Performance Evaluation of State-of-the-Art Linear Iterative Solvers Based on IDR Theorem for Large Scale Electromagnetic Multiple Scattering Simulations

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Abstract ─ The present paper treats with the latest linear iterative solver IDR(*s*) method and its variants proposed by P. Sonneveld and M. van Gijzen. We derive preconditioned algorithms of the solvers based on right preconditioning and list them. The solvers are numerically tested in terms of convergence and accuracy for the computation of electromagnetic wave scattering from over 10^4 dielectric cylinders. Consequently, minimization schemes for residual vectors refine not only convergence but also accuracy for the original IDR(*s*) method. However, a spurious convergence may be confirmed and its influence is *1* or *2* digit error independently of parameter *s*.

Index Terms ─ Boundary element method, electromagnetic multiple scattering, IDR(*s*) method.

I. INTRODUCTION

The aim of the authors' work is to simulate electromagnetic (EM) wave scattering from a medium composed of several kinds of objects in shape, size, and material. As a basic issue, we try to develop fast techniques for the computation by means of the boundary element method (BEM) [1]. In the BEM computation, the most time-consuming part rises from solving the dense linear system of equations followed by the discretization of boundary integral equations. Our previous work presented that both the computational and memory complexities for the multiplication of a vector by the coefficient matrix can be drastically reduced by a wideband fast multipole algorithm [2]. Then, we use an iterative solver based on the Krylov subspace method [3] because the operation of matrix-vector multiplication is in its algorithm. The propositions of high performance iterative solvers and preconditionings are important to the fast computation today.

The authors have used a generalized minimal residual method with restart process (GMRES(*m*) method [4]) for solving the linear system of equations [5]. A restart cycle *m* should be chosen a very large value for high convergence but it needs much memory space. P. Sonneveld and M. van Gijzen proposed the IDR(*s*) method which belongs to the Krylov subspace method [6]. "IDR" is an abbreviation of "Induced Dimension Reduction". The authors' numerical experiments revealed that the IDR(*s*) method was better than the GMRES(*m*) method in terms of convergence and memory efficiency for the BEM analyses of electromagnetic wave scattering from many dielectric cylinders. As the parameter *s* of the IDR(*s*) method is larger, however, convergence is more improved but the accuracy deteriorates drastically [7].

$$E_z^{\mathrm{inc}}(\boldsymbol{\rho}_i) = \frac{1}{2}E_z(\boldsymbol{\rho}_i) + \frac{\mathrm{j}}{4}\sum_{n=1}^{N}\left[\int_{C_n}\left\{E_z(\boldsymbol{\rho}'_n)\frac{\partial \mathrm{H}_0^{(2)}(k_0|\boldsymbol{\rho}_i - \boldsymbol{\rho}'_n|)}{\partial n'_n} - \mathrm{H}_0^{(2)}(k_0|\boldsymbol{\rho}_i - \boldsymbol{\rho}'_n|)\frac{\partial E_z}{\partial n'_n}(\boldsymbol{\rho}'_n)\right\}\mathrm{d}l'_n\right] \quad (1)$$

$$0 = \frac{1}{2}E_z(\boldsymbol{\rho}_i) - \frac{\mathrm{j}}{4}\int_{C_i}\left\{E_z(\boldsymbol{\rho}'_i)\frac{\partial \mathrm{H}_0^{(2)}(k_i|\boldsymbol{\rho}_i - \boldsymbol{\rho}'_i|)}{\partial n'_i} - \mathrm{H}_0^{(2)}(k_i|\boldsymbol{\rho}_i - \boldsymbol{\rho}'_i|)\mu_r^{(i)}\frac{\partial E_z}{\partial n'_i}(\boldsymbol{\rho}'_i)\right\}\mathrm{d}l'_i \quad (i = 1, 2, \cdots, N). \quad (2)$$

Such a prejudicial phenomenon concerning accuracy is called "a spurious convergence" in this paper. The authors managed the spurious convergence by using Sleijpen and van der Vorst's convergence refinement method [8]. This technique is very simple and easy to implement but is not complete solution. Sakurai *et al.* proposed an auto corrected (AC) IDR(*s*) method that perfectly dissolves the problem of the spurious convergence [9]. In this method, however, a computation of matrix-vector multiplication may be added for one iteration. Then the net computation time may become much longer for the AC-IDR(*s*) method than for the original one in solving a large scale dense linear system of equations. On the other hand, Sonneveld and van Gijzen modified the algorithm and proposed two variants of the IDR(*s*) method in order to expedite convergence [10, 11].

