Efficient Solution of Linear Systems in Microwave Numerical Methods

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Abstract— **A common bottle-neck, limiting the performance of many electromagnetic numerical methods, is the solution of sparse linear systems. Until now, this task has been typically solved by using iterative sparse solvers, whose require heavy computational efforts, especially when the problem is not well-conditioned.**

An alternative strategy is based on the use of banded solvers, which numerical complexity is quadratical with respect to the matrix bandwidth. Of course, these methods are efficient provided that the matrix bandwidth is sufficiently small. In this paper, a method (called WBRA) for the bandwidth reduction of a sparse matrix is presented: it is here specifically customized to typical electromagnetic matrices. The approach is superior to all the previous algorithms, also with respect to commercial well-known packages, and is suitable also for non-symmetric problems.

As demonstrated by results, the use of WBRA, in conjunction with common banded solvers, substantially improves (up to one order of magnitude) the solution times in several electromagnetic approaches, such as Mode-matching, FEM, and MoM analysis of microwave circuits. In conclusion, it is proved that the high efficiency and effectiveness of WBRA turns the strategy of bandwidth reduction combined with a banded solver into the most profitable way of solving linear systems in electromagnetic numerical methods.

I. Introduction

The use of numerical methods is nowadays commonly accepted as the most effective and efficient way to attack and solve electromagnetic (EM) problems. Many numerical codes are routinely used in the CAD of EM circuits, in complex scattering analyses, and in electromagnetic compatibility evaluations, just to mention some possible industrial and research tasks which are currently mainly performed via numerical approaches.

The obvious consequence of the continuous growth of numerical methods in daily work is an increasing demand for numerical efficiency and performance. As the problems get more complex, the issue of optimum memory exploitation and CPU-time reduction is crucial, provided that suitable numerical accuracy be guaranteed.

A common bottle-neck limiting the performance of many numerical approaches is represented by the solution of linear systems, usually sparse, which is very often one of the strongest numerical tasks for many EM methods. Mode-matching (MM), Method of Moments (MoM), both in its standard formulation and when using wavelet expansions, and Finite Element Methods (FEM) are just some examples demonstrating this.

In previous papers [17], [3] it was demonstrated that a very efficient strategy to improve the performance of many EM codes is the enhancement of the linear system solution time by an appropriate transformation of the system matrix. The matrix, generally sparse, is transformed into a banded one, with reduced bandwidth, this paving the way for a very effective use of high-performing banded solvers. The performance of the algorithm to reduce the bandwidth of a sparse matrix is, in this perspective, a key-point. In this paper, a new method is proposed to accomplish this task. It is suited to every kind of sparse matrix, but specifically tuned to achieve maximum performance on typical matrix patterns of EM problems. It is proved to outperform all the previous approaches, including commercial packages, on several real EM cases. The availability of such an efficient bandwidth reducer turns its use, in conjunction with a banded solver, into the most effective solution strategy, differently from before, when the bandwidth minimization effort was not so profitable, and the use of a sparse solver without matrix preprocessing was sometimes the winning approach.

The paper is structured as follows. In section 1 an overview of the problem is proposed. In section 2 the new algorithm for bandwidth reduction is proposed. In section 3 results are given. Finally, conclusions are drawn.

II. Bandwidth reduction used in conjunction with direct solvers vs. iterative sparse methods

Let

$$\mathbf{Ax} = \mathbf{B} \tag{1}$$

be a sparse system.

We know from very basic matrix algebra that, considered a permutation matrix $\mathbf{P}$ so that $\mathbf{P} \cdot \mathbf{P}^{\mathbf{T}} = \mathbf{I}$ (where $\mathbf{I}$ is the identity matrix), the system

$$\mathbf{PAP^T x' = PB} \qquad (2)$$

has the same numerical stability as (1), as its condition number is not changed. Moreover, the solution $\mathbf{x}$ of (1) is easily found from the solution $\mathbf{x}'$ of (2), as

$$\mathbf{x = P^{-1} x'} \qquad (3)$$

Therefore, if the permutation matrix is appropriately chosen, so that the transformed system matrix $\mathbf{A}' = \mathbf{PAP^T}$ is banded, with small bandwidth, banded solvers can be used with very high performance, as their complexity depends quadratically on the matrix bandwidth [5]. Unfortunately, the task of finding the transformation $\mathbf{P}$ that gives the minimum possible bandwidth is a very difficult problem; in [12] it is shown to be an NP-hard problem.

