On the Physical Interpretation of the Sobolev Norm in Error Estimation

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***Abstract*— Error estimates for the moment method have been obtained in terms of Sobolev norms of the current solution. Motivated by the historical origins of Sobolev spaces as energy spaces, we show that the Sobolev norm used in these estimates is related to the forward scattering amplitude, for the case of 2D scattering from a PEC circular cylinder and for 3D scattering from a PEC sphere. These results provide a physical meaning for solution error estimates in terms of the power radiated by the error in the current solution. We further show that bounds on the Sobolev norm of the current error imply a bound on the error in the computed backscattering amplitude.**

***Index Terms*— Sobolev space, error analysis, method of moments**

I. Introduction

Since the introduction of the method of moments for solving electromagnetic radiation and scattering problems, error analysis of numerical methods has received much attention in the mathematics literature. This effort has led to fundamental results on the convergence of the method of moments. Typical of this work are proofs that under various assumptions about the algorithm and scattering problem, as the mesh is refined or the number of degrees of freedom of the approximate solution increases, numerical solutions converge to exact solutions. Theorems of this kind have been obtained for 2D smooth closed curves and screens [1], dielectric polygons [2], and have been verified by numerical studies [3]. For smooth screens in 3D, similar results are available for scalar fields [2], [4], [5]. These results are of great importance because they place the algorithms of computational electromagnetics on solid theoretical ground.

The approach taken in this work by the numerical analysis community is to place the integral operators of radiation and scattering in a Sobolev space setting. This leads to asymptotic solution error estimates of the form

$$\|\Delta u\|_{\mathcal{H}^s} \le Ch^r \tag{1}$$

where the norm is defined on the Sobolev space $\mathcal{H}^s$, with $s = -1/2$ for the TM polarization and $s = 1/2$ for TE [6]. Δu is the difference between the exact current solution and a numerical solution, and h is the mesh element width or discretization length. The convergence rate r is typically $1/2$ for low order basis functions. All dependence on the physical problem and implementation details of the numerical method, including the incident field, frequency, scatterer geometry, and choice of basis functions, is lumped into the unknown constant C.

While the estimate (1) shows in an abstract sense that a numerical solution converges as the discretization length becomes small, it cannot be used to determine the error in a specific numerical solution because the Sobolev norm $\|\cdot\|_{\mathcal{H}^s}$ can be difficult to compute [7] and the constant C is unknown. Furthermore, it is not obvious how the Sobolev norm may relate to a directly measurable, physical quantity.

Motivated by the historical origin of Sobolev spaces as energy spaces, we show in this paper that the Sobolev norm in Eq. (1) is related to a readily computable, physical quantity: the power supplied by a surface current to its surroundings. Heuristically, a Sobolev space for fields in a volumetric region consists of those functions which have finite energy, where the energy measure is induced by a particular partial differential equation. Sobolev spaces of surface currents are defined slightly differently, as they consist of functions on the surface that radiate finite energy [7]. This definition is motivated by Poynting's theorem,

$$\int_{\mathcal{S}} \boldsymbol{E}^* \cdot \boldsymbol{J}_s \, d\mathcal{S} = \frac{i\omega}{2} \int_V \epsilon |\boldsymbol{E}|^2 - \mu |\boldsymbol{H}|^2 \, dV - \oint_{\partial V} \boldsymbol{S}^* \cdot \hat{n} \, dV$$

where the terms are defined as is usual in electromagnetic theory. The Sobolev space of fields $\boldsymbol{E}$ and $\boldsymbol{H}$ is essentially defined by requiring that the volume integral on the right-hand side be finite. In order to obtain consistent function spaces for fields and surface currents, at least nonrigorously, the Sobolev space of surface currents should include all functions on S for which the left-hand side is finite. If the surface current is produced by an incident field illuminating a PEC scatterer, then the left-hand side of Poynting's theorem with a suitable normalization becomes the forward scattering amplitude of the scatterer. This suggests a connection between the Sobolev norm in (1) and the forward scattering amplitude.