This paper investigates the performance of the variant IDR(*s*) methods for the computation of EM wave scattering from many dielectric cylinders by means of the BEM. We solve the linear system of equations of order 10^5 by the IDR(*s*) method and its variants and compare their convergence and accuracy. After this introductory Section I, Section II presents two-dimensional boundary integral equations and discretization of them. We explain the structures of the coefficient matrix and the unknown and right-hand side vectors. Section III outlines the variant IDR(*s*) methods with a right preconditioning. Performance evaluations are done in Section IV. Finally, Section V summarizes conclusions of this study. Throughout this paper, $\mathrm{e}^{\mathrm{j}\omega t}$ time convention is used and suppressed.

II. FORMULATION

Let us consider the two-dimensional problem of EM wave scattering by *N* infinitely long cylinders in a vacuum. The relative permittivity and permeability of the *i*th cylinder are $\varepsilon_{\mathrm{r}}^{(i)}$ and $\mu_{\mathrm{r}}^{(i)}$, respectively. The wave numbers of the vacuum and the *i*th cylinder are represented by k_0 and $k_i = k_0\sqrt{\varepsilon_r^{(i)}\mu_r^{(i)}}$, respectively. Each axis of the cylinders is parallel to the *z*-axis of the cylindrical coordinate system. We formulate this problem in the electrical field integral equations (EFIEs) for TM wave. The *z*-components of unknown electric fields E_z and their normal derivatives $\partial E_z/\partial n$ are given by Eqs. (1) and (2). Here, the C_i is the boundary of the *i*th cylinder, and $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}'_i$ are the observation and integration points on C_i, respectively. The $\mathrm{H}_0^{(2)}$ is the zero order Hankel function of the second kind, and $\partial/\partial n_i$ is the outward normal derivative on C_i. The E_z^{inc} is an incident wave.

The integral equations can be discretized through the BEM [1]. We divide each boundary into M_i boundary elements and choose the rectangular pulse function as a basis function. Using the point matching method, we obtain a dense linear system of *L* equations $\mathrm{A}\boldsymbol{x} = \boldsymbol{b}$, where $L = 2(M_1 + M_2 + \cdots + M_N)$. The linear system is composed of blocks and subvectors:

$$\mathcal{A} = \left[\begin{array}{ccc|ccc} a_{11}^{\mathrm{out}} & \cdots & a_{1N}^{\mathrm{out}} & b_{11}^{\mathrm{out}} & \cdots & b_{1N}^{\mathrm{out}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{N1}^{\mathrm{out}} & \cdots & a_{NN}^{\mathrm{out}} & b_{N1}^{\mathrm{out}} & \cdots & b_{NN}^{\mathrm{out}} \\ \hline a_{11}^{\mathrm{in}} & & 0 & b_{11}^{\mathrm{in}} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_{NN}^{\mathrm{in}} & 0 & & b_{NN}^{\mathrm{in}} \end{array}\right] \quad (3)$$

$$\boldsymbol{x} = \left[\begin{array}{ccc|ccc} \boldsymbol{\alpha}^{(1)} & \cdots & \boldsymbol{\alpha}^{(N)} & \boldsymbol{\beta}^{(1)} & \cdots & \boldsymbol{\beta}^{(N)} \end{array}\right]^T \quad (4)$$

$$\boldsymbol{b} = \left[\begin{array}{ccc|ccc} \boldsymbol{e}_z^{(1)} & \cdots & \boldsymbol{e}_z^{(N)} & \mathbf{0}^{(1)} & \cdots & \mathbf{0}^{(N)} \end{array}\right]^T. \quad (5)$$

Here, the blocks a_{ij} and b_{ij} are the matrix of M_i by M_j. The notations "out" and "in" mean outer and internal fields of cylinders, respectively. The subvectors of M_i order $\boldsymbol{e}_z^{(i)}$, $\boldsymbol{\alpha}^{(i)}$ and $\boldsymbol{\beta}^{(i)}$ indicate the incident wave, unknown field, and its normal derivative on the boundary elements, respectively. The *T* and $\mathbf{0}^{(i)}$ mean matrix transpose and the M_i dimensional zero vector, respectively. Equation (3) implies that the lower half of coefficient matrix is comparably sparse. Then, we reduce the number of unknowns in half and obtain the linear system given by Eq. (6) [5].