Many efforts have been made until now in order to solve this problem both in an exact way providing the optimal permutation, and in an approximated way, so that good permutations that sensibly reduce the bandwidth can be found with a much smaller computational effort. A very thorough review of approaches is proposed in [5], which must be integrated with more recent contributions [4], [7], [2].

Due to this difficulty, instead of looking for an optimum transformation, alternative approximated but faster strategies are usually followed in practical applications, as illustrated in the next section where the method for bandwidth reduction proposed in [6] referred as WBRA will be reviewed, putting forward its peculiar features with respect to other previous approaches.

Of course, the development of an effective bandwidth reduction algorithm does not guarantee an a-priori enhancement of performance whatever the EM numerical method might be. Its impact must be benchmarked by comparing it with respect to the current existing approaches. Among them, a usual approach is the use of iterative sparse solvers, which are generally applied directly to the original matrix, without any previous preprocessing. Iterative solvers are not very efficient, and require large numbers of iterations especially on non-well-conditioned problems. Nonetheless, they are largely used, due to their easy availability and to their strong stability. In fact, direct sparse solvers, which could be in principle more performing, are often prone to the risk of dense LU factors [5], highly degrading memory and computing-time requirements, and are therefore not considered a viable solution.

Therefore, on the basis of the previous considerations, when presenting results, we generally propose a comparison between a strategy using a bandwidth reduction, and a more standard one, using an iterative sparse solver.

III. The WBRA Approach for Bandwidth Reduction

One of the most effective classes of algorithms specifically devoted to bandwidth reduction is the one derived from the Cuthill-McKee (CM) method [8]. The main idea of this class of algorithms is related to the graph representation of the matrix (see Fig. 1). Consider a symmetrically structured matrix $\mathbf{A}$ of order n, let $G = (N, E)$ be the undirected adjacency graph related to $\mathbf{A}$, where each node $i \in N = \{1, \ldots, n\}$ represents the i-th row/column of the matrix, and there is an edge (i, j) between two nodes i and j $(i \neq j)$ if and only if the element of matrix $\mathbf{A}$ $a_{ij} \neq 0$. The basic idea of the computation can be summarized by the following steps:

1) *partitioning phase*: select a root node r, and partition N into subsets called *levels* $\{L_0, L_1, \ldots, L_p\}$ with $r \in L_0$, so that there are edges only between nodes belonging to the same level or to two adjacent levels; a partition into levels is called *level structure.* In Fig. 2, the level structure obtained with root equal to 6 is shown.

2) *numbering phase*: sort the nodes by increasing level index, and inside each level number them according to a particular criterion.

As the other algorithms in the CM class, WBRA follows this general scheme, but in the two phases exploits the structure of the combinatorial optimization problem underneath.

In the partitioning phase a partition into levels, whose larger subset has minimum cardinality, is sought. In fact the bandwidth is directly affected by the size of the largest subset. However, as this problem is as difficult as finding the minimum bandwidth, an approximated algorithm is applied. In Fig. 3 a possible redistribution of nodes between levels is provided. In this case the width of the largest level in the new structure is reduced to 2.

Now consider the numbering phase. WBRA applies the numbering to a set of "promising" level structures determined during the partitioning phase, that is to a set of level structures whose largest level is small.

Let $\{L_0, L_1, \ldots, L_p\}$ be the partition under consideration during the generic iteration:

Numbering the nodes, we assign the first numbers (i.e. the first positions in the permuted matrix) to the nodes in L_0, then the other positions are assigned following the precedence between levels, that is any element of level L_{h-1} always has a smaller number with respect to any other element of level $L_h, h = 1, \ldots, p$. Thus the numbering phase is carried out by a sequence of steps, one for each level.

Determining the optimal numbering of level L_h WBRA considers:

- (i) the numbers assigned to the elements of L_{h-1}, but in addition it considers also
- (ii) the possible effects of the numbering in level L_{h+1}.

