The Newton-Raphson Method for Complex Equation Systems

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***Abstract*—A formulation for the complex Newton-Raphson method is proposed. The derivation is obtained on the assumption of a nonanalytical equation system. The method is applied to the finite element calculation of shielding problems with sinusoidal excitation and ferromagnetic material. An application example for which measurement data are available is given in order to judge the convergence characteristics and the reliability of the proposed method.**

I. Introduction

The Newton-Raphson method is a well-known and powerful method for solving nonlinear real equation systems which is required for example in the calculation of nonlinear magnetostatic problems with the finite element method. In the case of sinusoidal excitation and nonlinear material behavior, e.g. in the case of shielding problems with ferromagnetic material, the application of a sinusoidal calculation based on a specialized material model leads to complex nonlinear equation systems [1], [2]. For the solution process the Newton-Raphson method is chosen and adapted to the complex case [3].

II. Basic Investigations

A. Analytical Systems

The complex equation system

$$F_i(A_1, A_2, \ldots, A_n) = U_i(A_1, A_2, \ldots, A_n) + \mathrm{j}\, V_i(A_1, A_2, \ldots, A_n) = 0 \quad (1)$$

with the complex variables

$$A_k = \xi_k + \mathrm{j}\,\eta_k$$

and $i, k = 1, 2, \ldots, n$ shall be considered. The expansion of (1) by Taylor series takes the form

$$F_i + \sum_{j=1}^{n} \frac{\partial F_i}{\partial A_j} \Delta A_j = 0. \quad (2)$$

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If the functions F_i are assumed to be analytical, the derivative $\partial F_i / \partial A_j$ can be expressed following the Cauchy-Riemann differential equations as

$$\frac{\partial F_i}{\partial A_j} = \frac{\partial U_i}{\partial \xi_j} + \mathrm{j}\,\frac{\partial V_i}{\partial \xi_j} \quad (3)$$

or as

$$\frac{\partial F_i}{\partial A_j} = \frac{\partial V_i}{\partial \eta_j} - \mathrm{j}\,\frac{\partial U_i}{\partial \eta_j}. \quad (4)$$

Equation system (2), which is a $n \times n$ complex system, would have to be solved using (3) or (4). Applying e.g. (3) and separating into the real and imaginary part, one obtains

$$U_i + \sum_{j=1}^{n} \left(\frac{\partial U_i}{\partial \xi_j} \Delta\xi_j - \frac{\partial V_i}{\partial \xi_j} \Delta\eta_j \right) = 0, \quad (5)$$

$$V_i + \sum_{j=1}^{n} \left(\frac{\partial V_i}{\partial \xi_j} \Delta\xi_j + \frac{\partial U_i}{\partial \xi_j} \Delta\eta_j \right) = 0. \quad (6)$$

B. Nonanalytical Systems

As it will be shown in the next section, the considered finite element formulation leads to a system which is not analytical. Consequently, the expressions (3) or (4) cannot be used. Therefore the following system shall be considered, which is equivalent to (1):

$$U_i(\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n) = 0,$$

$$V_i(\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n) = 0.$$

The Taylor expansion now is performed for the real and imaginary parts ξ_k and η_k:

$$U_i + \sum_{j=1}^{n} \left(\frac{\partial U_i}{\partial \xi_j} \Delta\xi_j + \frac{\partial U_i}{\partial \eta_j} \Delta\eta_j \right) = 0, \quad (7)$$

$$V_i + \sum_{j=1}^{n} \left(\frac{\partial V_i}{\partial \xi_j} \Delta\xi_j + \frac{\partial V_i}{\partial \eta_j} \Delta\eta_j \right) = 0. \quad (8)$$

As the considered system is not analytical, (2) is not applicable to this system. Thus the system to solve will not be a complex system with the unknowns ΔA_k, but a real valued $2n \times 2n$ system with the unknowns $\Delta\xi_k$ and $\Delta\eta_k$. For analytical systems, (7) and (8) are equivalent to (5) and (6), which can be seen if the Cauchy-Riemann differential equations are applied.

