The behavior of smart obstacles in electromagnetic scattering: mathematical models as optimal control problems

Lorella Fatone[1], Maria Cristina Recchioni[2], Adina Scoccia[2], Francesco Zirilli[3]

[1] Dipartimento di Matematica Pura e Applicata, University of Modena e Reggio Emilia, Modena, Italy
[2] Dipartimento di Scienze Sociali "D. Serrani", University Politecnica delle Marche, Ancona, Italy
[3] Dipartimento di Matematica "G. Castelnuovo", University of Roma "La Sapienza", Roma, Italy

Abstract

We consider a bounded obstacle characterized by a boundary electromagnetic impedance contained in the three dimensional real Euclidean space filled with a homogeneous isotropic medium. When an incoming electromagnetic field illuminates the obstacle a scattered field is generated. A smart obstacle is an obstacle that in the scattering process, circulating a surface electric current density on its boundary, tries to achieve a given goal. We consider four possible goals: making the obstacle undetectable (i.e.: furtivity problem), making the obstacle to appear with a shape and impedance different from its actual ones (i.e.: masking problem), making the obstacle to appear in a location different from its actual one eventually with a shape and impedance different from its actual ones (i.e.: ghost obstacle problem) and finally one of the previous goals limited to a given subset of the frequency space (i.e.: definite band problems). We consider the problem of determining the optimal electric current density to achieve the given goal. The relevance in many application fields (i.e. stealth technology, electromagnetic noise control, etc.) of these problems is well known. The previous problems are modelled as optimal control problems for the Maxwell equations. Some numerical results on test problems obtained solving the optimal control problems proposed are shown.

1. Introduction

In recent years the development of new technologies has made possible to build a vast class of "smart" objects. This wave of innovation has moved from cutting edge military applications to everyday life objects such as, for example, washing machines. In this paper we consider the problem of formulating adequate mathematical models of smart obstacles in the context of electromagnetic scattering. The general mathematical model that we have in mind to describe the behavior of a "smart" object is an optimal control problem. The problems considered in electromagnetic obstacle scattering are described by partial differential equations so that we deal with optimal control problems for partial differential equations. Optimal control problems are widely used in engineering as mathematical models. However their use is mainly limited to the control of systems governed by ordinary differential equations and their use in electromagnetic scattering is rather uncommon. The development of computer technology and numerical methods occurred in the last decades makes possible now to consider optimal control problems for systems of partial differential equations such as the Maxwell equations, that is makes possible the use of optimal control to model electromagnetic scattering problems. We consider four examples: furtivity problem (i.e.: the obstacle wants to be undetectable), masking problem (i.e.: the obstacle wants to appear with a shape different from its actual shape eventually with a boundary impedance different from its actual one), ghost obstacle problem (i.e.: the obstacle wants to appear in a location different from its actual location eventually with a shape and impedance different from its actual ones) and finally definite band problems (i.e.: the obstacle pursues one of the previous goals on a given subset of the frequency space). Recently similar problems in the context of time dependent acoustic and electromagnetic obstacle scattering have been studied from the point of view of formulating adequate mathematical models and of developing highly parallelizable numerical methods to solve them (see [1], [2], [3], [4], [5] and the websites: *http://www.econ.univpm.it/recchioni/w6*, *http://www.econ.univpm.it/recchioni/w8*, *http://www.econ.univpm.it/recchioni/w9*, *http://www.econ.univpm.it/recchioni/w10*, *http://www.econ.univpm.it/recchioni/w11*). Note that in these papers "smart" and "active" obstacles are synonyms. More in detail in [1] the furtivity problem in acoustic time dependent obstacle scattering has been modelled as an optimal control problem and the first order optimality conditions to solve it have been obtained as a system of coupled partial differential equations, finally a highly parallelizable numerical solver for this system of partial differential equations has been developed. Later in [2], in [4] and in [5] the masking problem in acoustics and the furtivity and the masking problems in electromagnetics have been studied and finally in [3] the definite band ghost obstacle problem in acoustics has been solved.

The practical interest of the mathematical models of smart obstacles proposed consists in the fact that these models can be used to design smart obstacles of practical value. Hence, for example, in the realization of radar absorbers the approach proposed can be a way of approaching the design of phase-switched screens (see for example [6], [7], [8]). In fact the phase-switched screen is an object that does not absorb the incident energy but shifts it in frequency using phase modulation so that the reflected energy falls outside the receiver bandwidth. That is, a phase-switched screen in our language can be seen as a smart obstacle that pursues the goal of being furtive in a given subset of the frequency space.

In Section 2 we formulate the mathematical models of the electromagnetic smart obstacles considered. In Section 3 we show some numerical results obtained solving the model proposed in Section 2 concerning the definite band furtivity problem.