These two criteria give rise to a well characterized combinatorial optimization problem whose particular structure allows us to determine the numbering of a level in almost linear time. The numbering obtained by applying the algorithm to the level structure of Fig. 3 is shown in Fig 4. A detailed discussion of the combinatorial structure of the problem, and of the numbering algorithm can be found in [6].

The rearranged matrix according to the permutation found by the algorithm is shown in Fig. 5.

A. The unsymmetric case

The method proposed until now, based on a matrix representation through an adjacency graph, is devoted to matrices with symmetrical patterns. On the contrary, as described in the result section, a method working also on matrices with an unsymmetric zero-non zero structure is needed quite often. One of the key-points of WBRA is its amenability to cope with this problem, without degradating the performance of the algorithm. This is achieved by symmetrizing the matrix structure in a cumbersome way.

Generally, the matrix pattern could be symmetrized in two possible fashions:

i) consider a matrix $\mathbf{A}^*$ where $a^*_{ij} = \max\{a_{ij}, a_{ji}\}$;

ii) symmetrize the matrix, that is consider $\mathbf{A}^* = \frac{1}{2}\mathbf{A}\mathbf{A}^\mathbf{T}$.

The latter approach is avoided as it may introduce some ill conditioning. In the former case, any symmetric bandwidth reduction algorithm can be applied to $\mathbf{A}^*$, then matrix $\mathbf{A}$ is permuted according to the obtained reordering. The drawback of this approach is evident when $\mathbf{A}$ is highly unsymmetric: many zero elements are dealt as they would be non zero. This is why devising ad hoc algorithms that can take advantage of the unsymmetry becomes important.

By contrast to the symmetric case, in the bandwidth minimization of unsymmetric matrices, we are not obliged to apply the same permutation to rows and columns. Thanks to this observation, we propose an algorithm divided into two phases. In the former phase a permutation is applied to rows only. Then the matrix is symmetrized according to method i). In the latter phase the bandwidth is reduced by applying the same permutation to rows and columns. Finally the matrix is "desymmetrized" by removing the nonzero elements introduced by the symmetrization step.

We adopted two methods for the first phase. The former method applies the row permutation that maximizes the number of non zero elements on the diagonal. This problem is known as the *transversal maximization* [5]. An alternative method (*symmetry maximization*) tries to maximize the number of symmetric elements, that is tries to permute the matrix rows so that in the final matrix the number of elements $a_{ij} \neq 0$ having the corresponding $a_{ji} \neq 0$ are as many as possible.

After the first phase (either transversal maximization or symmetry maximization), the WBRA is applied. The final performance (as demonstrated by results) is quite attractive. The algorithm is publically available through Internet at the web site `http://dvorak.istel.ing.unipg.it`.

IV. Results

Four main areas of applications are proposed: the MM analysis of rectangular waveguide circuits, the FEM analysis of planar circuits with metalic boxes, the MoM analysis of planar circuits with a Mixed-Potential Integral Equation (MPIE) approach, and a MoM using wavelet expansions. In the case of MM and MoM, the use of a bandwidth reduction in couple with a banded solver is compared with a standard iterative biconjugate-gradient sparse solver with preconditioning. For FEM, the performance of a standard package is compared with an implementation taking advantage from bandwidth reduction.

A. MM Analysis of MW Circuits

The MM analysis of MW circuits is an efficient and rigorous method, often used in CAD packages. Among its several attractive issues, the amenability to MW circuit optimization, via the Adjoint Network Method is one of the most interesting, as well as a paramount impulse to improve its performance.

As already discussed and demonstrated in [17], the solution of a sequence of linear systems, with different right-hand-sides and same matrix, is the numerical core of the approach. The pattern of the sparse system matrix $\mathbf{A}$ depends on the numbering adopted for the physical and electrical ports, as well as on the number of modes selected in every section of the circuit.

In this paper, we compare the performance of WBRA with other methods to reduce the bandwidth of the system matrix, and show the corresponding performance of the MM analysis. More specifically, we compare WBRA with a proprietary routine performing the Modified- Reversed-Cuthill-McKee (MRCM) approach (one of the most efficient implementation of the CM method [8]), with a commercial routine for bandwidth reduction proposed by MATLAB, and with a previous novel technique proposed by the authors in [3], called Tabu Search (TS). In Tab. I results are given on three matrices coming from the MM analysis of a complex 4x4 Butler matrix (Fig. 6), with different spatial resolutions. In Tab. I the reader can note: the system size N, the bandwidth of the original matrix $\mathbf{A}$ (IB), the final bandwidth achieved by the different algorithms and, in brackets, the time necessary to evaluate the permutation matrix. Times are given in seconds, on an IBM RS6000 250 T.