III. Formulation for the Finite Element Method

The application of the Newton-Raphson method for complex equation systems is related to nonlinear eddy current problems with sinusoidal excitation and ferromagnetic material. It is assumed that an appropriate material model is used which allows a sinusoidal calculation in spite of the nonlinearity. Such a model is the effective permeability model: it uses the rms value of the magnetic flux density, which can be obtained from the magnetization curve, to define an effective value of the permeability and the reluctivity, respectively. A detailed description can be found in [4], [5].

A. Basic Formulation

From Maxwell's equations one obtains

$$\nabla \times (\nu \nabla \times \vec{A}) + \kappa \mathrm{j}\,\omega \vec{A} = \vec{J}_0.$$

For two-dimensional problems, which shall be considered, the only component of the vector potential is the z-component. Thus it holds $\vec{A} = A\,\vec{e}_z = (\xi + \mathrm{j}\,\eta)\,\vec{e}_z$ and $\vec{J}_0 = J_0\,\vec{e}_z$, where J_0 is the exciting current density.

Applying Galerkin's method with the shape functions α, one gets

$$F_i = \sum_{k=1}^{n} \Bigg(\underbrace{\int_\Omega \nu_{\text{eff}}(\xi_\lambda, \eta_\lambda) \nabla\alpha_i \nabla\alpha_k \,\mathrm{d}\Omega}_{Q_{ik}(\xi_\lambda,\eta_\lambda)} + \mathrm{j}\,\omega \underbrace{\int_\Omega \kappa\, \alpha_i \alpha_k \,\mathrm{d}\Omega}_{P_{ik}} \Bigg) A_k - \int_\Omega \alpha_i J_0 \,\mathrm{d}\Omega \tag{9}$$

with $i, \lambda = 1 \ldots n$. $\nu_{\text{eff}}(\xi_\lambda, \eta_\lambda)$ is the effective reluctivity obtained by the mentioned material model. The nonlinearity of these equations consists in the dependence of ν_{eff} on the nodal values of the real and imaginary part of the vector potential. Separating (9) into its real and imaginary part results in

$$U_i = \sum_{k=1}^{n} Q_{ik}(\xi_\lambda, \eta_\lambda)\xi_k - \sum_{k=1}^{n} P_{ik}\eta_k - \int_\Omega \alpha_i J_0 \,\mathrm{d}\Omega, \tag{10}$$

$$V_i = \sum_{k=1}^{n} P_{ik}\xi_k + \sum_{k=1}^{n} Q_{ik}(\xi_\lambda, \eta_\lambda)\eta_k. \tag{11}$$

The system (10), (11) is not analytical which follows from the Cauchy-Riemann differential equations. It holds

$$\frac{\partial U_i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \sum_{k=1}^{n} Q_{ik}\xi_k = Q_{ij} + \sum_{k=1}^{n} \frac{\partial Q_{ik}}{\partial \xi_j}\xi_k, \tag{12}$$

$$\frac{\partial V_i}{\partial \eta_j} = \frac{\partial}{\partial \eta_j} \sum_{k=1}^{n} Q_{ik}\eta_k = Q_{ij} + \sum_{k=1}^{n} \frac{\partial Q_{ik}}{\partial \eta_j}\eta_k, \tag{13}$$

where it is obvious that

$$\frac{\partial U_i}{\partial \xi_j} \neq \frac{\partial V_i}{\partial \eta_j}.$$

For analytical systems (12) and (13) would agree. In an analogous way it can be shown that $\partial U_i/\partial \eta_j$ and $-\partial V_i/\partial \xi_j$ do not agree, either.

B. Taylor Expansion

The expansion by Taylor series of (10) and (11) in accordance with (7) and (8) leads to

$$U_i + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \left(\frac{\partial Q_{ik}}{\partial \xi_j}\xi_k \right) + Q_{ij} \right] \Delta\xi_j + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \left(\frac{\partial Q_{ik}}{\partial \eta_j}\xi_k \right) - P_{ij} \right] \Delta\eta_j = 0, \tag{14}$$

$$V_i + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \left(\frac{\partial Q_{ik}}{\partial \xi_j}\eta_k \right) + P_{ij} \right] \Delta\xi_j + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \left(\frac{\partial Q_{ik}}{\partial \eta_j}\eta_k \right) + Q_{ij} \right] \Delta\eta_j = 0. \tag{15}$$