Based on this connection, we derive a direct relationship between the forward scattering amplitude and the Sobolev norm $\|\cdot\|_{\mathcal{H}^s}$. Proofs of the result are given for the specific

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cases of the circular cylinder and sphere, and we conjecture that similar relationships hold for more general geometries. This relationship between the Sobolev norm and the forward scattering amplitude is used to provide a physical interpretation for error estimates of the form of (1). We further show that the bound (1) implies a bound on the error in the computed backscattering amplitude solution. These results provide a link between abstract results of numerical analysis and the physical quantities that are the desired results of practical CEM simulations. Some of the results in this paper for the 2D case were presented in [8].

II. PEC Infinite Circular Cylinder

For a plane wave incident in the $-x$ direction on a PEC circular cylinder, the induced current u may be written as a Fourier series, $u = (2\pi)^{-1/2} \sum_q U_q e^{iq\phi}$, where ϕ is the azimuthal angle and the Fourier coefficients are given by

$$U_q = \sqrt{\frac{2}{\pi}} \frac{2}{\eta k a} \frac{i^{-q}}{H_q^{(1)}(ka)} \tag{2}$$

for the TM case, and the TE case is identical by replacing the Hankel function $H_q^{(1)}(ka)$ with its derivative. Here, η is the characteristic impedance of free space, k is the wavenumber, and a is the cylinder radius.

In general, the currents induced on 2D PEC scatterers lie in fractional order Sobolev spaces, where the order s is $-1/2$ for the TM polarization and $1/2$ for the TE polarization. For closed surfaces, the Sobolev norm is computable and is given by [9], [10] as

$$\|u\|_{\mathcal{H}^s}^2 = \sum_q |U_q|^2 \left(1+q^2\right)^s. \tag{3}$$

When s is an integer, the Sobolev norm (3) reduces to the more usual definition in terms of the L^2 norm of the current and its derivatives up to order s. For example, for $s = 0$, $\|u\|^2 = \sum_q |U_q|^2$, which by Parseval's relation is the L^2 norm of the current. The relationship between Eq. (3) for fractional s and a physical quantity is not immediately apparent.

Extrapolating the foundational relationship between Sobolev spaces and physical energy, we will show that the Sobolev norm (3) is equivalent to the forward scattering amplitude, which is given by

$$P(u) = -\frac{k\eta}{4} \int_{\mathcal{C}} E^{s*} u \, dl = \frac{k\eta}{4} \int_{\mathcal{C}} (\mathcal{L}u)^* u \, dl, \tag{4}$$

where E^s is the tangential component of the scattered field and $\mathcal{L}$ is the EFIE operator so that $\mathcal{L}u = E^i$. Note that P is the left-hand side of Poynting's theorem (2) scaled by $-k\eta/4$. If E^i is a plane wave, P is the power scattered in the direction of the plane wave. Otherwise, it may be viewed as a generalized forward scattering amplitude. It will be convenient to express the forward scattering amplitude in series form. This may be done for an arbitrary current u by decomposing the Green's function in $\mathcal{L}$ as a sum over Bessel functions [11]:

$$\mathcal{L}u = \frac{2}{\pi k a \eta} \sum_q {\alpha_q}^* \int_0^{2\pi} u(\phi') e^{iq(\phi-\phi')} d\phi' \tag{5}$$

where the coefficients are given by

$$\alpha_q = \frac{\pi (ka\eta)^2}{8} \times \begin{cases} J_q(ka) H_q^{(2)}(ka) & \text{TM} \\ J_q'(ka) H_q^{(2)\prime}(ka) & \text{TE} \end{cases}, \tag{6}$$

and $J_q(ka)$ is the usual Bessel function. Substituting Eq. (5) into Eq. (4) yields

$$P(u) = \sum_q \alpha_q |U_q|^2. \tag{7}$$

To establish a rigorous relationship between the Sobolev norm and the forward scattering amplitude, we will use the notion of equivalent norms. If two norms are equivalent, then if $x \to 0$ in either norm, then it will vanish in both norms. Formally, two norms $\|\cdot\|$ and $\|\cdot\|'$, defined for the same space X, are said to be equivalent if there exists constants $c_1, c_2 > 0$ such that

$$c_1 \|x\| \le \|x\|' \le c_2 \|x\| \tag{8}$$

for every $x \in X$. We will also need the definition of a quasinorm. A quasinorm is a functional with the following properties:

1) $\|x\| \ge 0$ with equality iff x is everywhere 0.
2) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{C}$.
3) $\|x_1 + x_2\| \le K(\|x_1\| + \|x_2\|)$ for all x_1, $x_2 \in X$ and for some $K \ge 1$.