$$\begin{pmatrix} -a_{11}^{\mathrm{out}}(a_{11}^{\mathrm{in}})^{-1}+b_{11}^{\mathrm{out}}(b_{11}^{\mathrm{in}})^{-1} & \cdots & -a_{1N}^{\mathrm{out}}(a_{NN}^{\mathrm{in}})^{-1}+b_{1N}^{\mathrm{out}}(b_{NN}^{\mathrm{in}})^{-1} \\ \vdots & \ddots & \vdots \\ -a_{N1}^{\mathrm{out}}(a_{11}^{\mathrm{in}})^{-1}+b_{N1}^{\mathrm{out}}(b_{11}^{\mathrm{in}})^{-1} & \cdots & -a_{NN}^{\mathrm{out}}(a_{NN}^{\mathrm{in}})^{-1}+b_{NN}^{\mathrm{out}}(b_{NN}^{\mathrm{in}})^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(1)} \\ \vdots \\ \boldsymbol{x}^{(N)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_z^{(1)} \\ \vdots \\ \boldsymbol{e}_z^{(N)} \end{pmatrix}. \tag{6}$$

III. THE IDR(*s*) METHOD AND ITS VARIANTS

The IDR(*s*) method is one of the iterative solvers for nonsymmetric linear system of equations and is derived by a new approach different from conventional BiCG-like and GMRES-like solvers. In the conventional solvers, the solution vector is iteratively refined thorough a bi-orthgonalization scheme and a minimal norm scheme over the Krylov subspace, respectively [3]. In the IDR(*s*) method, whereas, spaces are iteratively generated from the complete Krylov space according to a rule. The solution vector is determined in order that the corresponding residual vector may belong to the spaces. From the IDR theorem, the dimensions of the spaces monotonically decrease, and the residual vector converges to the zero vector [6]. Thus, the IDR(*s*) method is a new solver which belongs to neither BiCG-like nor GMRES-like solvers.

The subsequent spaces are generated every *s+1* iteration, and we actually compute *s+1* basis vectors of corresponding space. Due to the induced dimension reduction, the residual vectors only for every *s+1*th iteration are expected to reduce monotonically. In order to accelerate convergence, Sonneveld and van Gijzen introduced the minimization scheme for residual vectors based on an orthonormalization. They also adopt a bi-orthogonalization scheme to minimize the residual vectors. The new solvers based on the above two techniques are called "MR-IDR(*s*) method" and "Bi-IDR(*s*) method", respectively.

A preconditioned MR-IDR(*s*) method and Bi-IDR(*s*) method are listed in Figs. 1 and 2, respectively. Here, we derive the preconditioned algorithms based on a right preconditioning. A pseudo code of a preconditioned IDR(*s*) method is already presented in [7]. The notations K^{-1} and *H* stand for a preconditioner and Hermitian adjoint, respectively.

