Finite elements in electromagnetics: A jubilee review

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Abstract: *This review describes the historical origins and the present state of the finite element method in electromagnetics. The foundation principles of finite elements are briefly reviewed. Applications are mainly to two and three dimensional problems of microwave and optical structures. The major types of scalar element are noted, and edge-interpolative vector elements are reviewed, firstly for the waveguide problem and then in three dimensions. An extensive bibliography is given, particularly stressing review papers.*

1. The Beginnings

When Courant prepared the text of his winter address to the American Mathematical Society for publication in 1943, he added a two-page Appendix to illustrate how the variational methods first described by Lord Rayleigh could be put to wider use in potential theory [1]. Choosing piecewise-linear approximants on a set of triangles which he called "elements", he dashed off a couple of two-dimensional examples and the finite element method was born.

Finite element methods remained dormant, perhaps waiting for computers to be invented, for more than a decade. They next appeared in the work of Duffin [2,3] in a form similar to that given by Courant but relying also on the mathematical ideas of Synge [4]. A variational approach was retained in this work, making it relatively hard of access to engineers, for whom variational methods were not then a part of the normal mathematical toolkit. Where variational methods were used at all by applied field analysts, they were viewed as ways of generating finite difference formulae [5].

Finite element activity in electrical engineering began in earnest about 1968–1969. A paper on waveguide analysis [6] was published in *Alta Frequenza* in early 1969, giving the details of a finite element formulation of the classical hollow waveguide problem. It was followed by a rapid succession of papers on magnetic fields in saturable materials [7], dielectric loaded waveguides [8], and other well-known boundary value problems of electromagnetics. The method was quickly applied to integral operators as well, both in electrostatics [9] and wire antenna problems [10].

In the decade of the eighties, finite element methods spread quickly. In several technical areas, they assumed a dominant role in field problems. An assessment of the finite element literature, and of its growth rate, may be obtained by examining the INSPEC bibliographic data base over the 1968–1992 period. In 1968 the number of extant finite element papers with electrical engineering content amounted to a mere handful. By 1993 the total had reached about 6000, with 600 or more additional papers published annually. The number of papers added to the literature each year, as recorded by INSPEC, is shown in Fig. 1.

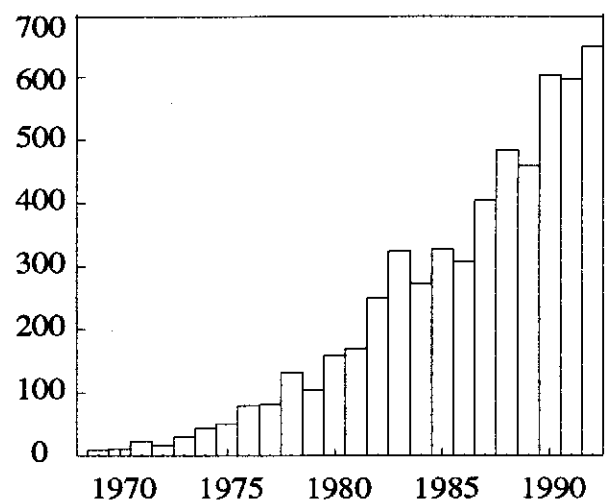

Fig. 1. Production of finite element papers in electrical engineering, 1968–1992.

Having initiated the method, mathematicians at first failed to grasp the significance of finite elements to practical analysis. Serious attention began to be paid to this technique about the same time as it gained a foothold in electrical engineering. Zlámal [11] published the first mathematical paper

explicitly devoted to the finite element method about the same time as electrical engineers began to use it seriously. He and Whiteman [12] recognized the value of this technique at an early date. Other able mathematicians, such as Ciarlet [13], quickly followed.

2. Theoretical Basis

Over the past 30 years the mathematics of finite elements has developed into a distinctive field of study, mainly based on projective and variational methods. This survey is directed to electromagnetics so a detailed treatment of the mathematics would be inappropriate. However, a brief overview may clarify what is special about electromagnetics problems.

2.1. Strong and weak solutions

The finite element method as used in electromagnetics is a special case of a general mathematical method [14] in which the differential or integral equation to be solved

$$\mathcal{P}u = v \tag{1}$$

is first replaced by an equivalent *weak form*, and this weak-form equation is subsequently solved by numerical approximation. The operator $\mathcal{P}$ may represent a boundary value problem (differential equations plus boundary conditions) or an integral operator, or a mixed (integrodifferential) operator.

The quantity u is said to be a *strong solution* of the operator equation (1). A *weak solution* $\overline{u}$ is obtained if the left and right sides of (1) have equal inner product projections onto all functions $w \in \mathbf{W}$, where $\mathbf{W}$ is some function space whose closure is the range of the operator $\mathcal{P}$,

$$\langle \mathcal{P}\overline{u}, w\rangle = \langle v, w\rangle, \qquad (\text{all } w \in \mathbf{W}). \tag{2}$$

The space $\mathbf{W}$ is open to choice so the weak reformulation permits a variety of approximate solutions. The general principle is to choose a finite-dimensional subspace $\mathbf{W}_N \subset \mathbf{W}$ and to solve the finite-dimensional (subspace) version of (2) in $\mathbf{W}_N$, a subspace which actually cannot contain the range of $\mathcal{P}$, and might not even intersect it! Such approximations are particularly attractive if the inner product $\langle a, b\rangle$ is an energy product [15] or a product integral, as illustrated by the following example.