A quasinorm differs from a norm in that for a norm, we have $K = 1$. We will prove that the quantity $\|\cdot\|_P \equiv \sqrt{|P(\cdot)|}$ is a quasinorm and is equivalent to the Sobolev norm (extending the notion of equivalency to include quasinorms).

To prove that $\|\cdot\|_P$ is a quasinorm, all three properties above must be shown. Property (1) is satisfied by further stipulating that there are no internal resonant modes, i.e., $\alpha_q \neq 0$ for all q. This is equivalent to saying that the interior Dirichlet (TM) or Neumann (TE) problem does not have a non-trivial solution. Satisfaction of property (2) is seen by substituting αU_q in for U_q in the scattering amplitude expression (7). Since $\|u\|_{\mathcal{H}^s}$ is a norm, it satisfies property (3) above with $K = 1$. Using this with the equivalency statement (8) yields the inequality

$$\|u_1 + u_2\|_P \le \frac{c_2}{c_1} \left(\|u_1\|_P + \|u_2\|_P\right). \tag{9}$$

This proves property (3) and $\|\cdot\|_P$ is a quasinorm. If $\mathcal{L}$ were self-adjoint, then $\|\cdot\|_P$ would be a norm. This is a minor point, since this paper relies on the equivalency relationship (8) to relate the Sobolev norm to a physical quantity, particularly in the sense as the Sobolev norm of an error current vanishes. The properties of norms and quasinorms are not used, except to couch the problem in a more familiar framework.

It remains to find constants c_1 and c_2 that satisfy Eq. (8) with $\|\cdot\| = \|\cdot\|_P$ and $\|\cdot\|' = \|\cdot\|_{\mathcal{H}^s}$. The constant c_1 is found by directly comparing the $\|\cdot\|_P$ norm with the Sobolev norm (3) term by term. This yields

$$c_1 = \left[\max_q |\beta_q|\right]^{-1/2} \approx \frac{2.5}{\eta} (ka)^{(4s-5)/6}, \tag{10}$$

where $\beta_q \equiv \alpha_q (1+q^2)^{-s}$. The approximation was made analytically using results of [12] for the TM case ($s = 1/2$)

and extended numerically to the TE case ($s = -1/2$), where estimating $\max_q |\beta_q|$ is more difficult.

A constant c_2 satisfying Eq. (8) can be derived by first classifying as low order modes those terms in (7) for which $|q| < q_0$, where q_0 is a positive integer to be specified. Splitting the sum in the scattering amplitude (7) into high and low order modes and also into real and imaginary parts, we define

$$\begin{aligned} R_l &\equiv \sum_{|q|<q_0} \mathrm{Re}(\alpha_q)|U_q|^2 \quad I_l \equiv \sum_{|q|<q_0} \mathrm{Im}(\alpha_q)|U_q|^2 \\ R_h &\equiv \sum_{|q|\geq q_0} \mathrm{Re}(\alpha_q)|U_q|^2 \quad I_h \equiv \sum_{|q|\geq q_0} \mathrm{Im}(\alpha_q)|U_q|^2 \end{aligned} \quad .(11)$$

We then rewrite the forward scattering amplitude (7) as

$$P(u) = R_l + R_h + i(I_l + I_h). \quad (12)$$

To compare the forward scattering amplitude to the Sobolev norm term by term, we will compare the low order terms of the Sobolev norm (3) with R_l since it allows us to guarantee that c_2 is finite except at resonance frequencies. We obtain the following relationship:

$$|R_l| \geq r_l \sum_{q:|q|<q_0} |U_q|^2(1+q^2)^s, \quad (13)$$

where $r_l = \min_{q:|q|<q_0} \mathrm{Re}(\beta_q)$. We can compare the high order terms of the Sobolev norm to I_h, since they both fall off at the same rate in q. Asymptotic expansions of $\mathrm{Im}(\alpha_q)$ show that the integer q_0 can be chosen sufficiently large such that the α_q have the same sign for all $|q| > q_0$. This allows us to bring the absolute value operator inside the summation in the definition of I_h and derive the relationship

$$|I_h| = \sum_{|q|\geq q_0} |\mathrm{Im}(\alpha_q)||U_q|^2 \geq i_h \sum_{|q|\geq q_0} |U_q|^2(1+q^2)^s, \quad (14)$$

