Obtaining Scattering Solutions for Perturbed Geometries and Materials from Moment Method Solutions.

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Abstract. In this paper, we present an efficient method for computing the solution to scattering problems using a perturbation scheme based on the solution of related original problems. Assuming the radar cross section has been computed for a particular scatterer associated with a moment method matrix B , we call the computation of the radar cross section of a slightly perturbed scatterer a "perturbed problem of B". If the original problem has n unknowns, and the perturbed problem is formed by changing p cells of the original problem, then our method requires an operation count of $O(n^2p + p^3)$ while a direct moment metbod solution requires an operation count of $O(n^3)$. Our method involves application of the Sherman-Morrison-Woodbury formula for inverses of perturbed matrices. We show that the method can be easily implemented in any moment method code, and the user does not have to learn a new input procedure.

Further, the modified code can provide a basis for a non-linear optimization procedure which minimizes the radar cross section of an obstacle by varying the surface impedances. An appropriate objective function in this problem depends on the radar cross section at the angles and frequencies of interest. Let n be the number of cells in the obstacle and let p be the number of cells with variable impedance, with $n >> p$. Then application of the Sherman-Morrison-Woodbury formula results in objective function evaluations requiring an $O(np + p^3)$ operation count. In contrast, application of the classical moment method results in objective function evaluations requiring an $O(n^3)$ operation count.

Numerical results from large practical problems demonstrate the efficiency and stability of the new method.

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1. Introduction

Let MOM be a generic moment method code that solves its matrix equation by Gaussian elimination. Suppose MOM has solved a certain scattering problem. A second scattering problem is called a perturbed scattering problem of the first if the scatterer of the second problem is a slight perturbation of the first, geometrically and/or electrically. In this paper, we present an easy modification of MOM, which we call UMOM, for the solution of perturbed scattering problems. The method employed is based on the Sherman- Morrison-Woodbury updating formula (which we will abbreviate as SMW in the rest of this paper).

We show that appropriate application of the SMW yields a method that is efficient and easy to use. If the original problem has n unknowns, and the perturbed problem is formed by changing p unknowns of the orginal problem, then our method requires an operation count of $O(n^2p + p^3)$ while a direct moment method solution requires an operation count of $O(n^3)$.

The SMW was the work of Sherman, Morrison [**1**],[**2**] and Woodbury [**3**], which is not well-known outside the community of numerical linear algebra. The formula was rediscovered and applied to different engineering disciplines. A partial list of references on the applications of the SMW is:

(1) R. Hockney 1970 [**4**],

(2) B.Buzbee, F.Dorr, J.George and G.Golub, 1971 [**5**],

(3) W.Proskurowski, O. Widlund 1976 [**6**],

(4) E.L. Yip 1986 [**7**],

(5) E.L. Yip and B. Tomas, 1987 [**9**],

(6) B.Tomas and E.L.Yip 1988 [**10**],

(7) R. Kastner 1988 [**11**].

Kastner's work on large planar structures uses a specialized form of the SMW, which is simpler than the application of the SMW to general moment method codes discussed in this section.

Section 2 contains a discussion of the classical theory of the SMW and one of its implementations. Section 3 presents its application to scattering problems. In Sections 4 and 5, we present the solution of two scattering problems: the perturbed problem and the optimal loading problem. Section 6 contains numerical results.

2 THEORY

This section presents the Sherman-Morrison-Woodbury updating formula, and an algorithm for its general implementation. The efficiency of the method in terms of operation count is also discussed.

If A and B are $n \times n$ matrices, and if $A - B$ is a rank p matrix, there exist $n \times p$ matrices U and V such that

$$A = B - UV^T, \tag{1}$$

(where the superscript T signifies the transpose of the corresponding matrix). The Sherman-Morrison-Woodbury updating formula expresses A^{-1} in terms of B^{-1}, U and V:

$$A^{-1} = B^{-1} + B^{-1}U(I - V^T B^{-1} U)^{-1} V^T B^{-1} \tag{2}$$

(For the sake of completeness, we derive the Sherman-Morrison-Woodbury formula in the Appendix.)

There are many different methods of implementing of the above equation for the solution of $Ax = b$. Algorithm 1 below is an implementation for the most general case, that is, when $B - A$ is an arbitrary rank p matrix.

Algorithm 1.

Step 1. Compute for $BZ = U$.

Step 2. Compute the matrix $K = (I - V^T Z)$, and its LU factors.

Step 3. Solve $By = b$ for y.

Step 4. Compute $w = V^T y$.

Step 5. Solve $Ks = w$ for s.

Step 6. The solution for $Ax = b$ can be computed as $x = y + Zs$.

Postmultiply both sides of equation (2) by b,

$$A^{-1}b = B^{-1}b + B^{-1}U(I - V^T B^{-1}U)^{-1}V^T B^{-1}b. \tag{3}$$

Substituting the matrices Z and K and the vectors y, w, and s which are defined in Algorithm 1 into equation (3), we see that, in the absence of numerical round-off, the vector x defined in Step 6 of Algorithm 1 satisfies $Ax = b$.

If A and B are full matrices, and if B has already been factored, then the amount of work in Algorithm 1 is of the order $p(n^2 + p^2/3)$; if p is small, this can be much less costly than factoring A.

3 Application to Scattering Problems

Let B and A be the coefficient matrices of the original and perturbed problems, respectively. If both the material and geometric properties of the two problems are different, then the matrix $B - A$ consists of a few non-zero rows and a few non-zero columns. If only the material properties of the two problems are different, the the matrix $B - A$ will consists of only a few non-zero columns. In order to apply the SMW, the matrices U and V in equation (1) need to be defined.