C. Evaluation of the Derivatives

The evaluation of the derivatives in (14) and (15) is done using the magnetization curve of the ferromagnetic material and the potential formulation. From (9) it follows

$$\frac{\partial Q_{ik}}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left(\int_\Omega \nu_{\text{eff}}(\xi_\lambda, \eta_\lambda) \nabla\alpha_i \nabla\alpha_k \,\mathrm{d}\Omega \right) = \int_\Omega \frac{\partial \nu_{\text{eff}}(\xi_\lambda, \eta_\lambda)}{\partial \xi_j} \nabla\alpha_i \nabla\alpha_k \,\mathrm{d}\Omega.$$

For the evaluation of the derivative of the effective reluctivity only *one* finite element shall be considered. The reluctivity is taken to be constant in that element. Thus the dependence on ξ_λ and η_λ expresses the dependence on the vector potential in that element:

$$A = \sum_\lambda (\xi_\lambda + \mathrm{j}\,\eta_\lambda)\alpha_\lambda, \tag{16}$$

where λ runs up to the number of nodes in the element. Thus one obtains

$$\frac{\partial \nu_{\text{eff}}(\xi_\lambda, \eta_\lambda)}{\partial \xi_j} = \frac{\partial \nu_{\text{eff}}}{\partial |B|} \left(\frac{\partial |B|}{\partial |B_x|} \frac{\partial |B_x|}{\partial \xi_j} + \frac{\partial |B|}{\partial |B_y|} \frac{\partial |B_y|}{\partial \xi_j} \right)$$

and

$$\frac{\partial \nu_{\text{eff}}(\eta_\lambda, \eta_\lambda)}{\partial \eta_j} = \frac{\partial \nu_{\text{eff}}}{\partial |B|} \left(\frac{\partial |B|}{\partial |B_x|} \frac{\partial |B_x|}{\partial \eta_j} + \frac{\partial |B|}{\partial |B_y|} \frac{\partial |B_y|}{\partial \eta_j} \right),$$

where the derivative $\partial \nu_{\text{eff}}/\partial |B|$ is determined from the magnetization curve. $|B|$ and $|B_i|$ are the rms values of the flux density and the component i of the flux density in one element, for which the relation

$$|B| = \sqrt{|B_x|^2 + |B_y|^2}$$

is valid. They are taken to be constant, e.g. they are the mean values in one element if no linear shape functions are used. It follows

$$\frac{\partial |B|}{\partial |B_x|} = \frac{|B_x|}{|B|}$$

and

$$\frac{\partial |B|}{\partial |B_y|} = \frac{|B_y|}{|B|}.$$

According to the relation $\vec{B} = \nabla \times \vec{A}$ one obtains for the complex value of the flux density under consideration of (16)

$$\begin{aligned}|B_x|^2 = B_x B_x^* &= \frac{\partial A}{\partial y}\frac{\partial A^*}{\partial y} = \\ &= \left[\frac{\partial}{\partial y}\sum_\lambda (\xi_\lambda + \mathrm{j}\,\eta_\lambda)\alpha_\lambda\right]\left[\frac{\partial}{\partial y}\sum_\lambda (\xi_\lambda - \mathrm{j}\,\eta_\lambda)\alpha_\lambda\right] = \\ &= \left[\sum_\lambda \xi_\lambda \frac{\partial \alpha_\lambda}{\partial y}\right]^2 + \left[\sum_\lambda \eta_\lambda \frac{\partial \alpha_\lambda}{\partial y}\right]^2.\end{aligned}$$

From this equation one gets directly

$$\frac{\partial |B_x|}{\partial \xi_j} = \frac{\mathrm{Re}\{B_x\}}{|B_x|}\frac{\partial \alpha_j}{\partial y}$$

and in an analogous manner

$$\frac{\partial |B_y|}{\partial \xi_j} = -\frac{\mathrm{Re}\{B_y\}}{|B_y|}\frac{\partial \alpha_j}{\partial x}.$$

The same procedure for $\partial \nu_{\text{eff}}/\partial \eta_j$ results in

$$\frac{\partial |B_x|}{\partial \eta_j} = \frac{\mathrm{Im}\{B_x\}}{|B_x|}\frac{\partial \alpha_j}{\partial y},$$

$$\frac{\partial |B_y|}{\partial \eta_j} = -\frac{\mathrm{Im}\{B_y\}}{|B_y|}\frac{\partial \alpha_j}{\partial x}.$$