2.2. Boundary-value problems.

The finite element method was first applied to boundary-value problems of differential equations, of a class illustrated by the mixed boundary value problem of Fig. 2. Here

$$\nabla^2 u = v \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial_D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial_N. \tag{3}$$

The solution u of (3) lies in a Hilbert space whose inner product is conveniently defined by the simple product integral

$$\langle a, b\rangle = \int_\Omega ab\, d\Omega. \tag{4}$$

Taking inner products on both sides with every possible $w \in \mathbf{W}$ in turn,

$$\int_\Omega w\nabla^2 u\, d\Omega = \int_\Omega wv\, d\Omega, \quad (\text{all } w \in \mathbf{W}), \tag{5}$$

and applying Green's second identity to the left-hand member, there results

$$\int_\Omega \operatorname{grad} w \cdot \operatorname{grad} \overline{u}\, d\Omega + \int_\Omega wv\, d\Omega - \oint_{\partial\Omega} w \operatorname{grad} \overline{u} \cdot \mathbf{dS} = 0. \tag{6}$$

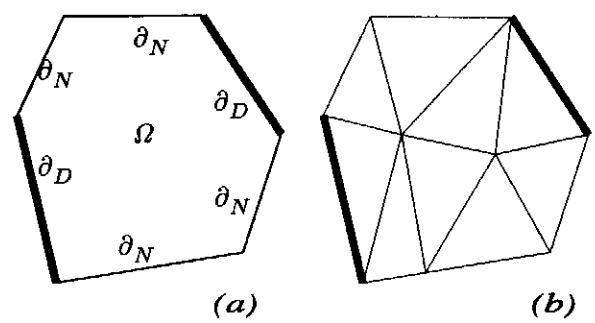

Fig. 2. (a) Simply connected two-dimensional region Ω. (b) One possible subdivision of Ω into triangular finite elements.

A key point to note is that $\langle \mathcal{P}\overline{u}, w\rangle$ is well defined by (6) even if $\overline{u}$ is only once differentiable, while $\nabla^2\overline{u}$ is not defined! This permits great freedom in choosing approximations for $\overline{u}$. However, to ensure the boundary conditions are satisfied it is still necessary to restrict the space $\mathbf{W}$ by requiring the boundary and bulk integrals in (6) to vanish independently,

$$\int_\Omega \operatorname{grad} w \cdot \operatorname{grad} \overline{u}\, d\Omega = -\int_\Omega wv\, d\Omega \qquad (\text{for all } w) \tag{7}$$

$$\oint_{\partial\Omega} w \operatorname{grad} \overline{u} \cdot \mathbf{dS} = 0, \qquad \partial\Omega = \partial_N \cup \partial_D. \tag{8}$$

Nothing in Green's second identity enforces the Dirichlet boundary condition $u = 0$; consequently, this condition must be explicitly imposed on the approximate solution. However, the surface integral (8) corresponding to the Neumann boundary condition appears naturally in the weak formulation. Consequently the Neumann boundary condition of eqn. (3) is termed a *natural* boundary condition, while the Dirichlet boundary condition is termed *principal* or *essential*.

Reformulation through Green's second identity weakens the continuity requirements on $\overline{u}$ but places stronger demands on w than on v: w must be once differentiable while v in the original boundary value problem (3) need not even be continuous. In fact the differentiability required by (6) is precisely equal for $\overline{u}$ and w. This suggests the weak solution may be sought in the space $\mathbb{W}$ itself. To summarize, $\mathbb{W}$ has three key properties in this problem: (1) its member functions are once differentiable, (2) they vanish on ∂_D, (3) the closure of $\mathbb{W}$ includes the range of $\mathcal{P}$. This weakened continuity requirement on u (which incidentally accounts for the strange name *weak form*) thus allows twice-differentiable solutions to be approximated by functions which do not themselves possess his property.

2.3. Finite element methods.

To obtain an approximate weak solution $\widetilde{u} \simeq \overline{u}$, the problem region Ω is partitioned into nonoverlapping, simply connected *finite elements* Ω_i,

$$\Omega = \bigcup_i \Omega_i. \tag{9}$$

A finite set $\{\alpha_k \mid k = 1, \ \ldots, \ N\}$ of approximating functions is defined to span $\mathbb{W}_N$, a finite subspace of $\mathbb{W}$. These functions must possess as high a degree of continuity as the weak form requires, and must satisfy all the principal boundary conditions. (In the example, $\alpha_k \in \mathbb{C}^0$, and $\alpha_k = 0$ on ∂_D, suffices). A large part of the literature of finite elements deals with systematic methods for partitioning Ω into elements and generating approximating functions on them. One popular method is to construct interpolation functions on an element-by-element basis so that the function values along the boundaries of each element will match the function values along the edges of its adjoining elements. Ensuring function continuity turns out to be quite easy, but it is surprisingly difficult to construct approximating functions with continuous derivatives.

Once the finite element function space $\mathbb{W}_N$ has been constructed, the weak solution is approximated by the finite summation

$$\widetilde{u} = \sum_k^N u_k \alpha_k. \tag{10}$$

Equation (7) then yields

$$-\sum_k^N \int_\Omega \operatorname{grad} \alpha_j \cdot \operatorname{grad} \alpha_k \, d\Omega \, u_k = \int_\Omega \alpha_j v \, d\Omega. \tag{11}$$

Since the integrals can be evaluated immediately, this may be regarded as a matrix equation and solved by the usual methods of numerical linear algebra. More generally, boundary-value problems in the scalar Helmholtz equation

$$(\nabla^2 + k^2)u = v \tag{12}$$

take the matrix form

$$(\mathbf{S} + k^2\mathbf{T})\,\mathbf{u} = \mathbf{v} \tag{13}$$

where

$$\begin{aligned} S_{jk} &= -\int_\Omega \operatorname{grad} \alpha_j \cdot \operatorname{grad} \alpha_k \, d\Omega, \\ T_{jk} &= \int_\Omega \alpha_j \alpha_k \, d\Omega, \\ v_k &= \int_\Omega v \alpha_k \, d\Omega. \end{aligned} \tag{14}$$

Most of the computing time is usually expended in solving the matrix equation, so a good deal of effort has been devoted to the development of methods able to exploit the algebraic and topological peculiarities of $\mathbf{S}$ and $\mathbf{T}$. These depend both on the physical problem class, and on the type of finite elements used.