1. Let $\boldsymbol{x}_0$ be an initial guess, and put $\boldsymbol{r}_0 = \boldsymbol{b} - \mathcal{A}\boldsymbol{x}_0$
2. $\mathcal{G} = \mathcal{U} = \mathcal{O},\ \mathcal{M} = \mathcal{I},\ \omega = 1,\ n = 0$
3. While $||\boldsymbol{r}_n||_2/||\boldsymbol{r}_0||_2 > \epsilon$ Do
4. For $i = 0, \ldots, s-1$ Do
5. Solve $\boldsymbol{c}$ from $\mathcal{M}\boldsymbol{c} = \mathcal{P}^H\boldsymbol{r}_n$
6. $\boldsymbol{v} = \boldsymbol{r}_n - \mathcal{G}\boldsymbol{c}$
7. $\boldsymbol{u}_n = \mathcal{U}\boldsymbol{c} + \omega\mathcal{K}^{-1}\boldsymbol{v},\ \ \boldsymbol{g}_n = \mathcal{A}\boldsymbol{u}_n$
8. For $j = 1, \ldots, i$ Do
9. $\boldsymbol{g}_n = \boldsymbol{g}_n - (\boldsymbol{g}_{n-j}^H\boldsymbol{g}_n)\boldsymbol{g}_{n-j}$
   $\boldsymbol{u}_n = \boldsymbol{u}_n - (\boldsymbol{g}_{n-j}^H\boldsymbol{g}_n)\boldsymbol{u}_{n-j}$
10. End Do
11. $\boldsymbol{g}_n = \dfrac{\boldsymbol{g}_n}{||\boldsymbol{g}_n||_2},\ \ \boldsymbol{u}_n = \dfrac{\boldsymbol{u}_n}{||\boldsymbol{g}_n||_2}$
12. $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - (\boldsymbol{g}_n^H\boldsymbol{r}_n)\boldsymbol{g}_n,$
    $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + (\boldsymbol{g}_n^H\boldsymbol{r}_n)\boldsymbol{u}_n$
13. $n = n + 1$
14. End Do
15. $\mathcal{G} = (\boldsymbol{g}_{n-1} \cdots \boldsymbol{g}_{n-s}),$
    $\mathcal{U} = (\boldsymbol{u}_{n-1} \cdots \boldsymbol{u}_{n-s})$
16. $\mathcal{M} = \mathcal{P}^H\mathcal{G}$
17. Solve $\boldsymbol{c}$ from $\mathcal{M}\boldsymbol{c} = \mathcal{P}^H\boldsymbol{r}_n$
18. $\boldsymbol{v} = \boldsymbol{r}_n - \mathcal{G}\boldsymbol{c},\ \ \boldsymbol{t} = \mathcal{A}\mathcal{K}^{-1}\boldsymbol{v}$
20. $\omega = \dfrac{\boldsymbol{t}^H\boldsymbol{v}}{\boldsymbol{t}^H\boldsymbol{t}}$
    $\left(\begin{array}{l} \rho = \dfrac{\boldsymbol{t}^H\boldsymbol{v}}{||\boldsymbol{t}||_2\,||\boldsymbol{v}||_2}, \\ \text{if } |\rho| < \kappa \text{ then } \omega = \dfrac{\kappa}{\rho}\omega \end{array} \right)$
21. $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \mathcal{U}\boldsymbol{c} + \omega\mathcal{K}^{-1}\boldsymbol{v}$
    $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \mathcal{G}\boldsymbol{c} - \omega\boldsymbol{t}$
22. $n = n + 1$
23. End While

Fig. 1. A preconditioned MR-IDR(*s*) algorithm.

1. Let $\boldsymbol{x}_0$ be an initial guess, and put
   $\boldsymbol{r}_0 = \boldsymbol{b} - \mathcal{A}\boldsymbol{x}_0$
2. $\boldsymbol{u}_i = \boldsymbol{g}_i = \boldsymbol{0}$ $(i = 0, \ldots, s-1)$,
   $\mathcal{M} = \mathcal{I}$, $\omega = 1$, $n = 0$
3. While $||\boldsymbol{r}_n||_2 / ||\boldsymbol{r}_0||_2 > \epsilon$ Do
4. $\boldsymbol{m} = \mathcal{P}^H \boldsymbol{r}_n$
5. For $i = 0, \ldots, s-1$ Do
6. Solve $\boldsymbol{c}$ from $\mathcal{M}\boldsymbol{c} = \boldsymbol{m}$
7. $\boldsymbol{v} = \boldsymbol{r}_n - \sum_{j=i}^{s-1} c_j \boldsymbol{g}_j$
8. $\boldsymbol{u}_i = \omega \mathcal{K}^{-1} \boldsymbol{v} + \sum_{j=i}^{s-1} c_j \boldsymbol{u}_i$, $\boldsymbol{g}_i = \mathcal{A}\boldsymbol{u}_i$
9. For $j = 0, \ldots, i-1$ Do
10. $\boldsymbol{g}_i = \boldsymbol{g}_i - \frac{\boldsymbol{p}_j^H \boldsymbol{g}_i}{M_{jj}} \boldsymbol{g}_j$
    $\boldsymbol{u}_i = \boldsymbol{u}_i - \frac{\boldsymbol{p}_j^H \boldsymbol{g}_i}{M_{jj}} \boldsymbol{u}_j$
11. End Do
12. $M_{ji} = \boldsymbol{p}_j^H \boldsymbol{g}_i$ $(j = i, \ldots, s-1)$
13. $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \frac{m_i}{M_{ii}} \boldsymbol{g}_i$,
    $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \frac{m_i}{M_{ii}} \boldsymbol{u}_i$
14. If $i \leq s-2$ then
15. $m_j = \begin{cases} 0 & (j = 0, \ldots, i) \\ m_j - \frac{m_i}{M_{ii}} M_{ji} & \\ & (j = i+1, \ldots, s-1) \end{cases}$
16. End If
17. $n = n + 1$
18. End Do
19. $\boldsymbol{t} = \mathcal{A}\mathcal{K}^{-1} \boldsymbol{r}_n$
20. $\omega = \frac{\boldsymbol{t}^H \boldsymbol{r}_n}{\boldsymbol{t}^H \boldsymbol{t}}$
    $\left(\rho = \frac{\boldsymbol{t}^H \boldsymbol{r}_n}{||\boldsymbol{t}||_2 \, ||\boldsymbol{r}_n||_2}, \right.$
    $\left. \text{if } |\rho| < \kappa \text{ then } \omega = \frac{\kappa}{\rho}\omega \right)$
21. $\boldsymbol{x}_n = \boldsymbol{x}_n + \omega \mathcal{K}^{-1} \boldsymbol{r}_n$, $\boldsymbol{r}_n = \boldsymbol{r}_n - \omega \boldsymbol{t}$
22. $n = n + 1$
23. End While