Before we proceed with our discussion, it is pertinent to indicate the following:

(1) If an $n \times n$ matrix C has only one non-zero column, say the j-th column, then if u is a vector of length n and equals the non-zero column of C, and v is a vector of length n with the value 1 at its j-th entry and zero everywhere else, then $C = uv^T$.

(2) If an $n \times n$ matrix R has only one non-zero row, say the i-th row, then if v is a vector of length n and its transpose equals the non-zero row R, and u is a vector of length n with the value 1 at its i-th entry and zero everywhere else, then $R = uv^T$.

Consider first the case when only the material properties are different in the two problems. Suppose there are q non-zero columns, $j_1, j_2, \ldots, j_q$, in $B - A$. Let U be a $n \times q$ matrix whose columns are the q non- zero columns of $B - A$. Let V be an $n \times q$ matrix whose k-th column, $k = 1, 2, \ldots, q$, has the value 1 at its j_k-th entry and zero everywhere else. Then $B - A = UV^T$.

The case when both the geometric and/or material properties are different is more complicated. Suppose $B - A$ has p non-zero rows, $i_1, i_2, \ldots, i_p$, and q non-zero columns, $j_1, j_2, \ldots, j_q$. Let R be a matrix whose rows are the non-zero rows of $B - A$. Let C be a matrix whose columns are the non-zero columns of $B - A$ minus the intersection of the non-zero rows and non-zero columns of $B - A$. (See Fig. 1) Then

$$B - A = R + C. \tag{4}$$

Let U be an $n \times (p + q)$ matrix of the form:

$$U = [P_1, U_1], \tag{5}$$

where P_1 is a $n \times p$ matrix whose k-th column, $k = 1, 2, \ldots, q$, has the value 1 at its i_k entry and zero everywhere else, and U_1 is an $n \times q$ matrix whose columns are the non-zero columns of C in equation (4). Similarly, let V be an $n \times (p+q)$ matrix whose transpose is of the form:

$$V^T = \begin{bmatrix} V_1 \\ Q_1 \end{bmatrix} \tag{6}$$

where V_1 is an $p \times n$ matrix whose rows are just the non-zero rows of R in equation (4), and Q_1 is an $q \times n$ matrix whose k-th row, $k = 1, 2, \cdots, q$ has the value 1 at its j_k-th row and zero everywhere else. Then from equations (4) to (6),

$$\begin{aligned} UV^T &= P_1 V_1 + U_1 Q_1 \\ &= R + C \\ &= B - A \end{aligned} \tag{7}$$

Once U and V are identified, Algorithm 1 may be applied.

4 Solving the Perturbed Problem

This section shows that the basic properties of the matrices U and V defined in section 3 provide a user-friendly and portable computer implementation for practical problems. In this implementation, the user describes the perturbed problem to UMOM in exactly the same way he describes the original problem to MOM. UMOM will figure out the differences between the two problems. This implementation is also portable in the sense that, in order to apply the SMW updating formula to another moment method code, one need only modify the subroutines of UMOM slightly.

For the convenience of discussion, we shall refer to the part of the scatterers which is different in the two problems as the "perturbed" part, and the other part as the "unperturbed" part.

The steps the user takes to solve his new problem are :

(1) Solve the original problem with MOM; specify that the problem and immediate computation information are to be saved.

(2) Use UMOM to solve the perturbed problem. The input process for UMOM includes defining the new problem and specifying the disk file on which the old problem is stored.

Generation of the pertinent information for the SMW requires the user's input to be processed by the routines: SORT, COMPARE, and INDEX. Each of these is explained in detail below.

The structural differences between the two problems are obtained first. UMOM SORTs (by the Shell sorting algorithm) the discrete points which describe the scatterers in the two problems, and then COMPAREs them. This is a very efficient procedure. Without the sorting, a brute force comparison requires $O(n^2)$ operations, where n is the number of points which describe the scatterering problem. With sorting , the comparison takes an average of $O(n \log n)$ operations.

Any sorting algorithm requires an ordering for the objects to be sorted. UMOM assumes the following ordering on the $x - y$ plane:

We say $(x_1, y_1) > (x_2, y_2)$ if and only if $x_1 > x_2$ or ($x_1 = x_2$ and $y_1 > y_2$).

For example, in Fig. 2, point 1 is less than point 2 which is less than point 3.

The INDEX process establishes the link between the matrices of the original and the perturbed problems. Write

$$A = A_0 + A_1 \tag{8.1}$$

$$B = B_0 + B_1 \tag{8.2}$$

where A_0 and B_0 contain matrix elements which correspond to the unperturbed part of the two scatterers, and A_1 and B_1 contain matrix elements which correspond to the perturbed part of the two scatterers.

Mathematically, INDEX generates a set of indices from which a permutation matrix P is defined with A_0 and B_0 related as:

$$P^T A_0 P = B_0. \tag{9}$$

(Note that the inverse of P is its transpose.)

The matrix equation we are interested in solving is

$$Ax = b \tag{10}$$

which is equivalent to

$$\begin{aligned} P^T A P P^T x &= P^T b \\ (P^T A_0 P + P^T A_1 P) P^T x &= P^T b \\ (B_0 + P^T A_1 P) P^T x &= P^T b \\ (B + P^T A_1 P - B_1) y &= P^T b, \end{aligned} \tag{11}$$

where $y = P^T x$. $P^T A_1 P - B_1$ in equation (11) has only non-zero rows and non-zero columns corresponding to the perturbed part of the scatterers. And according to section

3, matrices U and V can be found so that

$$B_1 - P^T A_1 P = UV^T.$$

The SMW can then be applied.