where $i_h = \min_{q:|q|\geq q_0} |\mathrm{Im}(\beta_q)|$. Since R_h and I_l are extraneous, we will discard them. By the definition of α_q, R_l and R_h have the same sign and R_h can be immediately eliminated from $P(u)$ to give the lower bound $|P(u)| \geq |R_l + i(I_l + I_h)|$. We can remove I_l by noting that

$$|I_l| \leq \max_{q:|q|<q_0} \left|\frac{\mathrm{Im}(\beta_q)}{\mathrm{Re}(\beta_q)}\right| |R_l| \leq M|R_l|, \quad (15)$$

where M is a constant given by

$$M \equiv \max\left(1, \max_{q:|q|<q_0} \left|\frac{\mathrm{Im}(\beta_q)}{\mathrm{Re}(\beta_q)}\right|\right). \quad (16)$$

We have defined M to guarantee that it satisfies $M \geq 1$. This allows us to apply inequality (43) from the appendix, yielding

$$|P(u)| \geq \frac{1}{3M}\left(|R_l| + |I_h|\right). \quad (17)$$

Substituting in the term by term comparisons (13) and (14) yields

$$3M|P(u)| \quad \geq \quad \min(r_l, i_h) \sum_q |U_q|^2(1+q^2)^s. \quad (18)$$

Simplifying and taking the square root of each side, gives $\|u\|_{\mathcal{H}^s} \leq c_2\|u\|_P$, where

$$c_2 = \sqrt{\frac{3M}{\min(r_l, i_h)}}. \quad (19)$$

Note that c_2 depends only on ka and not on the current u, as required. It can be proved from the definition of the α_q that c_2 is finite, except at resonance frequencies. This is consistent with Eq. (8) where, as ka approaches a resonance, $\|u\|_P$ vanishes if u is a resonant mode, but $\|u\|_{\mathcal{H}^s}$ does not. Thus, if $\|u\|_{\mathcal{H}^s} \leq c_2\|u\|_P$ is to be maintained, we must have $c_2 \to \infty$ at these frequencies. We have thus obtained

$$c_1\|u\|_P \leq \|u\|_{\mathcal{H}^s} \leq c_2\|u\|_P \quad (20)$$

which relates the Sobolev norm of a current to the forward scattering amplitude, a physically meaningful quantity.

III. PEC SPHERE

We now derive a similar relationship between a 3D Sobolev norm and the forward scattering amplitude for scattering from a PEC sphere. In general, any function tangential to a surface may be expressed in terms of its surface Helmholtz decomposition

$$\boldsymbol{J} = \boldsymbol{J}^{cf} + \boldsymbol{J}^{df} \quad (21)$$

where $\boldsymbol{J}^{cf}$ is curl-free (irrotational) and $\boldsymbol{J}^{df}$ is divergence-free (solenoidal). In [7], it is shown that $\boldsymbol{J}^{cf}$ and $\boldsymbol{J}^{df}$ on a sphere can be expanded as

$$\boldsymbol{J}^{cf} = \nabla^t \sum_{n=1}^{\infty}\sum_{m=-n}^{n} d_{mn}^{1/2} u_{nm}^{cf} P_n^{|m|}(\cos\theta)e^{im\phi} \quad (22)$$

and

$$\boldsymbol{J}^{df} = \hat{n}\times\nabla^t \sum_{n=1}^{\infty}\sum_{m=-n}^{n} d_{mn}^{1/2} u_{nm}^{df} P_n^{|m|}(\cos\theta)e^{im\phi}, \quad (23)$$

where the normalizing factor is

$$d_{mn} = \frac{(n-|m|)!\ (2n+1)}{(n+|m|)!\ 4\pi n(n+1)}. \quad (24)$$

Here, the $P_n^{|m|}(\cdot)$ is the associated Legendre function of the first kind ∇^t is the surface gradient. A Sobolev space for currents on 3D bodies is denoted by $H_{div}^{-1/2}$ and for a sphere the norm is given by

$$\|\boldsymbol{J}\|^2_{H_{div}^{-1/2}} \equiv \sum_{n=1}^{\infty}\left[U_n^{cf}(1+n^2)^{1/2} + U_n^{df}(1+n^2)^{-1/2}\right], \quad (25)$$

where

$$U_n^{cf} = \sum_{m=-n}^{n} |u_{nm}^{cf}|^2,\ U_n^{df} = \sum_{m=-n}^{n} |u_{nm}^{df}|^2. \quad (26)$$