3. Element Types

Finite elements used for electromagnetics initially resembled those used in structural mechanics, but differences in the underlying physical problems rapidly led to development of distinctive element types. It is probably fair to say that almost all work in electromagnetics has used distinctive elements since about 1970. The basic differences between scalar and vector problems, as well as between those of spatially infinite or finite extent, have led to

several families of fundamentally different elements.

3.1. Scalar Lagrangian simplexes

Scalar potential or wave problems have traditionally been solved using approximating functions based on Lagrangian interpolation polynomials. A family of interpolation polynomials ψ_i on some finite element Ω is associated with a point set $\mathbf{P}: \{P_i \mid P_i \in \Omega, i = 1, \ldots, K\}$ such that $\psi_i(P_j) = \delta_i^j$ (Kronecker delta). Aside from their computational advantages, interpolation polynomials are esthetically pleasing because all computed numbers represent physically significant quantities, e.g., local potential values. On the finite element Ω of Fig. 3(a), for example, the function ϕ is modeled by

$$\phi = \sum_i \phi(P_i)\psi_i, \tag{15}$$

so the coefficient that accompanies ψ_i is the value of ϕ at point P_i.

Interpolative finite element approximating functions are defined on an element-by-element basis so as to satisfy the principal continuity requirements at element boundaries. Suppose an interelement boundary is shared by two distinct elements Ω_i and Ω_j used to model some scalar function ϕ. Function continuity is assured if the function value at every interface point is determined entirely by the nodal values on that interface. For example, along the (one-dimensional) edge between two-dimensional elements of Fig. 3(b), the approximated function ϕ is a cubic polynomial in the distance s along the edge; the four coefficients of this cubic function are determined by the four nodal potential values associated with the edge.

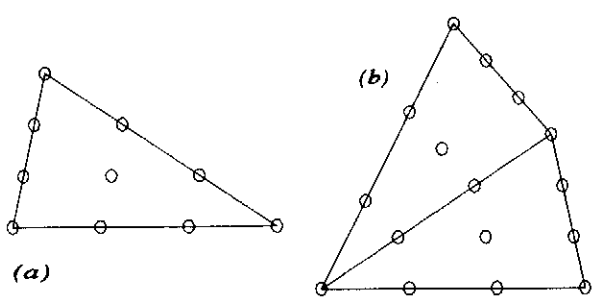

Fig. 3. (a) Triangular finite element with cubic interpolation node set. (b) Shared nodes of two cubic elements allow cubic interpolation on the interelement boundary.

Interpolation functions for simplex elements (lines, triangles, and tetrahedra) are readily derived by an extension into d dimensions of the classical Lagrangian interpolation theory [16]. Interpolation nodes are placed on the element in a regular array, as in Fig. 3(a). This procedure is best carried out in homogeneous coordinates $\zeta_1, \zeta_2, \ldots, \zeta_{d+1}$ attached to the simplex. These span the range $0 \leq \zeta_i \leq 1$ in any simplex and allow producing "universal" matrices that only need to be multiplied by a few geometric properties to produce the complete matrix representation of any simplicial finite element.

Establishing finite element matrices involves extensive algebraic manipulation of polynomials, an almost embarrassingly obvious application of computer algebra, which has indeed been used since the days of the Formac language [17]. By the late 1980s MACSYMA was routinely used for element generation in applied mechanics but electrical engineers rarely took any interest in this approach. Early developments in mechanics were reviewed by Noor and Andersen [18], and the state of the art in electromagnetics is covered by a more recent review paper [19]. Symbolic algebra will probably have a major impact on finite elements for electrical engineering applications in years to come.

3.2. Flexible elements

Although simplicial elements permit modeling of quite complicated geometric shapes, they are necessarily restricted to rectilinear interfaces and do not represent curved surfaces very well. The conventional solution to this problem, due to Ergatoudis, Irons and Zienkiewicz [20], is based on the simple observation that any Cartesian space coordinate, or indeed any linear measure of distance s, is (trivially) a linear function of the Cartesian coordinates x, y, z. It may therefore be expressed in terms of the finite element interpolation functions,

$$s(P) = \sum_i s(P_i)\,\psi_i(x_P, y_P, z_P), \tag{16}$$

where x_P, y_P, z_P are the coordinates of the point P. If the functions ψ_i are quadratic or higher polynomials, then (16) can express coordinate transformations which allow derivation of curvilinear elements from rectilinear ones. For example, the triangular element of Fig. 4(a) is mapped into the curved triangle of Fig. 4(b) by such a transformation. These elements are known as *isoparametric* because the *same* approximating functions are used to model the geometric shape and the fields to be determined. Isoparametric elements were used in magnetic field problems at an early date [21], and have remained in use by several working groups.

Their element matrices are obtained by the same process as previously. However, their evaluation is computationally more demanding because the integrations must be carried out numerically.

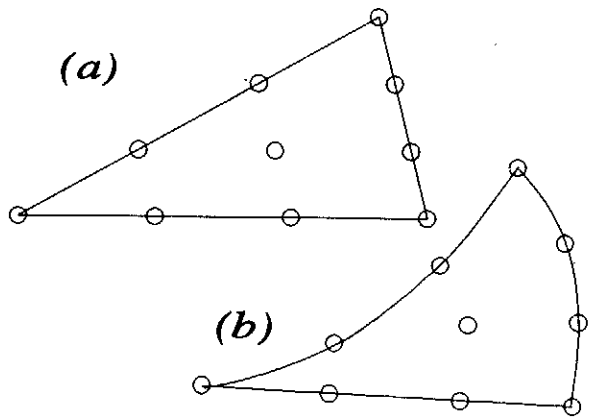

Fig. 4. Isoparametric element generation. (a) Rectilinear (simplicial) parent element. (b) Isoparametric triangle derived by coordinate mapping.