2. Mathematical models of electromagnetic smart obstacles

Let us begin introducing some notations. Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}^3$ be the three dimensional real Euclidean space and $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbf{R}^3$ be a generic vector, where the superscript T means transposed. We denote with $(\cdot, \cdot)$ the Euclidean scalar product in $\mathbf{R}^3$, with $\|\cdot\|$ the corresponding Euclidean vector norm and with $[\cdot, \cdot]$ the usual vector product. Let $\mathbf{R}^3$ be filled with a homogeneous isotropic medium of constant electric permittivity $\epsilon > 0$, constant magnetic permeability $\upsilon > 0$ and zero electric conductivity. Moreover we assume that there are no free electric charges or currents. Let us suppose that $\mathbf{R}^3$ contains an obstacle Ω given by a bounded set without holes and internal cavities, more technically, a bounded simply connected open set Ω, with locally Lipschitz boundary $\partial\Omega$. Let $\overline{\Omega}$ denote the set

$\Omega \cup \partial\Omega$ and $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x}), n_3(\mathbf{x}))^T \in \mathbf{R}^3$, $\mathbf{x} \in \partial\Omega$ be the outward unit normal vector to $\partial\Omega$ in $\mathbf{x} \in \partial\Omega$. In the following Ω will be the scatterer, that is the obstacle responsible for the scattering of the incoming electromagnetic field. We assume that Ω has a known constant real boundary electromagnetic impedance $\chi \geq 0$. The limit case of perfectly insulating obstacles (i.e. $\chi = +\infty$) can be treated with straightforward modifications of the material presented here.

We begin modelling the standard direct obstacle scattering problem that is, the scattering problem relative to an obstacle that does not pursue any goal. We refer to this obstacle as a "passive" obstacle.

We consider an incoming electromagnetic field $(\boldsymbol{E}^i(\mathbf{x},t), \boldsymbol{B}^i(\mathbf{x},t))$, $(\mathbf{x},t) \in \mathbf{R}^3 \times \mathbf{R}$. The electric vector field $\boldsymbol{E}^i(\mathbf{x},t) \in \mathbf{R}^3$, $(\mathbf{x},t) \in \mathbf{R}^3 \times \mathbf{R}$ and the magnetic induction vector field $\boldsymbol{B}^i(\mathbf{x},t) \in \mathbf{R}^3$, $(\mathbf{x},t) \in \mathbf{R}^3 \times \mathbf{R}$ associated to the incoming electromagnetic field satisfy the Maxwell equations, that is equations (1), (2), for $(\mathbf{x},t) \in \mathbf{R}^3 \times \mathbf{R}$. We use the M.K.S. unit system to write equations (1), (2) (see [9], p. 16). When the incoming electromagnetic field $(\boldsymbol{E}^i(\mathbf{x},t), \boldsymbol{B}^i(\mathbf{x},t))$, $(\mathbf{x},t) \in \mathbf{R}^3 \times \mathbf{R}$, hits the scatterer Ω generates a scattered electromagnetic field $(\boldsymbol{E}^s(\mathbf{x},t), \boldsymbol{B}^s(\mathbf{x},t))$, $(\mathbf{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$, solution of an exterior problem for the Maxwell equations. That is the scattered electric vector field $\boldsymbol{E}^s(\mathbf{x},t) \in \mathbf{R}^3$, $(\mathbf{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ and the scattered magnetic induction vector field $\boldsymbol{B}^s(\mathbf{x},t) \in \mathbf{R}^3$, $(\mathbf{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ satisfy the following equations,

$$\left(\operatorname{curl} \boldsymbol{E}^s + \frac{\partial \boldsymbol{B}^s}{\partial t}\right)(\mathbf{x},t) = \mathbf{0},$$
$$\left(\operatorname{curl} \boldsymbol{B}^s - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^s}{\partial t}\right)(\mathbf{x},t) = \mathbf{0},$$
$$(\mathbf{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \qquad (1)$$

$$\operatorname{div} \boldsymbol{B}^s(\mathbf{x},t) = 0, \operatorname{div} \boldsymbol{E}^s(\mathbf{x},t) = 0,$$
$$(\mathbf{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \qquad (2)$$

with the boundary condition,

$$[\mathbf{n}(\mathbf{x}), \boldsymbol{E}^s(\mathbf{x},t)] - c\chi\,[\mathbf{n}(\mathbf{x}), [\mathbf{n}(\mathbf{x}), \boldsymbol{B}^s(\mathbf{x},t)]] =$$
$$[\mathbf{n}(\mathbf{x}), \boldsymbol{b}(\mathbf{x},t)]\,, (\mathbf{x},t) \in \partial\Omega \times \mathbf{R}, \qquad (3)$$

where,

$$\boldsymbol{b}(\mathbf{x},t) = -\boldsymbol{E}^i(\mathbf{x},t) + c\chi\left[\mathbf{n}(\mathbf{x}), \boldsymbol{B}^i(\mathbf{x},t)\right],$$
$$(\mathbf{x},t) \in \partial\Omega \times \mathbf{R}, \qquad (4)$$