Fig. 2. A preconditioned Bi-IDR(s) algorithm.

The matrix P = ($\boldsymbol{p}_0$ $\boldsymbol{p}_1$ ⋯ $\boldsymbol{p}_{s-1}$) is an orthogonal matrix followed by [6]. The large bracket is Sleijpen and van der Vorst's convergence refinement technique. We fix the parameter κ at *0.7* that is recommended by them [8]. This technique has the effect to reduce a spurious convergence for the IDR(*s*) method [7]. As shown in the pseudo codes, solving a linear system of *s* equations is necessary for the IDR(*s*) methods. We do it by using the direct method with a LU factorization in the original IDR(*s*) method and the MR-IDR(*s*) method. In the Bi-IDR(*s*) method, the coefficient matrix M is a lower triangular matrix, and we obtain solution vector $\boldsymbol{c}$ by a forward substitution.

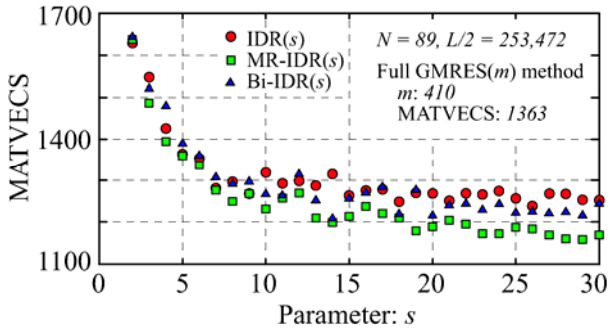
IV. PERFORMANCE EVALUATION

We treat EM wave scattering from regularly placed $N \times N$ dielectric circular cylinders whose configurations are listed in Tab. 1. The incident wave is assumed as a plane wave. Comparison among a various kinds of IDR(*s*) methods is carried out in terms of the convergence and accuracy. It notes that the computation time is directly connected with the number of matrix-vector multiplications to convergence (abbreviate to ``MATVECS", hereafter) [5]. Computations are performed on Intel Core2Duo E6700 processor and 2GB of main memory. An iterative process is begun with $\boldsymbol{x}_0 = \boldsymbol{0}$, and the stopping criterion ε is put at 10^{-10}. A matrix-vector multiplication in an iterative process is expedited by the wideband fast multipole algorithm with the tolerance of 10^{-10} [2]. We exploit the block Jacobi preconditioning.

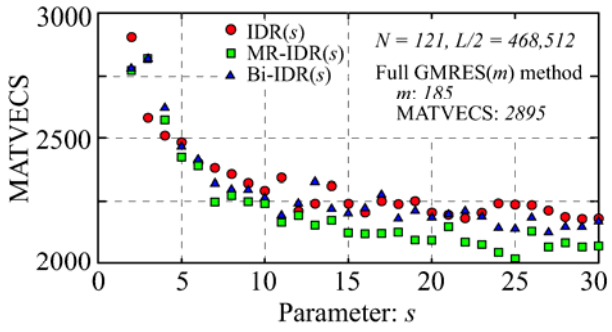
Results of performance evaluation are displayed in Figs. 3 and 4. Here, the accuracy for converged solution vector $\boldsymbol{x}_n$ is estimated by $||\mathrm{A}\boldsymbol{x}_n - \boldsymbol{b}||_2 / ||\boldsymbol{b}||_2$. Performances of a full GMRES(*m*) method are exhibited in each figure. The full GM-