Accuracy improvement in finite element analysis may be sought by refining the mesh. Alternatively, the analysis may be repeated using elements of higher order. In this *hierarchal* elements are valuable. Their approximating functions ψ_i are constructed in nested families, so that the functions of polynomial order n are a proper subset of the functions of order $n+1$. Such functions were first pioneered by Rossow and Katz [22] and have developed considerably since [23]. Because the approximating functions form nested families, programs can be organized to compute only the projection of the weak solution onto the newly added functions, not to repeat the entire calculation, when moving to a higher-order approximation.

3.3. Geometrically infinite elements

Electromagnetic fields often extend over infinite geometric regions. Even static field problems often lack clearly defined finite boundaries, while an infinite region is the very essence of radiation and propagation problems. Several methods have accordingly been developed for handling what might be called "infinite finite elements", i.e., elements that encompass finite energy or power in a geometrically infinite space. These, and related methods from civil engineering practice, were reviewed by Emson [24].

Consider the ribbon transmission line of Fig. 5(a). To find the fields surrounding it, all the infinite-element methods encase the line in an artificial delimiting surface that subdivides all space into a finite interior and an infinite exterior region. The interior is handled by conventional finite element techniques. For the exterior region several techniques are available, of which at least four are properly convergent: (1) hybrid representations, (2) recursive growth, (3) inversion mapping, (4) special boundary conditions. All may be viewed as ways to find *boundary elements* which correctly (but implicitly) represent the effect of the exterior field on values in the interior region.

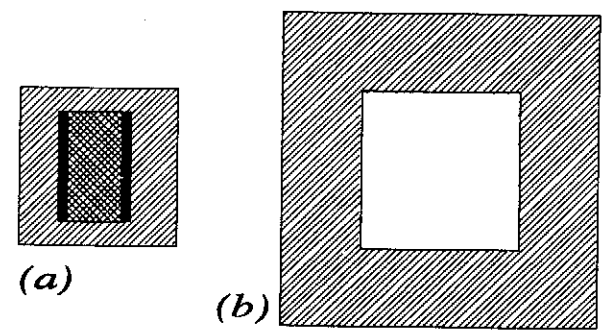

Fig. 5. (a) An artificial boundary encases the region of interest. (b) Space exterior to the artificial boundary is represented as an infinitely-extending element.

The hybrid technique describes the field by differential equations in the interior portion, integral equations in the exterior. This means choosing a set of approximating functions $\{\beta_i \mid i = 1, \ldots, M\}$ to satisfy the field equations exactly in the exterior and to match the interior element functions α_j along the exterior-interior interface. This procedure [25] was introduced and implemented around 1970 and has been extended variously since. For example, the exterior functions may be chosen to satisfy the field equations exactly, but match the interior elements only at the element nodes, not everywhere on the interface [26]; in this case, they may be expressed as orthogonal series expansions [27] or (in two dimensions) circular harmonics. The major difficulty in element formation is the evaluation of integrals containing Green's function singularities. These may be dealt with by geometric transformations [28] or by generating special weighted quadrature formulae [29]; or alternatively, by an ingenious scheme of using double boundaries [30].

Recursive growth algorithms model the exterior as a nested sequence of convex shells or annuli. They alternately add a shell and eliminate unwanted exterior variables, so that the memory required during construction of the exterior element with N_b boundary nodes never exceeds $9N_b^2$. The result is an extremely large, though still finite, exterior region [31]. Its growth rate in recursion can be doubly exponential, so that only a few recursion steps suffice to achieve immensely large exterior regions. The method is applicable to propagating-wave problems, provided an approximate radiation condition is attached to the outermost boundary and the growth of element size is controlled according to certain stability rules [32]. It has been used to solve waveguide problems [33,34] as well as two or three dimensional propagation.

Inversion mappings first appeared in Maxwell's *Treatise*. A circular or spherical boundary of radius R is drawn around the interior region and the exterior is mapped into a finite region by inverting all radial distances r with respect to this radius:

$$r' = \frac{R^2}{r}. \tag{17}$$

The differential equation governing the exterior region is transformed accordingly, and finite elements are constructed for this transformed equation. Two coupled boundary-value problems of the interior type thus result [35]. The restriction on boundary shape can be removed; non-circular boundary shapes have been proposed by Imhoff *et al.* [36], and a fairly complete theory of alternative shapes was developed by Stochniol [37].

In propagating-wave problems, the method of *absorbing boundary conditions* has recently gained great popularity. This method was initially developed by Bayliss and his associates [38,39], and augmented by other workers [40]. A review of the available variants of this method was given by Cooray and Costache [41]. Here a boundary element is created on which the wave function and its first derivatives are so related as to minimize the local reflection coefficient. A normally-directed outgoing wave is then *absorbed* by the boundary, much as a wave is absorbed by dissipative material in an anechoic room.

3.4. Vector operators and elements

A difficulty encountered with weak-form equivalents to boundary-value problems, and apparently peculiar to electromagnetic field problems, is the existence of *spurious modes*. These physically impossible solutions arise because weak solutions of the Helmholtz equation need not satisfy all four Maxwell equations even though the corresponding strong solutions do. Koshiba, Hayata and Suzuki [42] reviewed the literature and methodology of this problem in the context of waveguides, where this problem was first observed in the 1960s. It has been pointed out more recently that deterministic problems can suffer from the non-physical modes equally well [43]. This should not be surprising, given that the matrix representations of the eigenvalue problem of guided waves, and the deterministic problem of forced fields, involve the same finite element matrices and therefore have the same eigenfunction spectra. The spurious mode problem and its solutions will be dealt with in detail further below.

