Macmillan Company,* New York, 1968.

[5] M. F. Catedra, E. Gago, and L. Nu˜no, "A numerical scheme to obtain the RCS of three-dimensional bodies of resonant size using the conjugate gradient method and the fast Fourier transform," *IEEE Trans. Antennas and Propagation,* vol. 37, no. 5, pp. 528-537, May 1989.

[6] A. P. M. Zwamborn and P. M. van den Berg, "The three-dimensional weak form of the conjugate gradient FFT method for solving scattering problems," *IEEE Trans. Microwave*

Theory and Techniques, vol. 40, no. 9, pp. 1757-1766, September 1992.

[7] A. P. M. Zwamborn, P. M. van den Berg, J. Mooibroek, and F. T. C. Koenis, "Computation of three-dimensional electromagnetic-field distributions in a human body using the weak form of the CGFFT method," *Applied Computational Electromagnetics Society Journal*, vol. 7, no. 2, pp. 26-42, Winter 1992.

[8] R. K. Murphy, H. A. Sabbagh, J. C. Treece, and L. W. Woo, "A volume-integral code for electromagnetic nondestructive evaluation," *Proceedings 11th Annual Review of Progress in Applied Computational Electromagnetics*, March 1995.

[9] J. R. Bowler, L. D. Sabbagh, and H. A. Sabbagh, "A theoretical and computational model of eddy-current probes incorporating volume integral and conjugate gradient methods," *IEEE Trans. Magnetics,* vol. 25, no. 3, pp. 2650-2664, May 1989.

[10] http://www.kiva.net/~sabbagh.

[11] R. K. Murphy, H. A. Sabbagh, A. Chan, and E. H. Sabbagh, "A volume-integral code for electromagnetic nondestructive evaluation," *Proceedings 13th Annual Review of Progress in Applied Computational Electromagnetics*, March 1997.

[12] H. A. Sabbagh, "Splines and their reciprocal - bases in volume-integral equations," *IEEE Trans. Magnetics,* vol. 29, no. 6, pp. 4142-4152., November 1993.

[13] H. A. Sabbagh, R. K. Murphy, and E. H. Sabbagh, "Advances in modeling eddy-current NDE of ferromagnetic bodies," *Review of Quantitative Nondestructive Evaluation*, *American Institute of Physics,* vol. 22A, pp. 383-389, 2003.

[14] H. A. Sabbagh, L. D. Sabbagh, and J. R. Bowler, "A volume-integral code for eddy-current nondestructive evaluation," *COMPEL,* vol. 9, Supplement A, pp. 67-70, 1990.

[15] J. R. Bowler, L. D. Sabbagh, and H. A. Sabbagh, "Eddy-current probe impedance due to a surface slot in a conductor," *IEEE Trans. Magnetics,* vol. 26, no. 2, pp. 889-892, March 1990.

[16] H. A. Sabbagh, J. R. Bowler, and L. D. Sabbagh, "A model of eddy-current probes with ferrite cores," *Nondestr. Test. Eval.,* vol. 5, pp. 67-79.

[17] H. A. Sabbagh and L. D. Sabbagh, "An eddy-current model for three-dimensional inversion," *IEEE Trans. Magnetics,* vol. MAG-22, no. 4, pp. 282-291, July 1986.

[18] H. A. Sabbagh, L. D. Sabbagh, and S. N. Vernon, "Verification of an eddy-current flaw inversion algorithm," *IEEE Trans. Magnetics,* vol. MAG-22, no. 6, pp. 1881-1886, November 1986.

[19] H. A. Sabbagh, E. H. Sabbagh, and R. K. Murphy, "Validation of a volume-integral eddy-current code by benchmark experiments," *Review of Progress in Quantitative Nondestructive Evaluation, American Institute of Physics*, vol. 21, pp. 396-402, 2002.

[20] H. A. Sabbagh, E. H. Sabbagh, and R. K.Murphy, "Recent advances in modeling eddy-current probes," *Review of Progress in Quantitative Nondestructive Evaluation*, *American Institute of Physics,* vol. 21, pp. 423-429, 2002.

[21] H. A. Sabbagh, E. H. Sabbagh, and R. K. Murphy, "Reconstruction of semielliptical and epicyclic cracks by eddy-current inversion," *Review of Progress in Quantitative Nondestructive Evaluation, American Institute of Physics,* vol. 21, pp. 715-721, 2002.

[22] H. A. Sabbagh, E. H. Sabbagh, R. K. Murphy, and J. Nyenhuis, "Assessing thermal barrier coatings by eddy-current inversion," *Review of Progress in Quantitative Nondestructive Evaluation, American Institute of Physics,* vol. 21, pp. 722-727, 2002.