Finally one gets

$$\frac{\partial Q_{ik}}{\partial \xi_j} = \int_\Omega \frac{\partial \nu_{\text{eff}}}{\partial |B|}\left(\frac{\mathrm{Re}\{B_x\}}{|B|}\frac{\partial \alpha_j}{\partial y} - \frac{\mathrm{Re}\{B_y\}}{|B|}\frac{\partial \alpha_j}{\partial x}\right)\nabla\alpha_i\nabla\alpha_k\,\mathrm{d}\Omega,$$

$$\frac{\partial Q_{ik}}{\partial \eta_j} = \int_\Omega \frac{\partial \nu_{\text{eff}}}{\partial |B|}\left(\frac{\mathrm{Im}\{B_x\}}{|B|}\frac{\partial \alpha_j}{\partial y} - \frac{\mathrm{Im}\{B_y\}}{|B|}\frac{\partial \alpha_j}{\partial x}\right)\nabla\alpha_i\nabla\alpha_k\,\mathrm{d}\Omega.$$

Herewith all derivatives are determined and the equation system can be solved.

D. *Structure of the Equation System*

As shown in Section III.A., the system (10), (11) is not analytical which leads to the described Taylor expansion and thus to a $2n \times 2n$ real valued system. Compared to the analytical complex valued system (2) with the dimension n, twice of the memory capacity is necessary.

The matrix consists of four $n \times n$ parts which are symmetric sparse matrices. The whole matrix however is a nonsymmetric sparse matrix. It takes the form

$$\begin{bmatrix} \mathbf{Q}_\xi + \mathbf{Q} & | & \mathbf{Q}_\eta - \mathbf{P} \\ \mathbf{Q}_\xi + \mathbf{P} & | & \mathbf{Q}_\eta + \mathbf{Q} \end{bmatrix}$$

which shall be considered as symbolic: $\mathbf{Q}$ – terms related to Q_{ik}; $\mathbf{Q}_\xi$ – terms related to $\partial Q_{ik}/\partial \xi$; etc.

IV. Application Example

A. *Problem Definition*

The example considered, is a shielding problem: a steel plate over a double line. This is a problem which was set up at the TU Berlin in order to study the shielding effect of constructional steel. A detailed description can be found in [5].

The principal arrangement is shown in Fig. 1. The dimensions are: plate: $b = 1390\,\text{mm}$ and $d = 3\,\text{mm}$; distance of the conductors: $a = 250\,\text{mm}$; distance between the plate and the cable: $h = 100\,\text{mm}$. The origin of the cartesian coordinate system is located in the middle on top of the plate.

The cable consists of three conductors, each one made of twisted copper wires and arranged in a triangle. In each of them flows one third of the current I. For the calculation the cable is modelled as *one* conductor with a cross-section of $555\,\text{mm}^2$.

The plate material is constructional steel St 37. The conductivity has been determined as $\kappa = 6.41 \cdot 10^6\,\text{S/m}$. The magnetization curve has been measured for the used plate and is given in Fig. 2.

The goal is to determine the rms value of the magnetic flux density at the heights $y = 5\,\text{cm}$ and $y = 100\,\text{cm}$ above the plate. The exciting current is sinusoidal with the frequency $f = 50\,\text{Hz}$ and the rms value $I = 900\,\text{A}$.