Table 1: Physical parameters for cylinders

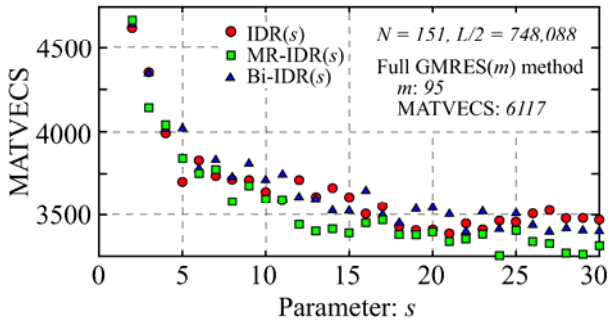
Normalized radius: $k_0 a$	*1.0*
Relative permittivity: $\varepsilon_r^{(i)}$	*2.0*
Relative permeability: $\mu_r^{(i)}$	*1.0*
Length between each cylinder: $k_0 d$ (in row and column directions)	$\sqrt{100\pi}\, k_0 a$
Fractional volume: *f*	*0.01*



(a) $N = 89,\ L/2 = 253,472$

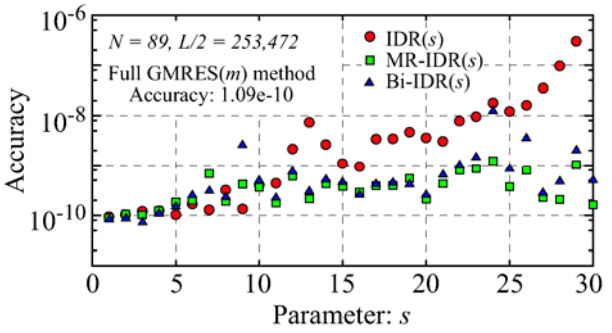


(b) $N = 121,\ L/2 = 468,512$

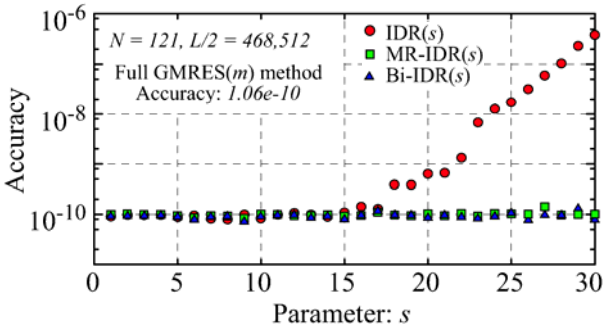


(c) $N = 153,\ L/2 = 748,088$

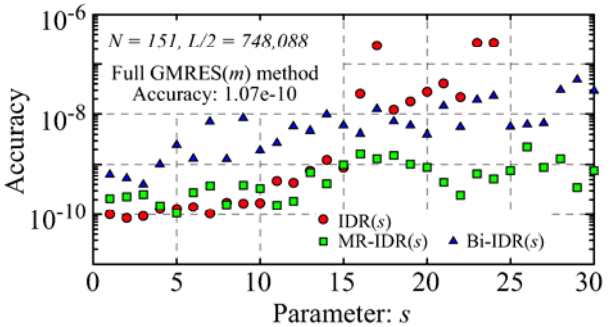
Fig. 3. Fluctuation of MATVECS for parameter *s* of the preconditioned IDR(*s*) methods.



(a) $N = 89,\ L/2 = 253,472$



(b) $N = 121,\ L/2 = 468,512$

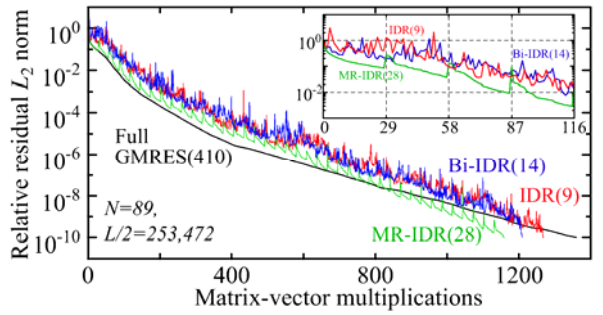


(c) $N = 153,\ L/2 = 748,088$

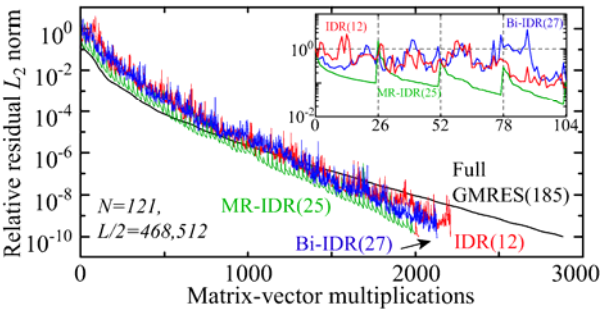
Fig. 4. Fluctuation of the accuracy for parameter *s* of the preconditioned IDR(*s*) methods.