[23] M. Hestenes, *Conjugate Direction Methods in Optimization,* New York: Springer Verlag, 1980.

[24] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, McGraw-Hill Book Company, New York, 1961.

[25] J. S. Knopp, H. A. Sabbagh, J. C. Aldrin, R. K. Murphy, E.H. Sabbagh, J. Hoffmann, and G.J. Steffes, "Efficient solution of electromagnetic scattering problems using spatial decomposition algorithms," *Review of Progress in Quantitative Nondestructive Evaluation, AIP,* vol. 25A, pp. 299-306, 2006.

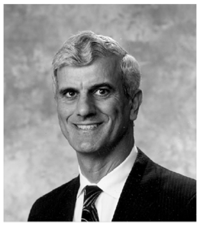
[26] B. M. Kolundzija and V. V. Petrovic, "Comparison of MoM/SIE, Mom/VIE and FEM based on topological analysis of two canonical problems," *IEEE Antennas and Propagation International Symposium,* vol. 1, pp. 274-277, June 1998.

[27] T. P. Sarkar, *Application of the Conjugate Gradient Method in Electromagnetics and Signal Processing*, Elsevier, New York, 1991.

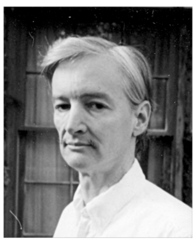
[28] M. F. Catedra, R. P. Torres, J. Basterrechea, and E. Gago, *The CG-FFT Method: Application of Signal Processing Techniques to Electromagnetics*, Artech House, Boston, 1995.

[29] A. Peterson, S. Ray, and R. Mittra, *Computational Methods for Electromagnetics*, IEEE, New York, 1998.

[30] W. C. Chew, J. M. Jin, E. Michielsssen, and J. M. Song, *Fast and Efficient Algorithms in Computational Electromagnetics*, Artech House, Boston, 2001.



Harold A. Sabbagh, received his BSEE and MSEE from Purdue University in 1958, and his Ph.D. from Purdue in 1964. In 1980, he formed Sabbagh Associates, Inc., and did research in the application of computational electromagnetics to nondestructive evaluation (NDE). This research evolved into the commercial volume-integral code, VIC-3D. In 1998, he formed Victor Technologies, LLC, in order to continue this research and further development of VIC-3D. His past professional activities have included a stint as president of ACES, and in 2010, he was elected to the grade of FELLOW in ACES.



R. Kim Murphy received his B.A. in physics from Rice University in 1978 and his Ph.D. in physics from Duke University in 1984. Since 1989, he has worked as a Senior Physicist for Sabbagh Associates Inc., and Victor Technologies, LLC. Dr. Murphy has been active in formulating models and coding in VIC-3D®, performing validation numerical experiments, and solving one-dimensional and three-dimensional inverse problems.

Elias H. Sabbagh received the B.Sc. in Electrical Engineering and the B.Sc. in Economics from Purdue University in 1990 and 1991. He has worked as system administrator, software engineer, and researcher for Victor Technologies since its inception. His interests include object-oriented programming, database administration, system architecture, scientific programming, and distributed programming.

John C. Aldrin received B.Sc. and M.Sc. degrees in mechanical engineering from Purdue University, West Lafayette, IN, USA, in 1994 and 1996 and the Ph. D. degree in theoretical and applied mechanics from Northwestern University, Evanston, IL, USA in 2001. Since 2001, he has worked as the principal of Computational Tools. His research interests include computational methods: modeling, data analysis and inverse methods, in ultrasonic and eddy current nondestructive evaluation.

Jeremy Knopp received the B. Sc. degree in engineering physics, M.Sc. degree in electrical engineering, and M. Sc. in applied statistics from Wright State University, Dayton, OH, USA, in 2001, 2005, and 2009, respectively. Since 2002, he has worked as a researcher at the nondestructive evaluation (NDE) branch of the Air Force Research Laboratory (AFRL). His research interests include eddy current NDE, computational electromagnetics, inverse problems, and model-assisted probability of detection. In 2009, he won the Charles J. Cleary award for basic research at AFRL.

Mark Blodgett received the B. Sc. in metallurgical engineering from Iowa State University, Ames, Iowa, USA in 1985 and M.Sc. and Ph.D. degrees in materials engineering from University of Dayton, Dayton Ohio, USA in 1992 and 2000, respectively. Since 1986, he has worked as materials research engineer in the Nondestructive Evaluation Branch at the Air Force Research Laboratory. His research interests include developing electromagnetic and multi-modal NDE approaches for materials characterization.