the condition at infinity and the radiation condition given respectively by,

$$[\boldsymbol{B}^s(\mathbf{x},t), \hat{\mathbf{x}}] - \frac{1}{c}\boldsymbol{E}^s(\mathbf{x},t) = o\left(\frac{1}{r}\right),$$
$$\boldsymbol{E}^s(\mathbf{x},t) = O\left(\frac{1}{r}\right),\ r \to +\infty,\ t \in \mathbf{R}, \quad (5)$$

where $\mathbf{0} = (0,0,0)^T$, $c = 1/\sqrt{\epsilon\upsilon}$, $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$, $\mathbf{x} \neq \mathbf{0}$, $r = \|\mathbf{x}\|$, curl $\cdot$ and div$\cdot$ denote respectively the curl and the divergence of $\cdot$ with respect to the $\mathbf{x}$ variable, $\partial \cdot /\partial t$ denotes the time derivative of $\cdot$, and $o(\cdot)$, $O(\cdot)$ are the Landau symbols. When we consider the case $\chi = +\infty$ the boundary condition (3) must be "rewritten". The two conditions contained in (5) imply the vanishing of the magnetic induction vector field at infinity, that is $\boldsymbol{B}^s(\mathbf{x},t) = O(1/r)$, $r \to +\infty$, $t \in \mathbf{R}$. Moreover we assume that the incoming electromagnetic field vanishes when $t \to -\infty$, that is $\boldsymbol{E}^i(\mathbf{x},t)$, $\boldsymbol{B}^i(\mathbf{x},t) \to \mathbf{0}$, $\mathbf{x} \in \mathbf{R}^3$, $t \to -\infty$, that implies that the scattered electromagnetic field vanishes when $t \to -\infty$ as well, that is $\boldsymbol{E}^s(\mathbf{x},t)$, $\boldsymbol{B}^s(\mathbf{x},t) \to \mathbf{0}$, $\mathbf{x} \in (\mathbf{R}^3 \setminus \overline{\Omega})$, $t \to -\infty$.

The scattering problem for a "passive" obstacle Ω can be stated as follows:

Scattering Problem (passive obstacle). Given the incident electromagnetic field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$, the obstacle Ω and its boundary electromagnetic impedance χ, solve the time dependent Maxwell equations (1)-(3), (5) in the unknowns $(\boldsymbol{E}^s, \boldsymbol{B}^s)$.

Let us study the possibility of transforming the "passive" obstacle into a "smart" obstacle.

Problem 1. Furtivity Problem. Given the incident electromagnetic field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$, the obstacle Ω and its boundary electromagnetic impedance χ choose a control vector field (i.e. a surface electric current density) defined for $(\mathbf{x},t) \in \partial\Omega \times \mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude" of the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) when hit by the incoming field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed.

To obtain a satisfactory formulation of the furtivity problem we modify the boundary condition (3) as follows,

$$[\mathbf{n}(\mathbf{x}), \boldsymbol{E}^s(\mathbf{x},t)] - c\chi\,[\mathbf{n}(\mathbf{x}), [\mathbf{n}(\mathbf{x}), \boldsymbol{B}^s(\mathbf{x},t)]] = [\mathbf{n}(\mathbf{x}), \boldsymbol{b}(\mathbf{x},t)] + (1+\chi)[\mathbf{n}(\mathbf{x}), \boldsymbol{\Phi}(\mathbf{x},t)], \quad (\mathbf{x},t) \in \partial\Omega \times \mathbf{R}. \tag{6}$$

The quantity $\boldsymbol{\Phi}(\mathbf{x},t)$, $(\mathbf{x},t) \in \partial\Omega \times \mathbf{R}$ has the dimension of an electric field and is related to the control variable that transforms the obstacle Ω from being passive to being smart. We assume that $\lim_{t\to\pm\infty} \boldsymbol{\Phi}(\mathbf{x},t) = \mathbf{0}$, $\mathbf{x} \in \partial\Omega$.

Let us define $\boldsymbol{\psi}(\mathbf{x},t) = \frac{\partial \boldsymbol{\Phi}}{\partial t}(\mathbf{x},t)$, $(\mathbf{x},t) \in \partial\Omega\times\mathbf{R}$ and let V be the space of the admissible controls, that we leave undetermined in this paper (see [5]) for a definition of V). Note that $\boldsymbol{\psi} = \frac{\partial \boldsymbol{\Phi}}{\partial t}$ has the dimensions of an electric (surface) current density. The furtivity problem can be formulated as the following optimal control problem,

$$\min_{\boldsymbol{\psi}\in V} \mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi})\,, \tag{7}$$

subject to the constraints (1), (2), (5), (6) and $\mathcal{F}_{\lambda,\mu}$ is the following functional,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\left\{\lambda|||\,[\mathbf{n}, \boldsymbol{E}^s]\,|||^2 + \lambda c^2|||\,[\mathbf{n}, \boldsymbol{B}^s]\,|||^2 + \mu\varsigma|||\,[\mathbf{n}, \boldsymbol{\psi}]\,|||^2\right\}. \tag{8}$$