N	IB	MRCM	MATLAB	TS	WBRA
116	75	6 (0.039)	6 (0.062)	6 (91.1)	6 (0.051)
245	115	83 (0.24)	55 (0.36)	25 (242)	27 (0.229)
503	452	78 (1.572)	61 (1.97)	64 (1000)	51 (1.234)

Tab. I: The effectiveness of different bandwidth reduction methods and (times in brackets) their efficiency.

As inferred from the table, TS is too slow to be used on serial platforms, and is therefore omitted. The WBRA is superior to both MATLAB and MRCM, as it is more effective and its time performance is better as the size of the problem increases. In Tab. II, for the same cases of Tab. I, the solution times are shown using: 1) a standard iterative sparse solver (BCG) with the original $\mathbf{A}$ matrix 2) a banded solver on the original $\mathbf{A}$ matrix (BNT) 3) a banded solver on the transformed $\mathbf{PAP^T}$ matrix ($\mathbf{P}$ evaluated with MATLAB (BTM)) 4) a banded solver on the transformed $\mathbf{PAP^T}$ matrix ($\mathbf{P}$ evaluated with WBRA (BTW)).

N	BCG	BNT	BTM	BTW
116	9.5	27.1	0.46	0.46
245	101.1	198	36.3	8.2
503	521	longer than 1000	442	296

Tab. II: Different simulation times for a MM code on a 4x4 Butler's matrix. A standard iterative solution (BCG) is compared with a banded solution (BNT), a banded solution with bandwidth minimization by MATLAB (BTM), and a banded solution with bandwidth minimization by WBRA (BTW).

As shown in Tab. I, the WBRA method outperforms the other approaches. Its use (as evidenced by Tab. II if you compare BTW with respect to BCG), allows a speed-up in the system solution of up to one order of magnitude with respect to the use of a standard iterative sparse solver, and (as evidenced by comparing BTW and BTM) of up to 4 times with respect to the use of previous bandwidth reduction methods.

B. FEM Analysis of Boxed Microstrip Lines

The analysis of microstrip lines surrounded by a metallic box is a common problem for the MW and EMC community. This problem has been attacked by using a public domain FEM package called EMAP [10]. In EMAP, a key-point in the analysis of the circuit is the repeated solution of a linear system. In FEM, the system is generally reduced to a banded structure. Therefore, EMAP uses a banded solver, and is quite amenable to be interfaced with the above mentioned modules for bandwidth reduction.

Several tests have been performed on circuits such as the one in Fig. 7, for different substrates and dimensions of both the box and microstrip. Some of the results are shown in two tables, with the same scheme as for the MM section: in Tab. III data report the efficiency of the different bandwidth reduction methods, whilst Tab. IV shows the effects of different bandwidth reductions (Matlab (BTM), and WBRA (BTW)) on the system solution time using the standard banded solver used in EMAP (BNT).

N	IB	MRCM	MATLAB	WBRA
291	107	106 (0.32)	106 (0.48)	62 (0.29)
615	156	165 (3.75)	163 (8.9)	88 (1.56)
1180	297	245 (24.8)	237 (142)	184 (5.9)

Tab. III: A comparison of performance for different bandwidth reduction methods for a FEM linear system.

N	BNT	BTM	BTW
291	34.1	33.8	8.8
615	243	289	59.6
1180	2048	1621	973

Tab. IV: Solution times for the FEM problem using a standard EMAP banded solver (BNT), with respect to 1) standard EMAP solver and Matlab bandwidth reduction (BTM) and 2) standard EMAP solver and WBRA bandwidth reduction (BTW).

It can be noted from Tabs. III and IV that the superior performance of WBRA, both for effectiveness, and for computing times, allows a speed-up in the system solution time of up to 4 times with respect to standard EMAP code.