Note that only A_1 and B_1 in equation (8) need to be generated. This can easily be accomplished by modifying the appropriate DO-loops in the code which generate the matrix elements in MOM.

Before applying the SMW, premultiply the right-hand-side vector b by P^T. Then in place of the regular linear equation solver, use the SMW updating formula to solve for y in equation (11). Then the true solution x is obtained as

$$x = Py \tag{12}$$

In summary, Fig. 3 illustrates the flow of MOM and UMOM and the structure of UMOM. The procedures in UMOM can be modified with minimal effort for adaptation to other moment method codes.

In our previous discussion, we assume the two problems generate matrices of the same dimensions. In the case in which they generate matrices of different dimensions, we show that minor modifications to the smaller matrix afford the use of the SMW.

In the case in which the dimension of B is greater than that of A, append an identity matrix to the right lower corner of A so that the two matrices have the same dimensions and replace equation (10) with

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{13}$$

Note that the solution of equation (13) is of the form

$$\begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is the solution of equation (10).

In the case where the dimension of A is greater than that of B, append an identity matrix to the lower right corner of B so that the two matrices have the same dimensions. Note that the necessary criterion for the application of the SMW updating formula is that $B^{-1}b$ can be computed efficiently. Note that

$$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

Thus the modified coefficient matrix for the original problem is as easy to "invert" as its unmodified form.

5 Optimal Loading

An immediate application of UMOM is to the optimal loading problem. We are interested in minimizing the scattering cross section of an obstacle by varying its surface impedance. The discrete approximation to this problem is a nonlinear optimization problem. This problem can be solved by applying UMOM. In the analysis below we consider a single angle and a single frequency; practical applications usually consider a range of angles and frequencies. For simplicity, only examples with real-valued impedances are considered in the analysis. The general case is solved by treating the real and imaginary parts as separate variables.

Let μ be the vector of discrete impedances of the cells in the scattering obstacle. Let B be the impedance matrix associated with the obstacle geometry and μ. Then the objective function for the optimization problem is

$$\sigma = \sigma(\mu)$$

where $\sigma = |L(x)|^2$, L is a linear functional independent of μ, and x solves $Ax = b$ with b the usual excitation vector. In practice the impedance varies over p cells with $n >> p$, where n is the total number of cells in the problem.

At each iteration, the bulk of computation in the optimal loading problem involves the computation of the objective function, which involves solutions to the matrix equation $Ax = b$, requiring $O(n^3)$ operations. We present an efficient method of computing of the objective function using the SMW which requires $O(np + p^3)$ operations: the first solution of the linear system is computed by MOM; subsequent solutions for different values of μ are computed by SMW.

For simplicity, assume below that the cells in the model are ordered so that the only the first p cells have variable impedance, and that these impedances vary independently. The general case can be handled by the INDEX process discussed in Section 3.

Let B be the impedance matrix associated with an initial impedance μ_0 and let A be the impedance matrix for an updated value of μ. Then from the discussion in section 2, $B - A = UV^T$ where U is an $n \times p$ matrix related to the basis and testing functions; in the case when both are pulse functions, the j-th column, $j = 1, 2, \ldots, p$, has the value 1 at its j-th entry and zero everywhere else, and

$$V^T = [D|0] \tag{14}$$

where D is a $p \times p$ diagonal matrix whose diagonal entries are the components of μ.

Note that U and b are independent of μ.

The flow of the optimization calculations proceeds as follows.

Algorithm 2.

Step 1 Initialization: Compute U, b, $\hat{U} = B^{-1}U$, $\hat{b} = B^{-1}b$ by MOM, with U and b overwritten by $\hat{U}$ and $\hat{b}$.

Step 2 Iteration: Compute the objective function, applying the SMW to compute

$$x = A^{-1}b = \hat{b} + \hat{U}(I - V^T\hat{U})^{-1}V^T\hat{b}. \tag{15}$$

Step 3 Test for Optimality: If the solution is optimal, stop. Otherwise recompute new μ and V^T as defined in equation (14) and go to Step 2.

Note that B^{-1} is no longer needed in Step 2, so the memory used by B^{-1} can be used to store $B^{-1}U$ if the appropriate I/O procedure is used.

The flow of the above calculations is summarized in Fig. 4.

From Step 2, it can be seen that using the SMW formula for the computation of $A^{-1}b$ results in an $O\left(np + p^3\right)$ operation count compared with an $O\left(n^3\right)$ operation count using MOM.

The use of objective function gradients in optimization algorithms is well known. We now investigate the computation of the objective function gradient using the SMW formula. A finite difference approximation to the gradient of σ requires $O\left(np^2\right)$ operations. The gradient can also be computed in $O\left(np^2\right)$ operations by applying the SMW formula and observing that required intermediate quantities are already stored for objective function evaluations.