The quantity ς is a positive dimensional constant and $\lambda \geq 0$, $\mu \geq 0$ are adimensional constants such that $\lambda + \mu = 1$. Moreover the norms $|||\cdot|||$ appearing in (8) are norms on a suitable space of functions defined on $\partial\Omega\times\mathbf{R}$ (see [5]). For example the square root of the integral over $\partial\Omega\times\mathbf{R}$ of the square of the vector norm of $\cdot$ is such a norm. Note that the solution of problem (7), (1), (2), (5), (6) when $\lambda = 0$, $\mu = 1$ is $[\mathbf{n}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x},t)] = \mathbf{0}$, $(\mathbf{x},t) \in \partial\Omega\times\mathbf{R}$, that is in this case $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ is the electromagnetic field scattered by the passive obstacle. On the other hand when $\lambda = 1$, $\mu = 0$ the solution of the same problem gives an obstacle completely undetectable since the minimization of (8) in this case gives $[\mathbf{n}(\mathbf{x}), \boldsymbol{E}^s(\mathbf{x},t)] = \mathbf{0}$, $[\mathbf{n}(\mathbf{x}), \boldsymbol{B}^s(\mathbf{x},t)] = \mathbf{0}$, $(\mathbf{x},t) \in \partial\Omega\times\mathbf{R}$ that implies $\boldsymbol{E}^s(\mathbf{x},t) = \mathbf{0}$, and $\boldsymbol{B}^s(\mathbf{x},t) = \mathbf{0}$, $(\mathbf{x},t) \in (\mathbf{R}\setminus\overline{\Omega})\times\mathbf{R}$. However when $\lambda = 1$, $\mu = 0$ the cost functional (8) does not contain a term that depends on the control employed $\boldsymbol{\psi}$. Note that $\boldsymbol{E}^s$, $\boldsymbol{B}^s$ depends implicitly on $\boldsymbol{\psi}$ through the boundary condition (6). The remaining cases, that is $0 < \lambda < 1$, correspond to nontrivial formulations of the furtivity problem.

Problem 2. Masking Problem. In the circumstances of Problem 1 given an obstacle $D \subseteq \Omega$, and its electromagnetic boundary impedance χ_D, choose a control vector field $\boldsymbol{\psi}(\mathbf{x},t)$, $(\mathbf{x},t) \in \partial\Omega\times\mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude of the difference" between the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) and the electromagnetic field $(\boldsymbol{E}^s_D, \boldsymbol{B}^s_D)$ scattered by D, χ_D when hit by the incoming field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed. The couple D, χ_D will be called the "mask". For simplicity we assume the mask to be a passive obstacle.

The Masking Problem can be modelled as the optimal control problem (7), (1), (2), (5), (6) if the functional $\mathcal{F}_{\lambda,\mu}$ that appears in (7) is defined as follows,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\left\{\lambda|||\,[\mathbf{n}, \boldsymbol{E}^s - \boldsymbol{E}^s_D]\,|||^2 + \lambda c^2|||\,[\mathbf{n}, \boldsymbol{B}^s - \boldsymbol{B}^s_D]\,|||^2 + \mu\varsigma|||\,[\mathbf{n}, \boldsymbol{\psi}]\,|||^2\right\} \tag{9}$$

Problem 3. Ghost Obstacle Problem. In the circumstances of Problem 1 given an obstacle G such that $G \neq \emptyset$, $\overline{G}\cap\overline{\Omega} = \emptyset$, its electromagnetic boundary impedance χ_G, and a bounded set without holes and internal cavities Ω_1 such that $\overline{\Omega}$, $\overline{G}$ are contained in Ω_1 and $\partial\Omega_1$ is a sufficiently regular surface, choose a control vector field $\boldsymbol{\psi}(\mathbf{x},t)$, $(\mathbf{x},t) \in \partial\Omega\times\mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures in $(\mathbf{R}^3 \setminus \overline{\Omega}_1) \times \mathbf{R}$ the "magnitude of the difference" between the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) and the electromagnetic field $(\boldsymbol{E}^s_G, \boldsymbol{B}^s_G)$ scattered by G, χ_G when hit by the incoming field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed. The couple G, χ_G will be called "ghost obstacle". For simplicity we assume the "ghost obstacle" to be a passive obstacle. The Ghost Obstacle Problem can be modelled as the optimal control problem (7), (1), (2), (5), (6) if the functional $\mathcal{F}_{\lambda,\mu}$ that appears in (7) is defined as follows,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\left\{\lambda\|\!|\,[\mathbf{n}, \boldsymbol{E}^s - \boldsymbol{E}^s_G]\,\|\!|_1^2 + \lambda c^2 \|\!|\,[\mathbf{n}, \boldsymbol{B}^s - \boldsymbol{B}^s_G]\,\|\!|_1^2 + \mu\varsigma \|\!|\,[\mathbf{n}, \boldsymbol{\psi}]\,\|\!|^2\right\}, \quad (10)$$

where $\|\!|\cdot\|\!|_1$ is a norm on a suitable space of functions defined on $\partial\Omega_1 \times \mathbf{R}$.