C. MoM Analysis of Microstrip Circuits

Recent enhancements in the analysis of planar circuits with an MPIE approach using a closed-form spatial-domain Green's function [11] allow a very efficient implementation of CAD tools. The MPIE can be discretized by using the MoM, thus generating a linear system whose solution allows the evaluation of the scattering parameters of the circuit. The system is generally dense, but very recently it has been demonstrated that it can be reduced to a sparse one, without affecting the accuracy of the simulation [1]. Therefore, also in this case the core of the numerical effort is a sparse linear system, and a bandwidth reduction is worth to be performed.

In Tabs. V and VI results are shown for two circuits reported in Fig. 8. As usual, Tab. V gives results concerning the effectiveness and efficiency of the WBRA with respect to other methods for bandwidth reduction. Tab. VI compares the solution time using a standard iterative sparse solver (BCG) with respect to using a banded solver, invoked straightly (BNT) or after performing a bandwidth reduction (using MATLAB (BTM) or WBRA (BTW)).

N	IB	MRCM	MATLAB	WBRA
220	208	120 (0.34)	119 (0.51)	72 (0.3)
401	310	141 (1.23)	136 (1.5)	82 (0.98)

Tab. V: A comparison of performance for different bandwidth reduction methods for a MPIE/MoM linear system.

N	BCG	BTM	BTW
220	71	14.8	6.4
401	214	49.6	18.8

Tab. VI: Solution times for the MPIE/MoM problem using standard sparse iterative solver (BCG), with respect to 1) banded solver and Matlab bandwidth reduction (BTM) and 2) banded solver and WBRA bandwidth reduction (BTW).

From Tabs. V and VI it can be noted that the use of WBRA enhances the efficiency of the system solution by up to 12 times with respect to a standard sparse iterative solver, and up to 3 times with respect to using a commercial bandwidth reducer before invoking a banded solver.

D. MoM using Wavelet expansions

In the past few years, the use of wavelet expansions in the solution of electromagnetic problems has become more and more frequent. Wavelet expansions have been introduced, for instance, in conjunction with the Method of Moments (MoM) discretization of integral equations [9], [20], in order to solve scattering problems with large-scale scatterers (thus containing a variety of length scales with respect to wavelength) [15]-[18], or to analize slot-apertures [19], microstrip floating line structures [21], as well as to study 2D and 3D dielectric structures [13], [14]. A common key-issue for all the above mentioned applications is the derivation of very sparse and well-conditioned linear systems, representing the numerical core of MoM approaches [1], [13], [14]. The moment matrix sparsity allows the use of very efficient iterative sparse solvers, and the good condition number guarantees a low number of iterations to converge, with a consequent dramatic improvement of performance.

Up to now, once the moment matrix has been sparsified using wavelet expansions, it has been assumed that iterative solvers are the best way to attack the linear system solution. We demonstrate here that, by means of appropriate matrix transformations, the use of a banded direct solver in conjunction with WBRA outperforms the iterative approach, especially when non-symmetric moment matrices are attained after split testing procedures in presence of compact-support functions [15], [16], [13], [14]. It must be put forward that the capability of dealing with non-symmetrical cases, without lost of effciency, is one of the most attractive features of WBRA. In fact, for previous bandwidth reduction approaches, the only way to face non-symmetric problems was represented by the matrix pattern symmetrization, with an obvious dramatic reduction of performance.

We refer, for the proposed results, to a MoM discretization of a Mixed-Potential Integral-Equation formulation for the analysis of planar microstrip circuits, as described in [1]. The MoM matrices are transformed in accordance with the use of Battle-Lemarie multiresolution expansions, as described in [15], [16], [13], [14], thus attaining non-symmetric matrices when splitting and truncations are performed to comply with boundary conditions. A double-layer microstrip waveguide has been studied, with different basis expansions, and different threshold values v_t have been applied onto the moment matrices, so that values having magnitude less than v_t per cent of the largest entry are considered as zeros. Of course, different approximations are attained on varying v_t, and errors have been estimated by comparing approximate results with the correct result attained without any thresholding. Table VII and VIII present results from two different cases of analysis of a double-layer microstrip (see Fig. 9), using different numbers of Battle-Lemarie wavelet functions. Different matrix sizes, respectively N=250 and N=478, have been attained. For different thresholds, the matrix sparsity S, the approximation error, and the results are reported, from two different strategies: i) a banded solver with WBRA (BTW), ii) an iterative sparse BCG solver (BCG) (the number of iterations to converge is also reported).