Recall that

$$\sigma = \left|L(x)\right|^2 = L(x)\,\overline{L(x)}$$

from which

$$\frac{\partial\sigma}{\partial\mu_k} = L\left(\frac{\partial x}{\partial\mu_k}\right)\overline{L(x)} + L(x)\,\overline{L\left(\frac{\partial x}{\partial\mu_k}\right)}$$

by the linearity of L. The derivatives of x are obtained by implicitly differentiating $Ax = b$:

$$\frac{\partial A}{\partial\mu_k}x + A\frac{\partial x}{\partial\mu_k} = 0$$

$$\frac{\partial x}{\partial\mu_k} = -A^{-1}\left(\frac{\partial A}{\partial\mu_k}x\right).$$

From the decomposition $B - A = UV^T$, the form of V^T as defined in equation (14), and the fact that U is independent of μ, it can be seen that $\frac{\partial A}{\partial\mu_k}$ has only one nonzero column, and it is just the k-th column of U which we call u_k. That is,

$$\frac{\partial A}{\partial\mu_k} = [0,\ldots,u_k,0,\ldots]. \tag{16}$$

If we write $x = (x_1,\ldots,x_n)^T$ then $\frac{\partial A}{\partial\mu_k}x = x_k u_k$. From this it follows that

$$\frac{\partial x}{\partial\mu_k} = -x_k A^{-1}(u_k). \tag{17}$$

Now the SMW formula can be applied to compute $A^{-1}(u_k)$ in $O(np)$ operations using the fact that $B^{-1}(u_k)$ is already stored for use in evaluating the objective function. Assuming that $\left(I - V^T B^{-1} U\right)^{-1}$ is also already stored, the gradient of σ can be calculated as above in $O\left(np^2\right)$ operations.

We have shown that the well known technique of impedance loading can be efficiently applied using moment method techniques. The implementation can use subroutines from existing moment method codes and existing optimization software. Unlike earlier techniques [12], a separate analysis is not required for each new geometry.

6 Numerical Examples

We have incorporated the SMW into three moment method codes currently in use:

(1) RAMZ - a modification of RAMVS [**13**] which is a 2D moment method code for scatterers treated with absorbing materials that satisfy an impedance boundary condition,

(2) DMS2 - a 2D volumetric moment method code internal to the Boeing Company,

(3) NEC - written by Burke and Poggio [**14**] which we use as a 3D moment method code for structures modeled as wire grid surfaces in free space or over a ground plane.

The modification made to NEC is for the optimal loading problem.

The following is a discussion of three examples, one corresponding to each of the the above codes and its corresponding modifications:

(1) The original problem is a perfectly conducting sheet of 10 wavelengths lying on the x-axis. The perturbed problem is obtained by replacing the leftmost 10 cells (1 wavelength) with a material whose electric and magnetic impedances are $(.5, .6)$ and $(1.65, 1.65)$. Both problems have 100 cells. The H-pol monostatic scattering pattern is computed from 0 to 30 degrees at 10 degree intervals. See Fig. 5 for a description of the geometry and angle orientation.

(2) The original problem is a perfectly conducting ellipse whose major and minor axes are respectively 4 wavelengths and 2 wavelengths. The major and minor axes lie respectively on the y- and x- axes. The perturbed problem is obtained by removing 4 cells on the right of the ellipse above the x-axis. The original problem has 101 cells and the perturbed problem has 97 cells. The E-pol monostatic RCS for incidence angles from zero to 10 degrees are computed. See Fig. 6 for a description of the

geometry and the angle orientation.

(3) This example is motivated by Schindler, Mack and Blacksmith [**12**]. A pair of parallel dipoles is viewed in the plane of the dipoles, polarized also in the plane of the dipoles. The dipoles are 1 meter long, spaced .2 meters apart, and are divided into 6 segments. The center two segments of each dipole are loaded with impedances of $200 + \jmath 200$ at initialization. The scattering is optimized over the sector from 0 to 40 degrees using 5 angles at a single frequency, 300 mhz. See Fig. 7 for a description of the geometry and the angle orientation.

All three examples were run on the VAX 11-785. The table below summarizes the matrix dimensions, perturbation order and CPU time. n and p are respectively the dimensions of the original matrix and the perturbed part. MOM and UMOM respectively represent the original moment methods code and their corresponding SMW modification.

TABLE 1. CPU SECOND PERFORMANCE COMPARISON

Example	n	p	MOM sec.	UMOM sec.
1	200	20	36.9	26.7
2	101	4	19.6	8.7
3	12	4	1.56	0.07

The timing for Example 3 is the timing per iteration. This example ran for 201 iterations.

The scattering patterns of the above examples verify the accuracy of the methods presented in this paper. Fig. 8 shows the scattering pattern of the perfectly conducting sheets and the treated sheet (example 1). The treated case was run through the original moment method code and the answers coincides with those obtained by the SMW-modified moment method code. Fig. 9 shows the scattering pattern of the closed elliptic conductor and the elliptic conductor with an aperture. The optimal impedances for the center 2 segments of each dipole in example 3 are $78 + 407j$ and $56 + 487j$. Fig. 10 illustrates the effect of this optimal loading.

7 Conclusion

The classical theory of the Sherman-Morrison-Woodbury updating formula and its application to scattering problems have been presented. Two examples have been considered: the perturbed problem and the optimal loading problem. It has been shown that an easy modification to a basic moment method code yields an efficient solution method for the perturbed problem and the optimal loading problem. Our numerical examples have demonstrated that the new method is numerically stable and is between 1.5 to 22 times faster than the classical approach.

Note that for any matrix X such that $I - X$ is nonsingular, we have

$$(I - X)^{-1} = I + X + X^2 + X^3 + \cdots . \tag{A.1}$$

The right hand side of equation (A.1) is the Taylor series of $(I - X)^{-1}$.