4. Applications to Magnetics

Finite elements have been applied to problems of applied magnetics, to waveguides and resonators, to antennas and scatterers; there is also a substantial literature in the area of semiconductor process modeling. This review concentrates mainly on the popular area of microwave and optical devices, but it must include certain parts of magnetics — most notably the eddy current problem — because they involve the same mathematical difficulties.

The early papers of Chari and Silvester [9,44] that dealt with finite element applications to magnetic field analysis were followed up by other workers fairly quickly. By 1990 this had become the dominant numerical method for magnetics problems, and now accounts for nearly half the finite element literature in electromagnetics. The central problem here is to solve the magnetic vector potential equation

$$\mathrm{curl}\left(\frac{1}{\mu_r}\mathrm{curl}\,\mathbf{A}\right) - \mu_0 g \frac{\partial \mathbf{A}}{\partial t} = \mu_0 \mathbf{J}, \tag{18}$$

subject to appropriate boundary conditions. Konrad [45] has reviewed this field well, though in view of recent rapid growth, that review has now become dated.

Although the magnetic flux density **B** due to a given set of currents is clearly unique, the accompanying vector potential **A** is open to choice of gauge and therefore not unique. Gauge transformations are introduced by specifying the divergence of **A**. Because the curl of **A** must always equal the flux density **B**, these two specifications define **A** unequivocally. A typical choice is

$$\operatorname{div} \mathbf{A} = -\mu g V - \mu\epsilon \frac{\partial V}{\partial t}, \tag{19}$$

where V is the electric scalar potential. Numerous other choices are possible. However, they all have the same form in classical two-dimensional magnetostatics, where $\mathbf{A}$ is time-invariant and possesses only one component. This situation characterizes a large class of useful problems in electromechanics and the electric machines community in particular adopted the finite element method at an early date. It was widely assumed that three-dimensional problems would be solvable by straightforward extensions of the techniques that worked so well in two dimensions, and that it was merely a matter of waiting for computing machines to grow large and powerful enough to handle three-dimensional problems. This supposition, however, proved false. Initial attempts to solve three-dimensional problems largely ignored the question of gauge [46]; in other words, they allowed the computer to choose the gauge through arithmetic chance and roundoff error. The resulting values of $\mathbf{A}$, of course, are irreproducible, though $\mathbf{B}$ is well defined. The gauge problem and the associated choice of potential formulation may now be regarded as solved [47,48] — so far as any problem in technology is ever solved — but it has taken almost a decade to establish what methods are actually useful and correct.

This area has produced a large number of software packages for general use by design engineers and analysts, experts in magnetics with little knowledge of finite element methods. A recent survey of the available techniques was given by Tsukerman, Konrad, Bedrosian and Chari [49] while both the methods and the available software packages were comprehensively reviewed by Tseng [50].

5. Microwave and Optical Components

Optical and microwave applications of finite elements are now considered in detail. Microwave devices were indeed the first class of electromagnetic field problem solved by finite element methods [6]. Daly [51] analyzed wave propagation in microstrip lines at an early date, and the hollow waveguide problem attracted sufficient analysis to merit a review paper [52] shortly thereafter. Stone [53] extended the methodology to acoustic guided waves and Konrad determined the fields in cavity resonators [54]. Antenna analysis by finite elements lay dormant after an early start [55], but has recently blossomed. The review by Glisson [56] indicates much valuable material, though unfortunately it is already becoming dated. More recently, finite elements have had particularly strong impact in the analysis of optical waveguides [57] and related devices. Recent texts on finite elements applied to microwave and optical frequencies are by Davies [58] and a comprehensive treatment by Koshiba [59].

The microwave and optical component problems [60] addressed here include *(a)* waveguide that is uniform in z, where a modal approach is appropriate, *(b)* resonators and *(c)* closed scattering problems (e.g., finding the scattering matrix for a transmission filter). Problem *(a)* is quasi-two-dimensional while *(b)* and *(c)* are very closely related and genuinely 3-D in nature.

5.1. Uniform waveguide

To consider item *(a)* of the list above, the following four examples will be examined:

(i) hollow conducting waveguide (e.g. hollow ridged waveguide)
(ii) conductors coexisting with $\epsilon(x,y)$ (e.g. microstrip and coplanar waveguide)
(iii) optical waveguide $\epsilon(x,y)$ (e.g. rib or channel waveguide)
(iv) nonlinear optical guide $\epsilon(x,y,|\mathbf{E}|)$ where permittivity ϵ depends on optical power level.

Firstly to clarify the problem being considered [61,62]: it is a uniform optical or microwave guide, where the structure is strictly uniform in z (as declared in (i) to (iv) above), and modes are being looked for, defined by

$$\begin{aligned} \mathbf{E}(x,y,z,t) &= \mathbf{e}(x,y) \exp j(\omega t - \beta z) \\ \mathbf{H}(x,y,z,t) &= \mathbf{h}(x,y) \exp j(\omega t - \beta z) \end{aligned} \tag{20}$$

Finite elements have been applied mostly by using a variational approach and many formulations have been used [61], generally in terms of one of the six: $\mathbf{H}$; $\mathbf{E}$; $\mathbf{H}$ and $\mathbf{E}$; H_z and E_z; $\mathbf{H}_t$; $\mathbf{E}_t$; or $\mathbf{H}_t$ and $\mathbf{E}_t$. There is no best choice, but the most common is the following variational form (with *s.v.* denoting "stationary value of") [63]:

$$\omega^2 = s.v. \frac{\iiint (\operatorname{curl} \mathbf{H})^* \| \epsilon \|^{-1} (\operatorname{curl} \mathbf{H})\, d\Omega}{\iiint \mathbf{H}^* \| \mu \| \mathbf{H}\, d\Omega}. \tag{21}$$