RES (*m*) method means that the restart cycle *m* is decided in order that we may use over *90%* of the amount of memory installed. These figures expose that both the MR-IDR(*s*) and Bi-IDR(*s*) methods refine not only the convergence but also accuracy for the original IDR(*s*) method. The MR-IDR(*s*) method is the best in the three types of IDR(*s*) method in terms of convergence and accuracy, and converges faster than the full GMRES (*m*) method. Unfortunately, the spurious convergence is confirmed in figure 4 but its feature is different among the solvers. Deterioration of accuracy for the MR-IDR(*s*) and Bi-IDR(*s*) methods do not always occur and its impact may be in *1* or *2* digit error independently of parameter *s*. The authors are considered that a mechanism of the spurious convergence for the MR-IDR(*s*) and Bi-IDR(*s*) methods are identical but differ from that for the original IDR(*s*) method.

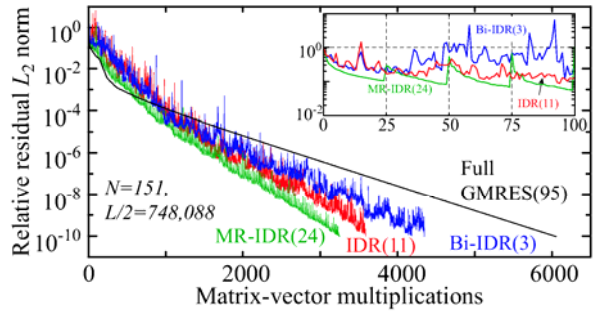
Figure 5 exhibits convergence behaviors of a variety of IDR(*s*) methods with an optimal parameter and the full GMRES (*m*) method. The optimal parameter means that MATVECS is minimum and the deterioration of accuracy is under *1* digit. We can find from Fig. 5 that the full GMRES (*m*) method rapidly converge at first but the convergence curve of it becomes slow due to restart.



(a) $N = 89,\ L = 253,472$



(b) $N = 121,\ L = 468,512$



(c) $N = 153,\ L = 748,088$

Fig. 5. Convergence of the preconditioned IDR(*s*) methods and the full GMRES (*m*) method.

The convergence curves of the IDR(*s*) methods have a thumping vibration, but it reaches the value of stopping criterion at a constant rate. The magnification of the curves for first several iterations is pasted on each figure. We can see from them that the convergence curve of the MR-IDR(*s*) method is smoother than other two types of the IDR(*s*) methods. The needles in the convergence curve for the MR-IDR(*s*) method are confirmed every *s+1* iterations. This may be caused by the update space in algorithm.

Table 2 is the comparison of the convergence, the memory used, and accuracy among the variant IDR(*s*) methods for the top three parameters and the GMRES (*50*), GMRES (*100*) and full GMRES (*m*) methods. Here, the top three parameters mean that MATVECS is minimum and the accuracy is around the stopping criterion.

Table 2: Comparison between the IDR(*s*) methods and the GMRES (*m*) methods

(a) $N = 89,\ L/2 = 253,472$

Method	MATVECS (ratio)	Mem. [MB] (ratio)	Accuracy
IDR(9)	1268 (0.93)	385 (0.21)	1.33e-10
IDR(7)	1280 (0.94)	362 (0.19)	1.27e-10
IDR(6)	1354 (0.99)	350 (0.19)	1.67e-10
MR-IDR(28)	1159 (0.85)	612 (0.33)	1.99e-10
MR-IDR(30)	1169 (0.86)	636 (0.34)	1.61e-10
MR-IDR(27)	1169 (0.86)	600 (0.32)	2.28e-10
Bi-IDR(14)	1209 (0.89)	445 (0.24)	5.40e-10
Bi-IDR(18)	1218 (0.89)	493 (0.26)	4.47e-10
Bi-IDR(20)	1216 (0.89)	517 (0.28)	2.64e-10
GMRES(50)	1783 (1.31)	468 (0.25)	1.10e-10
GMRES(100)	1662 (1.22)	662 (0.35)	1.06e-10
GMRES(410)	1363 (1.00)	1865 (1.00)	1.09e-10