			Sol. time (s.)	
v_t	S	Error	BTW	BCG (n.iterations)
2%	99%	5.4%	0.08	0.09 (8)
1%	96%	2.4%	0.12	0.23 (17)
0.5%	91%	0.7%	0.21	0.48 (31)

Tab. VII : Results for a matrix of dimension

N=250. Computing times for a banded solver and WBRA strategy versus BCG strategy are shown, for different threshold values, and the corresponding matrix sparsity S and solution error due to thresholding effects. For BCG the number of iterations needed to converge is shown in the brackets.

			Sol. time (s.)	
v_t	S	Error	BTW	BCG (n.iterations)
2%	98%	4.8%	0.30	0.1 (11)
1%	94%	2.1%	0.64	1.4 (14)
0.5%	88%	0.6%	0.80	6.0 (28)

Tab. VIII : Results for a matrix of dimension N=478. Computing times for a banded solver and WBRA strategy versus BCG strategy are shown, for different threshold values, and the corresponding matrix sparsity S and solution error due to thresholding effects. For BCG the number of iterations needed to converge is shown in the brackets.

As apparent from Tab. VII and VIII, an appropriate value for thresholding is 0.5%, so that the approximation error is smaller than 1%. In this case, for N=250, a speed-up of nearly 2.3 is achieved, when using the BTW strategy with respect to BCG, whilst for N=478 a speed-up of nearly 7.5 is observed.

V. Conclusions

In this paper a new method, called WBRA, to perform the bandwidth reduction of a sparse matrix has been presented. It generally works for every kind of sparse matrix, but it is specifically tuned to achieve maximum performance on typical matrix patterns encountered in electromagnetic numerical problems, as well as to manage also with non-symmetric zero-non-zero-pattern matrices (this being crucial when attacking some wavelet problems).

The efficiency and effectiveness of the method is superior to all the other commercial and public domain packages available on the market, as demonstrated on matrices generated by the mode-matching, FEM, MoM/MPIE, and MoM/wavelet analysis of rectangular waveguide and planar circuits.

The huge advantages of the WBRA's use in the analysis of MW circuits is also proved for the above mentioned problems. It is demonstrated that enhancements of one order of magnitude can be achieved, with respect to the use of classical iterative sparse solvers, by using WBRA in conjunction with banded solvers. This is attained thanks to the high efficiency of WBRA, which reduces the bandwidth reduction times, improves the effectiveness of bandwidth reduction, and substantially decreases the numerical complexity of the banded solution.

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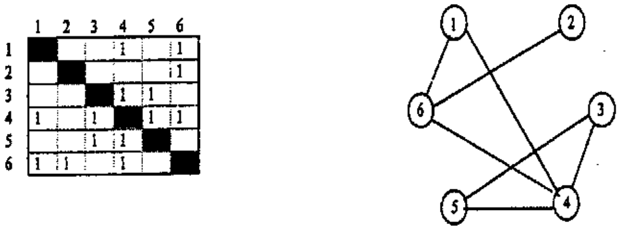

Fig. 1. The input matrix (bandwidth=5) and the relative adiacency graph.

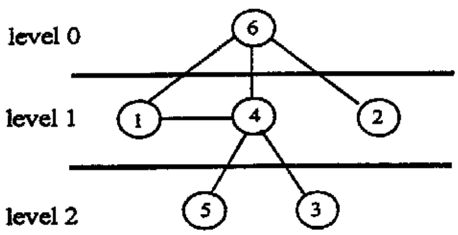

Fig. 2. The adiacency graph partitioned into levels.

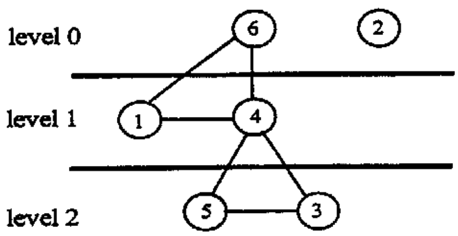

Fig. 3. The level structure after the enhancement phase.