Recall that

$$\begin{aligned} A &= B - UV^T \\ &= B(I - B^{-1}UV^T). \end{aligned}$$

Thus the inverse of A can be written as

$$\begin{aligned} A^{-1} &= [B(I - B^{-1}UV^T)]^{-1} \\ &= [I - B^{-1}UV^T]^{-1}B^{-1}. \end{aligned} \tag{A.2}$$

Note that since A is nonsingular, the inverse of $I - B^{-1}UV^T$ exists and can be written as an infinite series:

$$\begin{aligned} [I - B^{-1}UV^T]^{-1} &= I + B^{-1}UV^T + B^{-1}UV^TB^{-1}UV^T + \cdots + (B^{-1}UV^T)^k + \cdots \\ &= I + B^{-1}U[I + V^TB^{-1}U + \cdots + (V^TB^{-1}U)^{k-1} + \cdots]V^T \\ &= I + B^{-1}U[I - V^TB^{-1}U]^{-1}V^T \end{aligned} \tag{A.3}$$

Substitute A.3 into A.2 we obtain:

$$A^{-1} = B^{-1} + B^{-1}U(I - V^TB^{-1}U)^{-1}V^TB^{-1}$$

which is the SMW.

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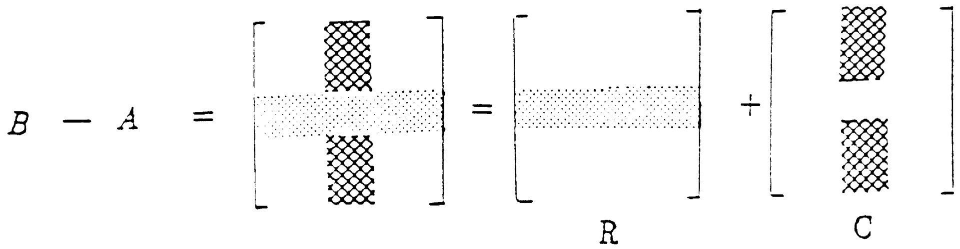