Finally we formulate the so called Definite Band Problems.

Let $K \subseteq \mathbf{R}$ be an assigned set of the frequency space that we assume to be an open interval symmetric with respect to the origin, let $\check{I}_K(t)$, $t \in \mathbf{R}$ be the inverse Fourier transform of the characteristic function of the set K and let us denote with $f * g$ the convolution product with respect to the time variable of the functions f and g. The set K is the definite band in the frequency space where the smart obstacle pursues its goal.

Problem 4. Definite Band Furtivity Problem. In the circumstances of Problem 1 given K choose a control vector field $\boldsymbol{\psi}(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude" in the frequency band K ($K \subset \mathbf{R}$) of the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) when hit by the incoming field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed.

The Definite Band Furtivity Problem can be modelled as the optimal control problem (7), (1), (2), (5), (6) if the functional $\mathcal{F}_{\lambda,\mu}$ that appears in (7) is defined as follows,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\left\{\lambda\|\!|\,\check{I}_K * [\mathbf{n}, \boldsymbol{E}^s]\,\|\!|^2 + \lambda c^2 \|\!|\,\check{I}_K * [\mathbf{n}, \boldsymbol{B}^s]\,\|\!|^2 + \mu\varsigma \|\!|\,[\mathbf{n}, \boldsymbol{\psi}]\,\|\!|^2\right\}. \quad (11)$$

Similarly we can consider the remaining goals on a definite band:

Problem 5. Definite Band Masking Problem. In the circumstances of Problem 1 given K, an obstacle $D \subseteq \Omega$, and its electromagnetic boundary impedance χ_D, choose a control vector field $\boldsymbol{\psi}(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude of the difference" in the frequency band K between the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) and the electromagnetic field $(\boldsymbol{E}^s_D, \boldsymbol{B}^s_D)$ scattered by D, χ_D when hit by the incoming field $(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed.

The Definite Band Masking Problem can be modelled as the optimal control problem (7), (1), (2), (5), (6) if the functional $\mathcal{F}_{\lambda,\mu}$ that appears in (7) is defined as follows,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\left\{\lambda\|\!|\check{I}_K * [\mathbf{n}, \boldsymbol{E}^s - \boldsymbol{E}^s_D]\|\!|^2 + \lambda c^2 \|\!|\check{I}_K * [\mathbf{n}, \boldsymbol{B}^s - \boldsymbol{B}^s_D]\|\!|^2 + \mu\varsigma \|\!|[\mathbf{n}, \boldsymbol{\psi}]\|\!|^2\right\}. \quad (12)$$

Problem 6. Definite Band Ghost Obstacle Problem. In the circumstances of Problem 1 given K, an obstacle G such that $G \neq \emptyset$, $\overline{G} \cap \overline{\Omega} = \emptyset$, its electromagnetic boundary impedance χ_G, and a bounded set without holes and internal cavities Ω_1 such that $\overline{\Omega}$, $\overline{G} \subset \Omega_1$, and $\partial\Omega_1$ is sufficiently regular choose a control vector field $\boldsymbol{\psi}(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \partial\Omega \times \mathbf{R}$ in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude of the difference" in the frequency band K between the electromagnetic field $(\boldsymbol{E}^s, \boldsymbol{B}^s)$ scattered by Ω, χ (when the control vector field is active) and the electromagnetic field $(\boldsymbol{E}^s_G, \boldsymbol{B}^s_G)$ scattered by G, χ_G when hit by the incoming field

$(\boldsymbol{E}^i, \boldsymbol{B}^i)$ and the "magnitude" of the control vector field employed.

The Definite Band Ghost Obstacle Problem can be modelled as the optimal control problem (7), (1), (2), (5), (6) if the functional $\mathcal{F}_{\lambda,\mu}$ that appears in (7) is defined as follows,

$$\mathcal{F}_{\lambda,\mu}(\boldsymbol{\psi}) = (1+\chi)\big\{\lambda \||\, \check{I}_K * [\mathbf{n}, \boldsymbol{E}^s - \boldsymbol{E}^s_G]\, \||_1^2 + \lambda c^2 \||\, \check{I}_K * [\mathbf{n}, \boldsymbol{B}^s - \boldsymbol{B}^s_G]\, \||_1^2 + \mu\varsigma \||\, [\mathbf{n}, \boldsymbol{\psi}]\, \||^2 \big\}. \quad (13)$$

Note that the Definite Band Problems formulated, that is Problems 4, 5, 6, are generalizations of Problems 1, 2, 3. In fact when we choose $K = \mathbf{R}$ the Definite Band Furtivity, Masking and Ghost Obstacle Problems reduce respectively to the Furtivity, Masking and Ghost Obstacle Problems. The advantage of solving the Definite Band Problems rather than the corresponding problems on the entire frequency space is that the "price" to be paid in term of the control variable employed is smaller when the Definite Band Problems are considered. In fact as shown in [10] in the acoustic case in the Definite Band Furtivity and Ghost Obstacle Problems the "quantity" of the control variable, measured by the norm used in the cost functional, required to get a given furtivity effect (or to get a given "ghost" effect) in the frequency band K is smaller than the "quantity" of the control variable needed to get the same effect on the entire frequency space (i.e. when $K = \mathbf{R}$).