(b) $N = 121,\ L/2 = 468,521$

Method	MATVECS (ratio)	Mem. [MB] (ratio)	Accuracy
IDR(10)	2291 (0.79)	765 (0.85)	8.38e-11
IDR(9)	2320 (0.80)	744 (0.82)	9.80e-11
IDR(8)	2351 (0.81)	723 (0.80)	7.96e-11
MR-IDR(25)	2020 (0.70)	1080 (0.58)	1.01e-10
MR-IDR(24)	2046 (0.71)	1059 (0.57)	9.99e-11
MR-IDR(29)	2067 (0.71)	1164 (0.62)	9.98e-11
Bi-IDR(27)	2129 (0.74)	1122 (0.60)	9.89e-11
Bi-IDR(25)	2143 (0.74)	1080 (0.58)	1.06e-10
Bi-IDR(24)	2146 (0.74)	1059 (0.57)	9.33e-11
GMRES(50)	3661 (1.26)	905 (0.48)	1.06e-10
GMRES(100)	3291 (1.14)	1262 (0.67)	1.07e-10
GMRES(185)	2895 (1.00)	1870 (1.00)	1.06e-10

(c) $N = 153,\ L/2 = 749,088$

Method	MATVECS (ratio)	Mem. [MB] (ratio)	Accuracy
IDR(10)	3639 (0.59)	1146 (0.63)	1.70e-10
IDR(5)	3702 (0.61)	977 (0.53)	1.30e-10
IDR(9)	3710 (0.61)	1112 (0.61)	1.64e-10
MR-IDR(24)	3249 (0.53)	1620 (0.89)	4.14e-10
MR-IDR(29)	3261 (0.53)	1789 (0.98)	3.45e-10
MR-IDR(28)	3269 (0.53)	1755 (0.96)	8.29e-10
Bi-IDR(3)	4349 (0.71)	909 (0.50)	4.08e-10
Bi-IDR(2)	4655 (0.76)	875 (0.48)	4.57e-10
Bi-IDR(1)	7827 (1.28)	841 (0.46)	6.46e-10
GMRES(50)	6647 (1.09)	1366 (0.75)	1.06e-10
GMRES(95)	6117 (1.00)	1820 (1.00)	1.07e-10
GMRES(100)	- (-)	- (-)	-

Here, we cannot execute the GMRES (*100*) method for *N = 153* case due to the shortage of memory space. The ratio to the results of the full GMRES (*m*) method is written in the columns of MATVECS and Mem. in Tab. 2.

Figures 2 and 3 and Tab. 2 disclose that the optimal parameters for the IDR(*s*) and MR-IDR(*s*) methods may be in $5 \le s \le 10$ and $25 \le s \le 30$, respectively. However, that for the Bi-IDR(*s*) method is hard to determine because the top three parameters are quite different in each problem. The amount of used memory of the IDR(*s*) methods increases proportional to parameter *s*. Then the MR-IDR(*s*) method with optimal parameters needs more memory space than the IDR(*s*) and Bi-IDR(*s*) methods.

The IDR(*s*) methods with the optimal parameter converge faster than the full GMRES (*m*) method, and the difference of MATVECS becomes larger accordingly to the enlargement of problem size. The amount of used memory is less for the IDR(*s*) methods than for the full GMRES (*m*) method. Therefore, we conclude that IDR(*s*) method is better than the GMRES (*m*) method in terms of not only convergence but also the amount of memory used in the large scale computation of EM wave scattering from many objects. Especially, the MR-IDR(*s*) method is the best among the three types of IDR(*s*) method in terms of convergence and accuracy. However, spurious convergence may occur, and the user has to check the accuracy for convergent solution.

V. CONCLUDING REMARKS

This paper remarks variants of IDR(*s*) method. Preconditioned MR-IDR(*s*) and Bi-IDR(*s*) algorithms are presented. Performance evaluations are done for the computation of the dense linear system of equations of order 10^5 followed by the BEM analysis of EM wave multiple scattering. As a consequence, the MR-IDR(*s*) method is the best among the three types of IDR(*s*) method in terms of convergence and accuracy. Optimal parameter for the MR-IDR(*s*) method may be in $25 \le s \le 30$. The MR-IDR(*s*) method with optimal parameter converges faster and spends less memory space than the full GMRES (*m*) method. However, the accuracy the MR-IDR(*s*) method may deteriorate in *1* digit independently of parameter *s*. The investigation and provision of it are important subjects.

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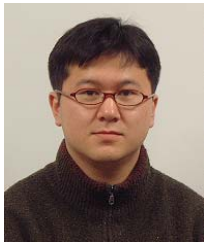

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