A feature of this formula for optics is that, although ϵ is inhomogeneous, μ is invariably constant, so that physically $\mathbf{H}$ is continuous everywhere. This eases the finite element representation, as all three components of $\mathbf{H}$ are now continuous. As the permittivity enters as a simple multiplying factor in the numerator, quite arbitrary $\epsilon(x,y)$ can be included with negligible additional computing cost. Anisotropic dielectric has commonly to be modeled in optics, and this gives no problem with formulation (21) providing only that the material is everywhere lossless. To use formulation (21) one chooses a propagation constant β which gives the z-dependence of all fields. Applying finite elements across the waveguide cross-section with the Rayleigh-Ritz method gives the following matrix equation:

$$\mathbf{Av} = \omega^2 \mathbf{Bv} \tag{22}$$

$\mathbf{B}$ is real, symmetric and positive-definite, while $\mathbf{A}$ is Hermitian but can usually be reduced to real, symmetric. One then has a range of quite excellent matrix algorithms for numerical solution.

By far the most serious difficulty with a finite element (or almost any other) procedure based on (21) is the occurrence of many spurious modes which come from the numerical procedure. These have already been referred to, and we will return later to this difficulty, but the chief trouble is with non-physical solutions which do not satisfy the divergence condition [64]. (Another class of 'spurious modes' cluster around zero frequency, but these are less troublesome, and again will be referred to later.) Unfortunately the number of spurious modes increases with finite element mesh density and matrix order, so that any desire for high accuracy is accompanied by persistent difficulties. For many years this failure has been rectified by adding a *penalty term* [65] to give [66,67]

$$\omega^2 = s.v.\left\{ \frac{\iiint (\operatorname{curl}\mathbf{H})^* \| \epsilon \|^{-1} (\operatorname{curl}\mathbf{H})\, d\Omega}{\iiint \mathbf{H}^* \| \mu \| \mathbf{H}\, d\Omega} + \frac{p\epsilon_0^{-1} \iiint (\operatorname{div}\mathbf{H})^* (\operatorname{div}\mathbf{H})\, d\Omega}{\iiint \mathbf{H}^* \| \mu \| \mathbf{H}\, d\Omega} \right\} \tag{23}$$

The penalty parameter p shifts the spectrum of all modes, but most strongly the spurious modes. The user can therefore 'filter out' the unwanted modes. Theoretically all now seems respectable — the required constraint has been added in a classic least-squares sense, with little additional computational effort. Although the penalty function is widely used in finite element work, in this context the whole procedure is neither numerically robust nor mathematically elegant.

Much effort has gone into the proper removal of spurious modes from the accurate approaches over the last decade. One method that retains sparsity, maintains reasonable computational economy and preserves a canonical matrix form such as eqn. (22) is to accept the original variational form of eqn. (21) but apply a more appropriate choice of basis vector functions from the armory of finite elements. This is by taking advantage of the recent introduction of edge elements [68], tangential vector finite elements (TVFE's) [69,70,71] and Whitney forms in general [72,73,74], which have transformed many applications of finite elements.

The method seems to have more relevance, and to be more comfortable, in three dimensions, as treated in section 6, rather than in two. Briefly, tangential vector finite elements (of which edge elements are the simplest subset) are a radical alternative to the nodal shape functions commonly used in finite elements. They represent the vector, not via three separate components, but as a vector which is arranged to have prescribed *tangential*, but not normal, values at the edges of the elements. Lee [72], Koshiba [75] and Hano [76] have in fact used tangential vector elements, a family of vector basis functions of which edge elements are the simplest and lowest order. It can be seen that their choice results in much more involved algebra and calculus (for evaluation of the usual element matrices) than nodal elements. Also a considerably more complicated matrix eigenvalue equation emerges, which arises from their introduction of a field transformation to obtain a canonical eigenvalue form in β^2. Some of these complications arise because of the disparity between *three*-dimensional vectors and *two*-dimensional fields.

In 1984 Hano [76] introduced a special set of finite elements. By using simple Cartesian rectangular elements, he was able to choose separate shape functions for the x, y and z components that achieved the basic aim of tangential vector elements. This was, in essence, a precursor of the edge element approach, but limited strictly to rectangular elements. Covariant projection elements were applied

to 3-D problems in 1988 [77] and applied to the waveguide problem in 1991 [78]. All eliminate the troublesome spurious modes, but leave a countable cluster of (strictly) spurious modes around zero frequency.

5.2. Uniform waveguide with nonlinear material

To illustrate the almost limitless versatility of finite elements, a more complicated problem is now taken up. In all of the above, it has been presumed that there can be a transversely varying permittivity $\epsilon(x,y)$. But *nonlinear* material, as described by $\epsilon(x,y,|\mathbf{E}|)$, is of considerable interest [79] and can similarly be solved [80,81] by using the above methods within an iterative loop, which seeks consistency between the linear solver and the specified nonlinear law for ϵ. A typical structure would have constant (with respect to field) ϵ over part of the cross-section, but a finite region would have a law such as [82]

$$\epsilon = \epsilon_{lin} + \Delta\epsilon_{sat}\left\{1 - \exp\left(-\frac{a\,|\mathbf{E}|^2}{\Delta\epsilon_{sat}}\right)\right\}. \qquad (24)$$

Clearly for any two specified values of ω, β and total power in the mode, the third parameter is an unknown eigenvalue, necessitating some iterative scheme for its solution. Finite element solutions are now being routinely produced for fairly arbitrary nonlinear laws as well as two-dimensional profiles of permittivity.