A straightforward mode to solve the six control problems formulated here is the use of an optimization routine and a numerical solver for the Maxwell equations. This approach is computationally very expansive since it implies the solution of the Maxwell equations (several times due to the necessity of estimating gradient and eventually "Hessian" of the cost functionals involved in the control problems) at each iteration of the optimization procedure. A computationally cheaper approach can be obtained using the Pontryagin maximum principle. In fact under some hypotheses using the Pontryagin maximum principle it is possible to write the first order optimality conditions corresponding to these control problems as a system of partial differential equations with the necessary boundary, initial and final conditions. Highly parallelizable numerical methods can be developed to solve these systems of partial differential equation. For brevity we refer the interested reader to [4], and [5], [3], [10]. The numerical results obtained in Section 3 have been obtained using the Pontryagin maximum principle. In fact, we have derived the first order optimality conditions, i.e. a system of partial of differential equations for Problem 1 and a similar system for Problem 6 and then we have solved these systems developing suitable solvers based on the operator expansion method presented in [5] and [11].

3. Some numerical results

We present some numerical results relative to two experiments involving smart obstacles. In both experiments we choose $c = 1$, $\varsigma = 1$ and the following electromagnetic incoming field,

$$\boldsymbol{E}^i(\mathbf{x}, t) = (1, 0, 0)^T e^{-[x_3 - t]^2},$$
$$\boldsymbol{B}^i(\mathbf{x}, t) = (0, 1, 0)^T e^{-[x_3 - t]^2},$$
$$(\mathbf{x}, t) \in \mathbf{R}^3 \times \mathbf{R}. \quad (14)$$

The smart obstacle of the first experiment is a sphere of center the origin and radius 2 (Figure 1a)) with boundary impedance $\chi = 2$ that pursues the goal of being undetectable (i.e. Furtivity Problem, Problem 1).

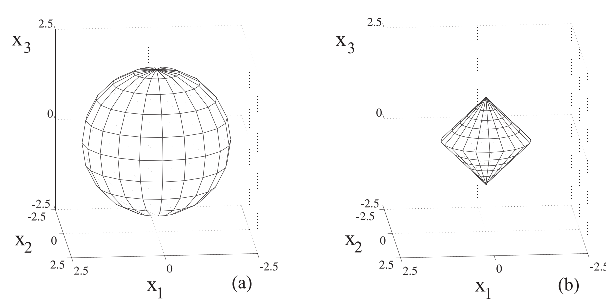

Figure 1. Obstacles.

The smart obstacle of the second experiment is a perfectly conducting (i.e. $\chi = 0$) double cone (see Figure 1b)) that pursues the goal of being undetectable in the subset $K = (-1, 1)$ of the frequency space (Definite Band Furtivity Problem, Problem 4). The double cone consists of two cones of the same height 1.2 and base (a circle having center the origin and radius 1.2) one upon the other through their bases.

The numerical results relative to the first experiment are shown in Table I. Let us describe these results. Let B_{R_i}, $i = 1, 2, 3$ be spheres having center the origin and radii $R_i = 2.0 + (i-1) * 0.5$, $i = 1, 2, 3$ and let $t_\nu = -2 + \nu$, $\nu = 1, 2, 3$ be three time values such that the incident field is beginning to hit the scatterer ($t = t_1 = -1$), is going through the body of the scatterer ($t = t_2 = 0$) and is leaving the scatterer ($t = t_3 = 1$) respectively. Note that the spheres B_{R_i}, $i = 1, 2, 3$ contain or coincide with the smart obstacle, $\Omega \subseteq B_{R_i}$, $i = 1, 2, 3$. Let $\boldsymbol{E}_a^s$ and $\boldsymbol{E}_p^s$ denote the electric field generated by the smart obstacle when the optimal surface electric current density is used, and by the same obstacle considered as a passive obstacle respectively. For $i = 1, 2, 3$, $\nu = 1, 2, 3$ we define the following quantities,

$$\epsilon_{E,R_i,\nu}^{a,\lambda} = \left[\int_{\partial B_{R_i}} \|\boldsymbol{E}_a^s(\mathbf{x}, t_\nu)\|^2 ds_{\partial B_{R_i}}(\mathbf{x}) \right]^{1/2}, \tag{15}$$