Fig. 1 The Perturbed Parts of the New Matrix

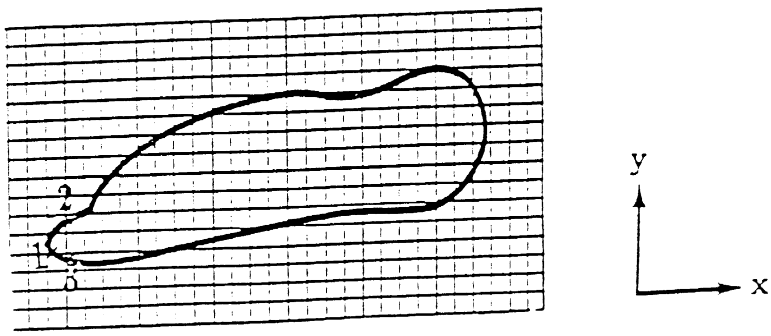

Fig. 2 An Example of Ordering Points in the $x - y$ Plane

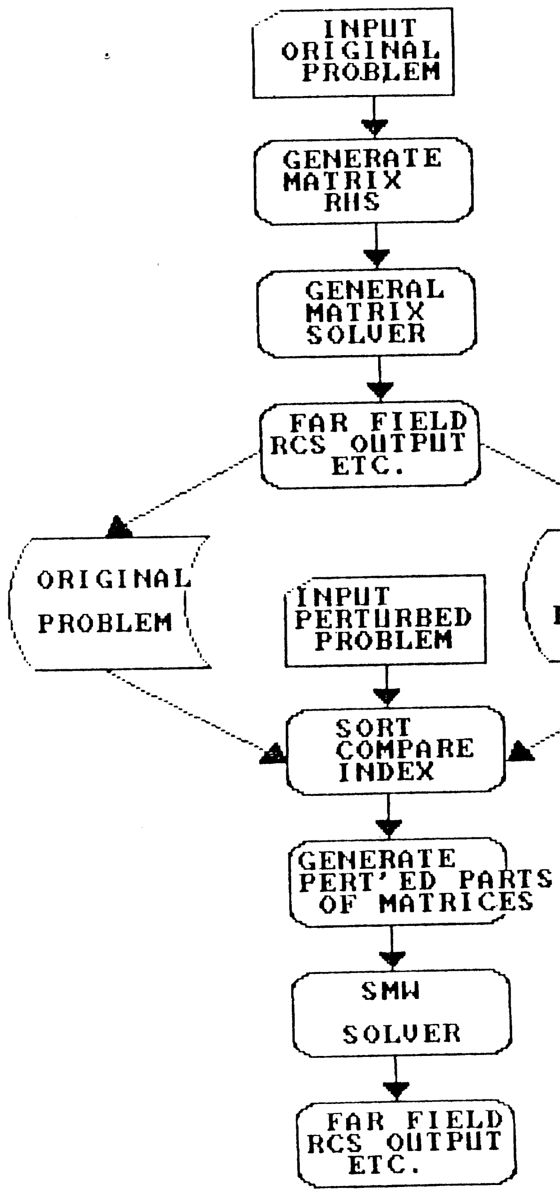

Fig. 3 Flow Diagram for MOM-UMOM

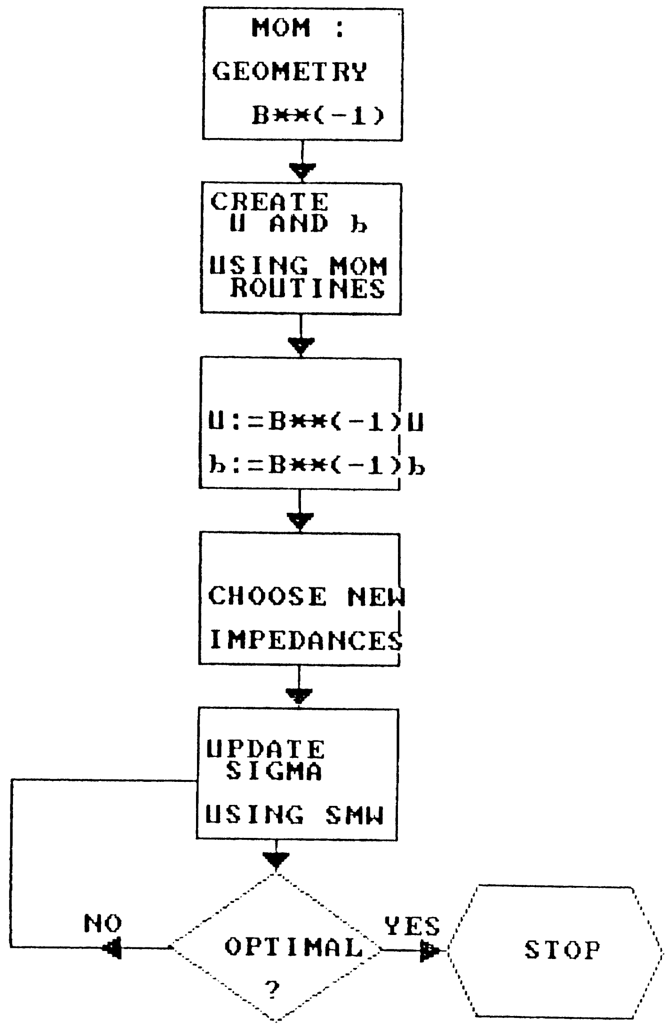