6. Three-Dimensional Structures

We now consider resonators and closed scattering problems. If we focus discussion here on the resonator, it is with the understanding that most of the aspects of finite element implementation are identical for the two classes of problem [60,83]. For any solution approach to the resonator, an analogous approach exists for the scattering problem. The main difference is that part of the resonator walls have to be removed and replaced by resistive boundary conditions (simulating a matched port) and/or by reactive impedances and/or by stipulating at the removed walls (say) the magnetic field and evaluating the consistent electric field — and so evaluating the scattering matrix. Though these considerations are not trivial, we by-pass them to consider the more fundamental issues of the finite element formulation and choice of basis functions.

The other major difference between the resonator and the evaluation of a scattering matrix is that the former leads inevitably to an eigenvalue problem — best couched into a canonical matrix equation with relevant eigenvalues. The scattering problem is essentially a deterministic, rather than eigenvalue, problem and so inevitably leads, instead of eqn. (22), to a matrix for solution of the usual deterministic form $\mathbf{Ax} = \mathbf{y}$. This is, of course, a fundamentally easier matrix equation to solve than the eigenvalue equation (22).

We therefore concentrate on solution of the typical resonator which consists of an enclosing conducting wall, with an interior of permittivity $\|\epsilon\|(x,y,z)$ and possibly (most likely at microwaves) permeability $\|\mu\|(x,y,z)$. Optical resonators can similarly be studied without metal walls, but some mathematical artifact (such as a magnetic or electric wall, if physically realistic, or an 'evanescent' infinite element [57]), is best used to avoid treatment of a lossy resonator, which loses the Hermitian formulation. As with the waveguide problem of section 5.1, finite elements have been implemented by generalized Galerkin [84,85] and variational [86,87] approaches. Unless some special requirement forces the issue, it is generally possible and advisable to use a variational approach. The same variational form of eqn. (21) can be used, or its dual form, with $\mathbf{H}$ replaced by $\mathbf{E}$, and ϵ and μ exchanged.

Again if inhomogeneity of material is purely dielectric, then the form in terms of $\mathbf{H}$ is easier by orthodox nodal elements, as the 3-vector $\mathbf{H}$ is continuous everywhere. By contrast any form using $\mathbf{E}$ would need extra care for continuity of appropriate fields and flux densities between elements However, separate use of each formulation has the advantage of bounding the true solution [70,87] and/or providing information for error signatures, relevant to the important matter of adaptive mesh generation [70].

Resonators have been studied for many years using 3-D versions of the traditional nodal elements. Just as in 2-D, where the simplest first degree elements are popular for their ultimate simplicity and ultimate sparsity of resulting matrix, so can first degree tetrahedral elements be used. Again, whether in 2-D or 3-D, first order elements fit very conveniently into a Delaunay-generated mesh of elements, especially with adaptive mesh-generators [70]. Higher order elements are also used [88], including their attractive subset, hierarchal elements, where extra nodes are added to elements, as one proceeds to higher order, without abandoning any nodes from the low orders.

Spurious modes have commonly been reported [64,89,90] with 3-D resonators just as with (quasi-)2-D problems. Clearly less work has been reported on 3-D numerical work than on 2-D, because of the heavier computing demands of the former. Spurious solutions can be avoided or alleviated by the use of edge elements, which will be briefly described next.

6.1. Edge and tangential *vs.* nodal elements

So far in the vast majority of finite element work, nodal or scalar shape functions have been used [85], as already described. For a vector field, whether **H**, **E**, **B**, **D**, **A** or **J**, the conventional scheme has been to represent the vector as three separate scalars in the standard nodal manner. Analytically exact and conventional as this procedure is, it grates — it is uneasy in this context. In contrast, there are alternative schemes for directly expanding vectors in terms of vector forms.

Whitney forms [70,74,75] span the relevant possibilities of continuity between elements related to the grad, curl and div operators. W^0 forms are spaces of scalar basis functions that are continuous, without continuous derivatives, between elements. W^1 forms are spaces of vector functions that have continuity of tangential components, but not normal components, between elements. W^2 forms are vector functions having continuity of their normal components only between elements. W^3 forms are of scalar functions without continuity between elements. Working with tetrahedral elements, as we progress through the forms W^0, W^1, W^2, W^3 we go through W^0 which is nodal-based, W^1 which is edge-based, W^2 which is facet-based, and W^3 which is volume-based. Their intimate connection also comes from the fact that the gradient of a 0-Whitney form is a (combination of) 1-Whitney forms, the curl of a W^1 form is a W^2 form, and the divergence of a W^2 form is a W^3 form.

We illustrate with just one particular Whitney form W^1 called the *edge element.* It is a first degree polynomial form. The objective is to develop a vector representation that assures that across any face between adjacent finite elements, the tangential part (only) of the vector is continuous. This will be perfectly appropriate for representing **H** or **E**.