$$\epsilon_{E,R_i,\nu}^{p} = \left[\int_{\partial B_{R_i}} \|\boldsymbol{E}_p^s(\mathbf{x}, t_\nu)\|^2 ds_{\partial B_{R_i}}(\mathbf{x}) \right]^{1/2}, \tag{16}$$

and

$$\epsilon_{E,R_i}^{\lambda} = \min_{\nu=1,2,3} \frac{|\epsilon_{E,R_i,\nu}^{p} - \epsilon_{E,R_i,\nu}^{a,\lambda}|}{|\epsilon_{E,R_i,\nu}^{p}|}, \ i = 1, 2, 3, \tag{17}$$

where $ds_{\partial B_{R_i}}$ is the surface measure on ∂B_{R_i}, $i = 1, 2, 3$. Note that the quantity $\epsilon_{E,R_i,\nu}^{a,\lambda}$, $\epsilon_{E,R_i,\nu}^{p}$ $i = 1, 2, 3$, $\nu = 1, 2, 3$ are a sample of the "magnitude" of the electric fields generated by the smart obstacle and by the passive obstacle respectively. The quantity $\epsilon_{E,R_i}^{\lambda}$, $i = 1, 2, 3$ is a measure of how the electric field generated by the smart obstacles is small when compared with the electric field generated by a passive obstacle that is, is a measure of the furtivity effect achieved. The results obtained

Table I . Furtivity Effect

R_i	ν	$\epsilon_{E,R_i}^{a,\lambda}$	ϵ_{E,R_i}^{p}	$\epsilon_{E,R_i}^{\lambda}$
$\lambda = 0.1, \quad \mu = 0.9$				
2.0	1	1.651	2.323	0.287
2.5	1	1.082	1.421	0.238
3.0	1	0.753	0.911	0.173
$\lambda = 0.5, \quad \mu = 0.5$				
2.0	1	0.808	2.323	0.652
2.5	1	0.525	1.421	0.630
3.0	1	0.367	0.911	0.597
$\lambda = 0.9, \quad \mu = 0.1$				
2.0	1	0.191	2.323	0.918
2.5	1	0.123	1.421	0.913
3.0	1	0.085	0.911	0.906

are satisfactory when $\epsilon_{E,R_i}^{\lambda}$ is close to one, in fact when $\epsilon_{E,R_i,\nu}^{a,\lambda} = 0$ we have $\epsilon_{E,R_i}^{\lambda} = 1$.

Note that the column denoted with ν in Table I contains the minimizer of formula (17). Results similar to those shown in Table I have been obtained for the magnetic induction vector field (see [5] for further details).

Note that the furtivity effect increases when λ increases and that it ranges from 17% when $\lambda = 0.1$ to 90% when $\lambda = 0.9$ (see Table I).

Finally Figures 2, 3 show the numerical results relative to the second experiment. In this experiment we choose $\lambda = 0.9$, $\mu = 0.1$, $K = (-1, 1)$. As above, let $\boldsymbol{E}_a^s$, $\boldsymbol{B}_a^s$ and $\boldsymbol{E}_p^s$, $\boldsymbol{B}_p^s$ be the electric vector field and the magnetic induction vector field scattered by the smart double cone when the optimal surface electric current density is employed and by the passive double cone respectively. Figure 2 shows from left to right in the colour scale shown the Euclidean norms of the convolution products $\check{I}_K * \boldsymbol{E}_a^s$, $\check{I}_K * \boldsymbol{E}_p^s$, $\check{I}_K * \boldsymbol{B}_a^s$, $\check{I}_K * \boldsymbol{B}_p^s$ on the sphere B_{R_2} as a function of the polar angles (θ, ϕ), for three different values of the time variables that is, $t = 0$, $t = 2$, $t = 3$. Note that $\Omega \subset B_{R_2}$ and that the norms of the vector fields $\check{I}_K * \boldsymbol{E}_a^s$, $\check{I}_K * \boldsymbol{B}_a^s$ are negligible compared to the corresponding norms $\check{I}_K * \boldsymbol{E}_p^s$, $\check{I}_K * \boldsymbol{B}_p^s$.

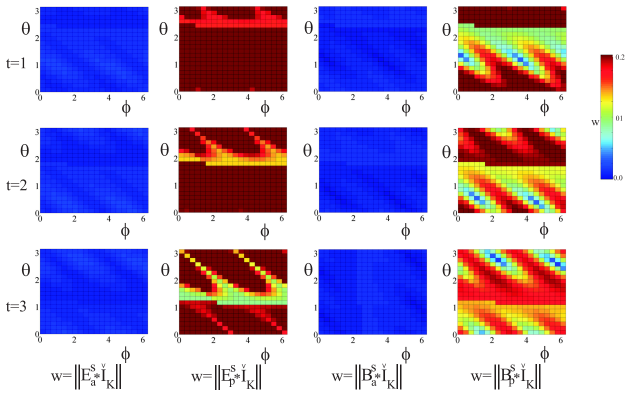

Figure 2. Furtivity effect in the frequency band K.