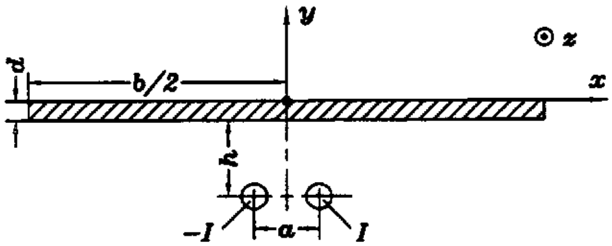

Fig. 1. Principal arrangement

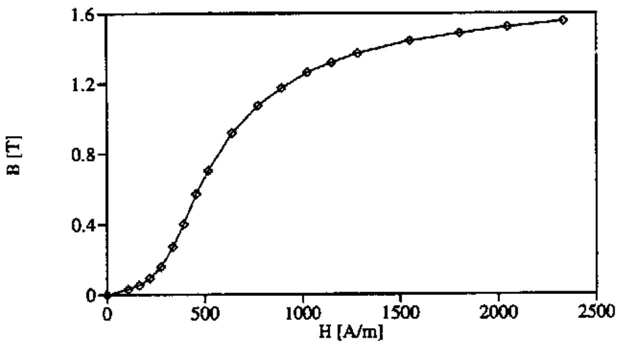

Fig. 2. Magnetization curve of the plate

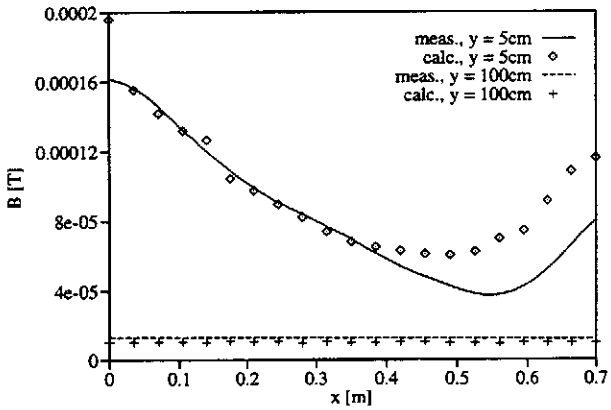

Fig. 3. Magnetic flux density (rms) above the plate

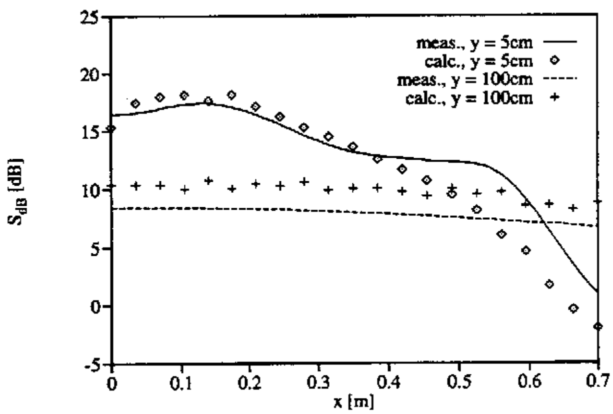

Fig. 4. Shielding factor

B. Results

In the following figures the result obtained by measurements is compared with the result obtained by the described calculation method. Figure 3 shows the rms value of the magnetic flux density whereas Fig. 4 shows the shielding factor, which is defined as

$$S_{\mathrm{dB}} = 20 \log \frac{B_{\text{not shielded}}}{B_{\text{shielded}}} \,\mathrm{dB}$$

and can be taken to judge the shielding efficiency. The calculated and measured values of the flux density either agree or the calculated ones are higher. As hysteresis losses were not taken into account for the calculation, this behavior was to be expected.

The calculation was performed on a HP 755 Workstation. The linear system was solved using the ILUBiCG method with a convergence criterion of 10^{-7}. Table I gives information about the data of the calculation process. It can be seen that the proposed Newton-Raphson method for nonlinear complex systems allows a considerably fast calculation of the interesting field quantities. Especially the number of Newton iterations is very low, i.e. the proposed determination of the derivatives leads to very fast convergence of the Newton process, for which the convergence criterion was set to $||\Delta A||/||A|| < 10^{-3}$. The total calculation time was 803 s.

TABLE I
DATA OF THE CALCULATION PROCESS

Refinement step	1	2	3	4	5
Newton iterations	2	3	3	3	3
Calculation time [s]	0.2	1.4	11.1	83.0	707.2
$\|\|\Delta A\|\|/\|\|A\|\|$	8e-4	9e-5	9e-5	8e-5	8e-5
Finite elements	112	448	1792	7168	28672
Matrix dim. ($2n$)	138	498	1890	7362	29058
Matrix elements	1716	6564	25668	101508	403716

V. Conclusions

A method for solving nonlinear complex equation systems, based on the Newton-Rahpson method for real systems, was proposed. It could be shown that the described method has very good convergence characteristics and thus allows a considerably fast calculation of nonlinear shielding problems. The reliability of the calculated results was proved by a comparison with measurements.

Summarizing, the proposed method is an appropriate tool for the fast determination of the basic tendency of the efficiency of shielding arrangements.

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