Fig. 4 Flow Diagram for The Optimal Loading Calculatio

Fig. 5 Geometry and Angle Orientation of Example 1

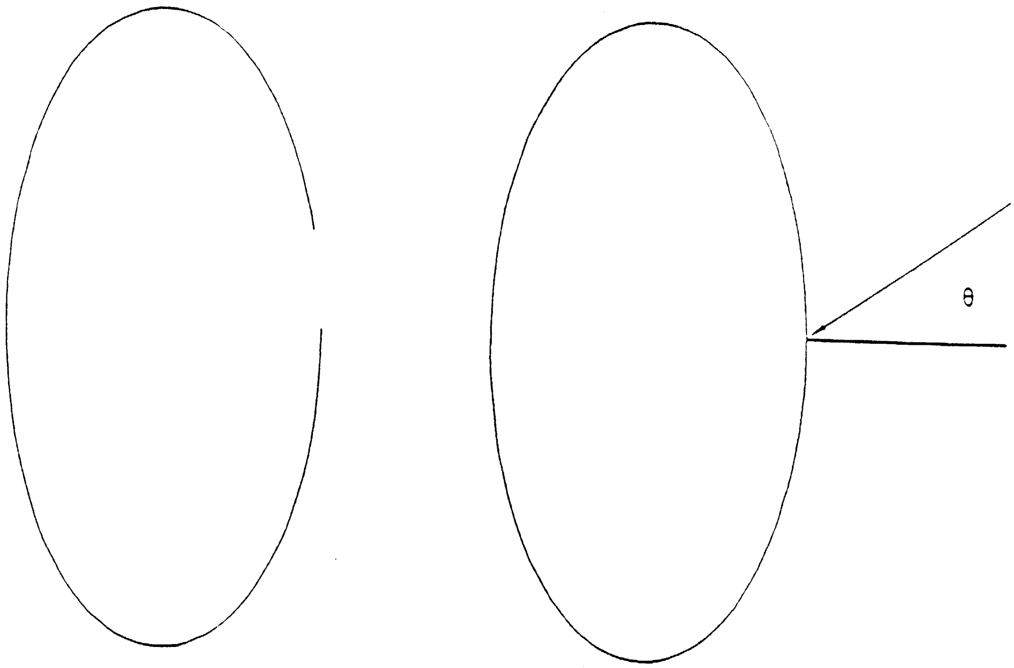

Fig. 6 Geometry and Angle Orientation of Example 2

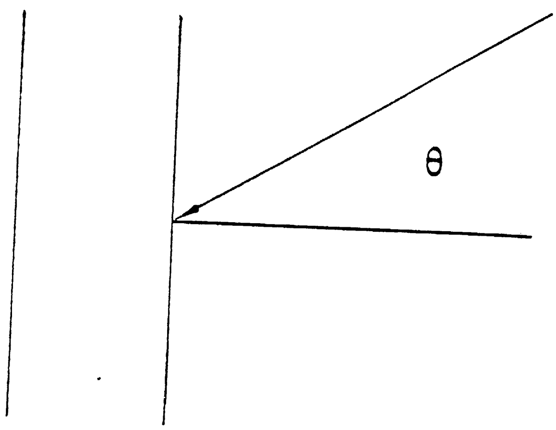

Fig. 7 Geometry and Angle Orientation of Example 3