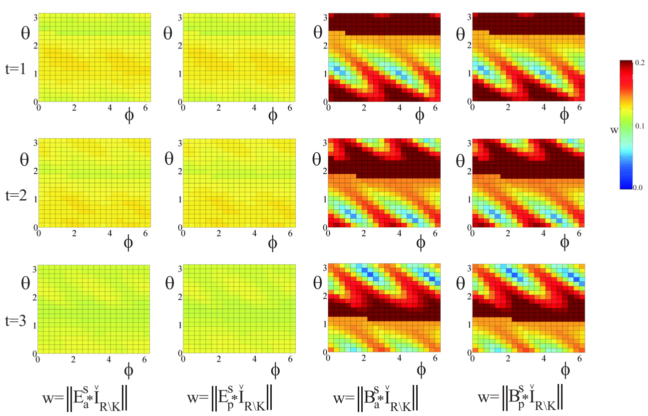

Figure 3. Furtivity effect outside the frequency band K.

Similarly Figure 3 shows from left to right in the colour scale shown the norms of the convolution products $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{E}_a^s$, $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{E}_p^s$, $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{B}_a^s$, $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{B}_p^s$ on the sphere B_{R_2} as a function of the polar angles (θ, ϕ) for $t = 0$, $t = 2$, $t = 3$. Note that the norms of the vector fields $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{E}_a^s$, $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{B}_a^s$ are similar to the corresponding norms of $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{E}_p^s$, $\check{I}_{\mathbf{R}\setminus K} * \boldsymbol{B}_p^s$. That is, outside of the frequency band K no furtivity effect is present. Note that the colour scales used in Figures 2 and 3 to represent the data are the same.

4. Conclusions

In this paper we have shown how mathematical models, such as optimal control problems, can be used profitably to design smart objects able to pursue non trivial goals. The main advantage of the mathematical formulation of the electromagnetic scattering problem involving smart obstacles proposed in this paper is that it allows to reduce the solution of the scattering problem to the solution of an optimal control problem whose optimal solution can be determined as the solution of a suitable system of coupled partial differential equations. This fact guarantees a great computational efficiency. In fact the most standard approaches solve the optimal control problem iteratively. That is at each step of the iterative procedure a system of partial differential equations must to be solved.

References

[1] F. Mariani, M. C. Recchioni, and F. Zirilli, "The Use of the Pontryagin Maximum Principle in a Furtivity Problem in Time-Dependent Acoustic Obstacle Scattering", *Waves in Random Media,* Vol. 11, 549-575, 2001.

[2] L. Fatone, M. C. Recchioni, and F. Zirilli, "A Masking Problem in Time Dependent Acoustic Obstacle Scattering", *ARLO - Acoustics Research Letters Online,* Vol. 5, Issue 2, 25-30, 2004.

[3] L. Fatone, M. C. Recchioni, and F. Zirilli, "Mathematical models of "active" obstacles in acoustic scattering", in *Control and Boundary Analysis*, J. Cagnol and J. P. Zolesio Editors, Marcel Dekker/CRC Press, Boca Raton, Fl. USA, Lecture Notes in Pure and Applied Mathematics, Vol. 240, 119-129, 2005.

[4] L. Fatone, M. C. Recchioni, and F. Zirilli, "Some Control Problems for the Maxwell Equations Related to Furtivity and Masking Problems in Electromagnetic Obstacle Scattering", in G.C. Cohen, E. Heikkola, P. Joly and P. Neittaanmaki editors, *Mathematical and Numerical Aspects of Wave Propagation. Waves 2003*, Springer Verlag, Berlin, 189-194, 2003.

[5] L. Fatone, M. C. Recchioni, and F. Zirilli, "Furtivity and Masking Problems in Time Dependent Electromagnetic Obstacle Scattering", *Journal of Optimization Theory and Applications,* Vol. 121, 223-257, 2004.

[6] B. Chambers, "A Smart Radar Absorber", *Smart Materials and Structures,* Vol. 8, 64-72, 1999.

[7] K. L. Ford, and B. Chambers, "A Smart Microwave Absorber", *IEE Electronics Letters,* Vol. 36, 50-52, 2000.

[8] B. Chambers, and A. Tennant, "Influence of Switching Waveform Characteristics on the Performance of a Single Layer Phase-Switched", *IEEE Transactions on Electromagnetic Compatibility,* Vol. 44, 434-441, 2002.

[9] J. A. Stratton, "Electromagnetic Theory", McGraw-Hill, New York, 1941.

[10] L. Fatone, G. Pacelli, M. C. Recchioni, and F. Zirilli, "The Use of Optimal Control Methods to Study Two New Classes of Smart Obstacles in Time Dependent Acoustic Scattering", submitted to *Journal of Engineering Mathematics,* 2005.

[11] M. C. Recchioni, and F. Zirilli, "A New Formalism for Time-Dependent Electromagnetic Scattering from a Bounded Obstacle", *Journal of Engineering Mathematics,* Vol. 47, 17-43, 2003.