Consider a tetrahedron as in Fig. 6 with vertices 1 to 4 and edges 1 to 6. For node i, we first use the so-called barycentric coordinate ζ_i which has value 1 at node i, is zero over all tetrahedra not containing node i, and varies linearly over tetrahedra that contain node i. These are the local or 'volume' coordinates within the tetrahedron, so that ζ_i has unit value at node i, and varies linearly to zero at the three other vertices. To introduce the edge elements, we focus on edge 1, denoted as e_1, which joins vertices 1 and 2; associated with e_1 we introduce the vector basis function $\mathbf{w}_1$ defined over the tetrahedron by:

$$\mathbf{w}_1 = \zeta_1 \operatorname{grad} \zeta_2 - \zeta_2 \operatorname{grad} \zeta_1 \tag{25}$$

$\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$ and $\mathbf{w}_6$, are similarly defined, each associated with one of the other five edges. Although $\mathbf{w}_1$ is a first degree polynomial vector field, it turns out that at all points *along* edge 1, the projection onto edge e_1 of vector $\mathbf{w}_1$has constant value. The projections onto e_1 of vectors $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, $\mathbf{w}_5$ and $\mathbf{w}_6$ are all zero. Equation (25) therefore gives a vector substitute for the interpolatory shape functions of nodal elements,

$$\mathbf{V}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = \sum_i v_i \mathbf{w}_i \tag{26}$$

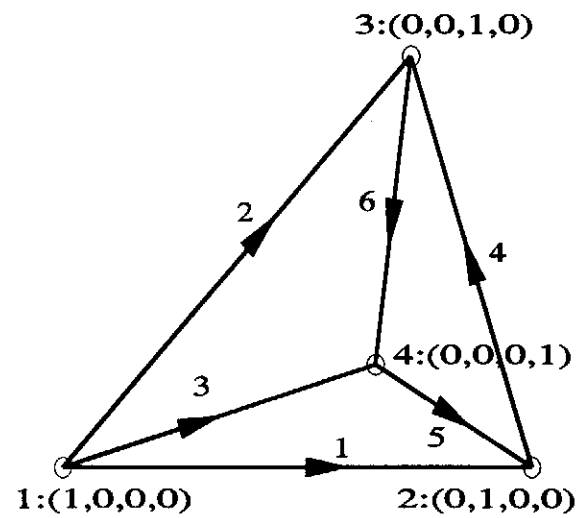

Fig. 6. Tetrahedron vertices (with local coordinates $\zeta_1,\zeta_2,\zeta_3,\zeta_4$) and directed edges.

On any edge e_i **V** will assume a value that varies; but the component (or projection) along e_i will remain constant and equal to $v_i\mathbf{u}_i$, where $\mathbf{u}_i$ is the unit vector directed along edge i with the sense of the arrow in Fig. 6. Clearly if we use the same specifications of the v_i (called the *edge values*) of eqn. (26) for different tetrahedra sharing the same edge, the overall vector field will have a continuous tangential component of vector across edges, and similarly across faces. The normal components will

have no such enforced continuity. A special property of edge elements is that their divergence is identically zero within the tetrahedra, though not across inter-element faces.

In this section we have concentrated on edge elements in 3-D. Their extension to 2-D is straightforward, although the fact that the physical quantities are fundamentally 3-vectors makes the situation less comfortable in 2-D than 3-D. For instance it is commonly found necessary [77] to use both edge and nodal elements because of the lack of parity between transverse and longitudinal components of field. Readers are referred to the literature [71,73,77,89,90,91] for details of these Whitney and related forms.

6.2. Tangential, edge and nodal elements for resonators

A number of workers have reported successful use of edge elements for the solution of resonators [84–89]. Motives have included belief in their superiority to nodal elements because of their lack of divergence-free related spurious modes, and their more efficient representation, via their fewer free variables and/or sparsity of global matrices. If one compares edge elements (with their built-in divergence-free vector) with nodal representation of the three Cartesian components of field, there can surely, be no arguing that the component version has a fundamental redundancy and therefore inefficiency. If they also give solutions free of a serious class of spurious modes, then the edge elements have very much in their favor. Recent evidence supports their fundamental advantage.

In electromagnetics, the use of edge elements was firstly with low frequency applications, especially concerning eddy currents [92]. With resonators, use with magnetic vector field formulations, eqn. (21) or its dual form in terms of **E**, has recently led to more satisfactory results [84–89]. Universally it is reported that the spurious modes associated with non-zero divergence are totally removed. However, as mentioned earlier with the waveguide problem, when the formulation used has frequency as the resulting spectrum (as in eqn. (21) and its matrix version (22)), other spurious modes appear clustered around zero frequency. Curiously, these non-physical solutions are ignored by many workers as not being serious — as not being worthy of the title of "spurious"! They are indeed less troublesome, as their location in the spectrum is very confined and away from the usual region of interest.

7. Conclusions

Finite elements in electrical engineering have had a varied and interesting history. At an early stage of development, several difficult problems were encountered, such as the determination of gauge in vector potential problems and the appearance of spurious modes in solutions of the Helmholtz equation. It has taken nearly twenty years to master these difficulties, and their solutions point the way for other, still more valuable, methods.

Incompletely solved problems of considerable importance now include general methods for (1) orthospectral elements, (2) field singularities, (3) boundary integrals, including absorbing boundaries. Several particular element types free of spurious modes are known and used, but the generation of families of such elements has still eluded analysts. Much the same can be said for singular elements. In the formation of boundary integrals, similar difficulties of integration are encountered as with field singularities. All three areas should prove fertile ground for the application of symbolic algebra. Given the current near-ubiquity of computer algebra systems, there appears to be every reason for optimism for their further development for finite element methods in electromagnetics.

In the use of finite elements for computation of high-frequency waveguides and 3-D structures there have been both major achievements and serious difficulties. As computing power has increased it has made more important the *robustness* of methods used, so that human intervention is eliminated from the inner shells of the computation. In this context it is believed that the introduction of vector finite element forms (or allied forms) will improve the performance of many codes when, as in electromagnetics, vector fields are inevitably involved. The case is more powerful in three dimensions than in two-dimensional problems. There are eloquent arguments for Whitney or equivalent vector forms with their basis in differential forms. Computational results have only appeared in the last three years or so, but hopefully the practical impact will soon be assessed, then these issues will be clarified or simplified, as were similar issues with nodal finite elements when they first appeared.

8. References

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