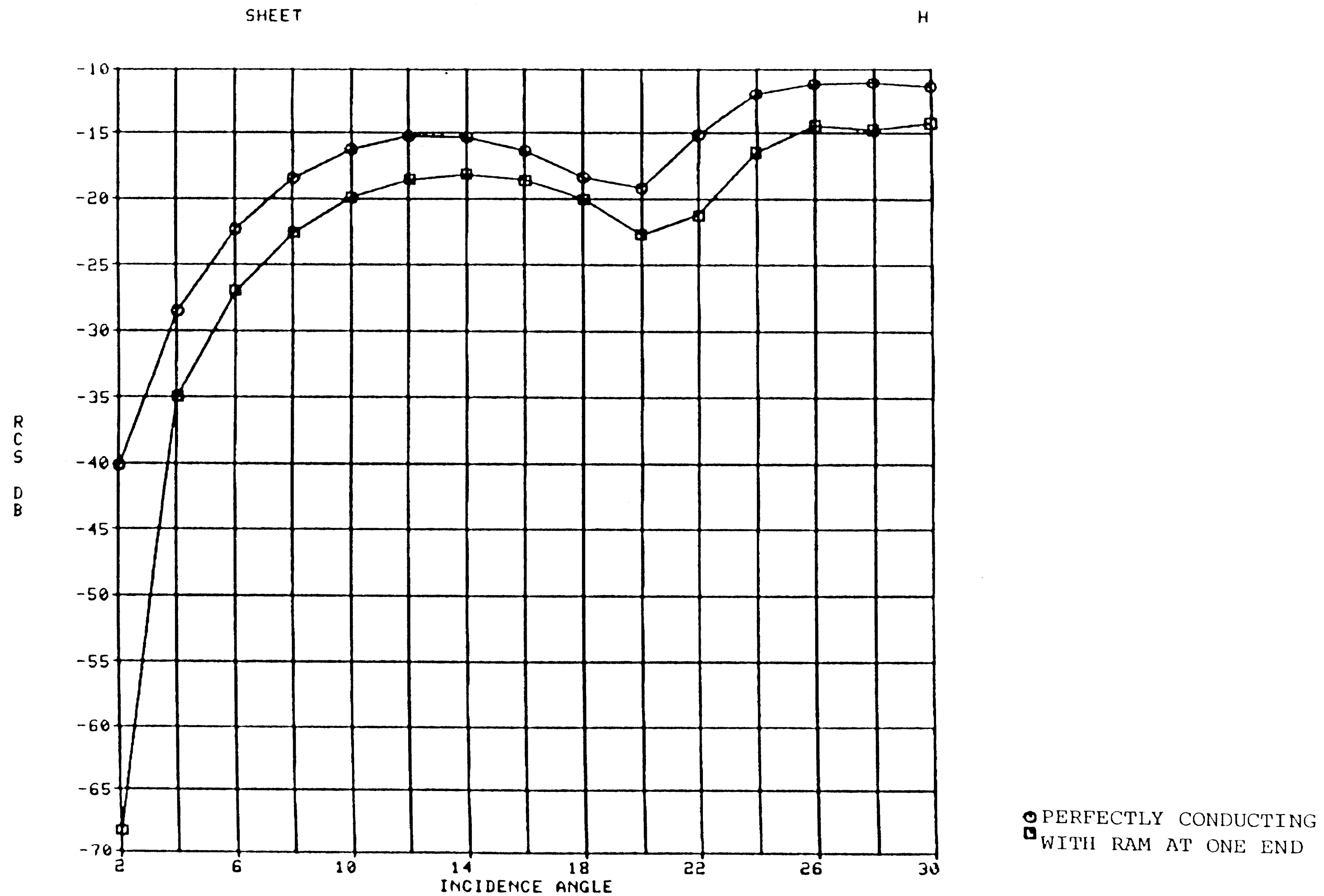

Fig. 8 Scattering Pattern for Example 1

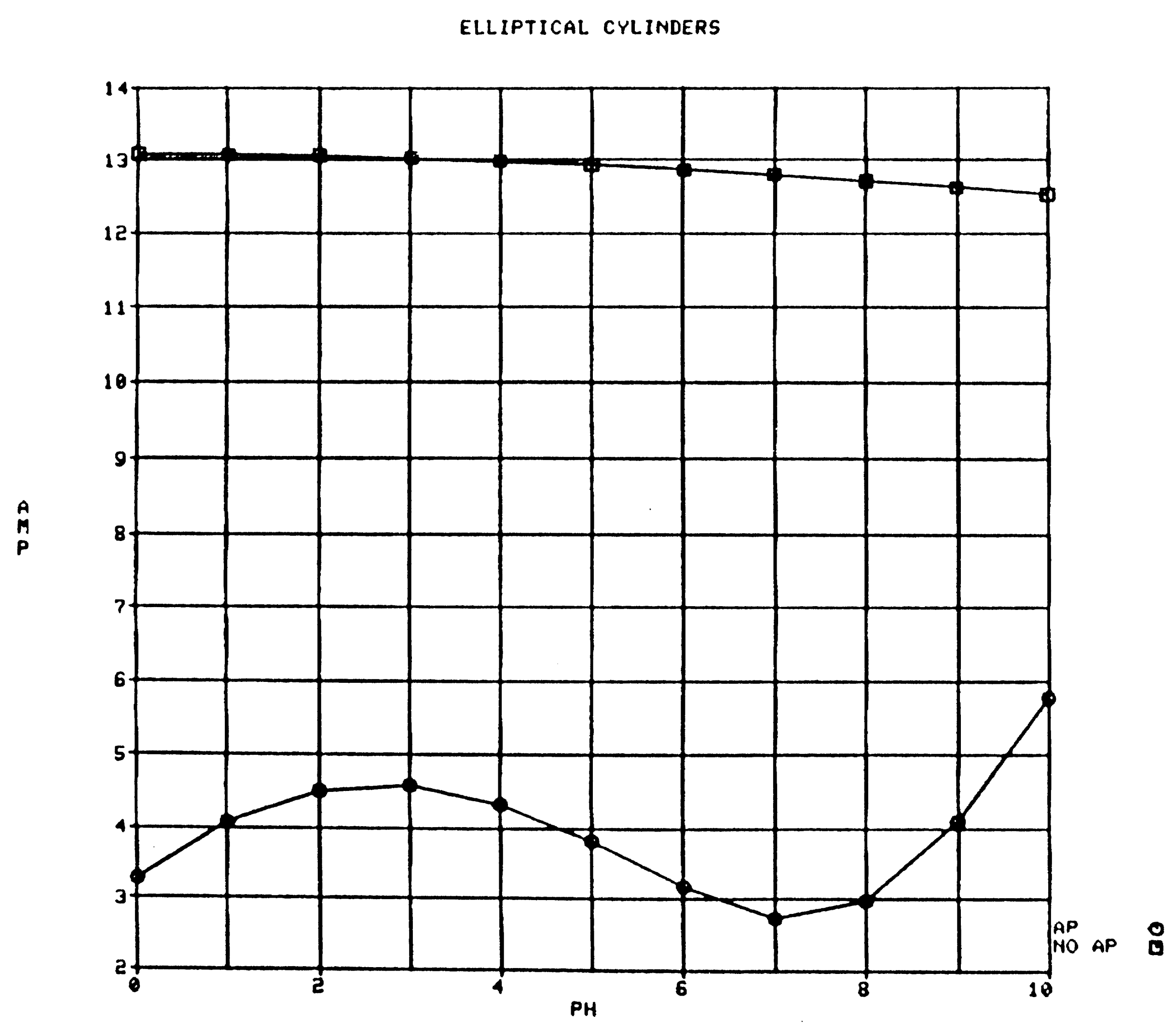

Fig. 9 Scattering Pattern for Example 2

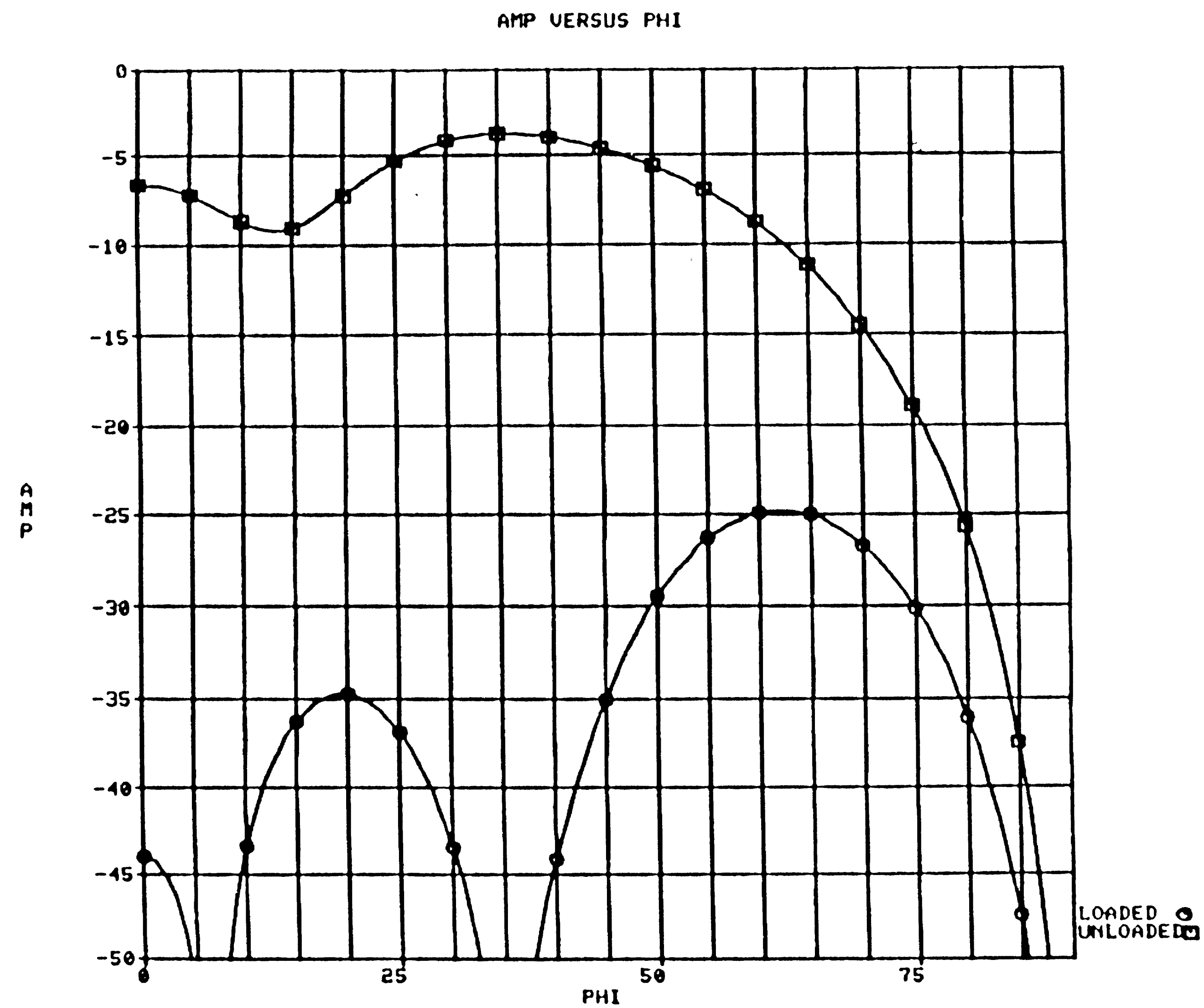

Fig. 10 Scattering Pattern for Example 3