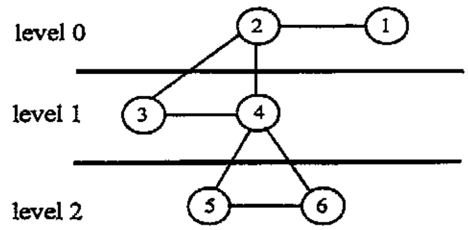

Fig. 4. The level structure after the renumbering phase.

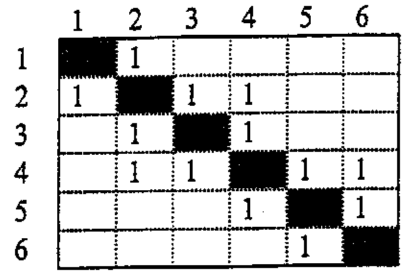

Fig. 5. The renumbered matrix (bandwidth=2).

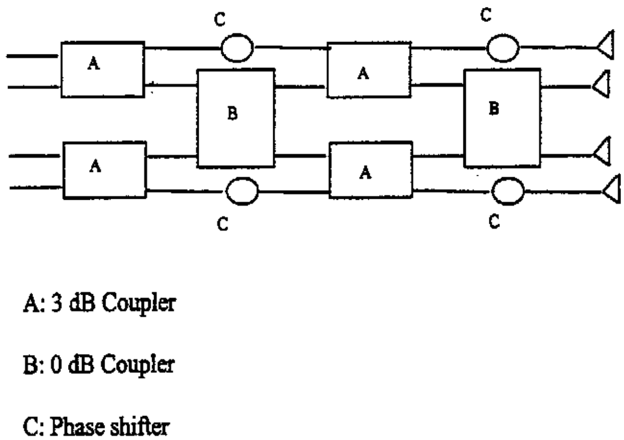

Fig. 6. The 4x4 Butler matrix simulated with the MM approach.

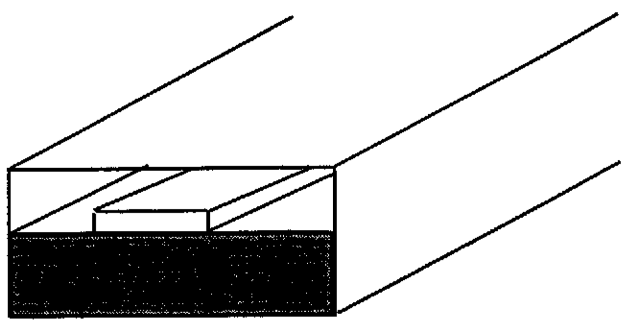

Fig. 7. The boxed microstrip waveguide simulated with the FEM approach.

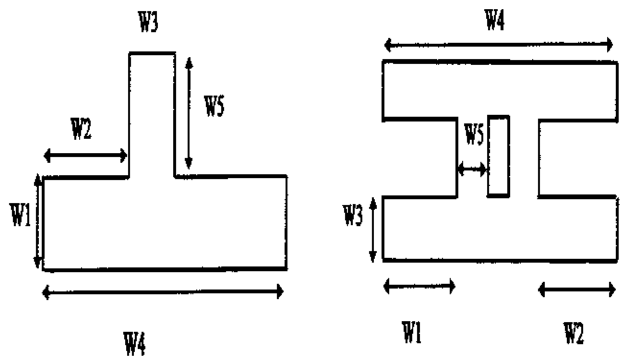

Fig. 8. The circuit simulated, with different values for w1, w2, w3, w4 and w5, with the MoM approach.

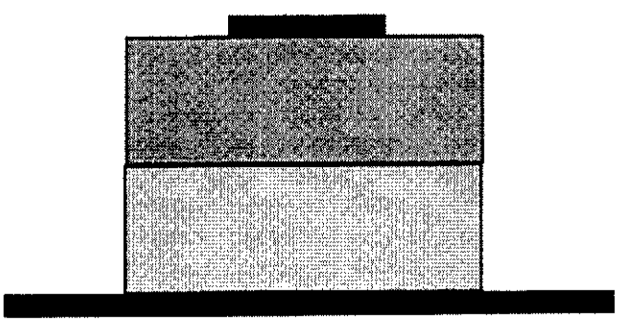

Fig. 9. The circuit simulated, with different wavelet basis functions, with the MoM approach.