Using the orthogonality relationships in [7, Sec. VIII], it can be shown that the forward scattering amplitude decomposes as

$$P(\boldsymbol{J}) = P^{cf}(\boldsymbol{J}^{cf}) + P^{df}(\boldsymbol{J}^{df}), \quad (27)$$

where

$$P^{cf} = \sum_{n=1}^{\infty} \alpha_n^{cf} U_n^{cf} \ , \ P^{df} = \sum_{n=1}^{\infty} \alpha_n^{df} U_n^{df}. \tag{28}$$

Here, P^{cf} is the forward scattering amplitude due to the curl-free component of the current and P^{df} is similarly defined for the divergence-free component. Note the similarity of Eq. (28) to the 2D expression (7). In these expressions,

$$\begin{aligned} \alpha_n^{cf} &= -\frac{i(k\eta)^2}{4\pi}[ka\ j_n(ka)]'[ka\ h_n^{(2)}(ka)]' \\ \alpha_n^{df} &= -\frac{i(k\eta)^2}{4\pi}[ka\ j_n(ka)][ka\ h_n^{(2)}(ka)], \end{aligned} \tag{29}$$

where $j_n(ka)$ and $h_n^{(2)}(ka)$ are the usual spherical Bessel and spherical Hankel functions, respectively. Performing the same term by term comparison as was done for the circular cylinder, we obtain

$$c_1^{3\text{D}} \sqrt{|P(\boldsymbol{J})|} \le \|\boldsymbol{J}\|_{H_{div}^{1/2}}, \tag{30}$$

where $c_1^{3\text{D}} = \left[\max\left(c_1^c, c_1^d\right)\right]^{-1/2}$ and

$$c_1^c = \max_{n:n\geq 1} |\alpha_n^c (1+n^2)^{-1/2}|. \tag{31}$$

The constant c_1^d is defined similarly, replacing α_n^c with α_n^d and $-1/2$ with $1/2$ in the exponent. Numerically, $c_1^{3\text{D}} \approx 4.5/(k\eta)\,(ka)^{-2/3}$. As in the two-dimensional problem, Eq. (30) implies that if the current $\boldsymbol{J}$ vanishes in the Sobolev norm, then the forward scattering amplitude must also vanish. For scattering from a circular cylinder, we proved a stronger equivalency relationship. Because of possible cancellation between radiation from curl-free and divergence-free modes, a constant analogous to c_2 cannot be obtained for the sphere. Fortunately, this stronger equivalency is not essential to provide a physical interpretation of the Sobolev norm in error estimates, as will be seen.

IV. NUMERICAL EXAMPLES

To illustrate the relationship between current measures in 2D, we consider two example TM currents. The current $u^{(1)}$ is induced by an incident plane wave and $u^{(2)}$ is a single mode $e^{iq'\phi}$ that is nearest to resonance ($|\alpha_{q'}| \le |\alpha_q|,\ |q| \le q_0$) for a given value of ka. The corresponding Fourier coefficients are given by Eq. (2) for $u^{(1)}$ and by $U_q^{(2)} = \sqrt{2\pi}\delta_{qq'}$, where $\delta_{qq'}$ is the Kronecker delta. Figure 1 shows the ratio $\|\cdot\|_{\mathcal{H}^s}/\|\cdot\|_P$ for $u^{(1)}$ and $u^{(2)}$ as a function of electrical size ka. We plot on the same axes c_1 and c_2. The ratio of norms is always bounded below by c_1 and above by c_2, as proved. Near resonances, the bound c_2 becomes large, but away from resonances it is on the order of 0.01.

Similarly, define a current $\boldsymbol{J}$ on a sphere that is induced by a plane wave of unit amplitude, $\hat{\mathbf{x}}$ polarized and traveling in the negative z direction. In this circumstance, the current coefficients can be obtained using results of [13] and are given by

$$u_{nm}^{cf} = \frac{\sqrt{\pi}}{k\eta} i^{-n-1} \frac{\sqrt{2n+1}}{ka\, h_n^{(1)}(ka)} (\delta_{m,-1} - \delta_{m,1}) \tag{32}$$

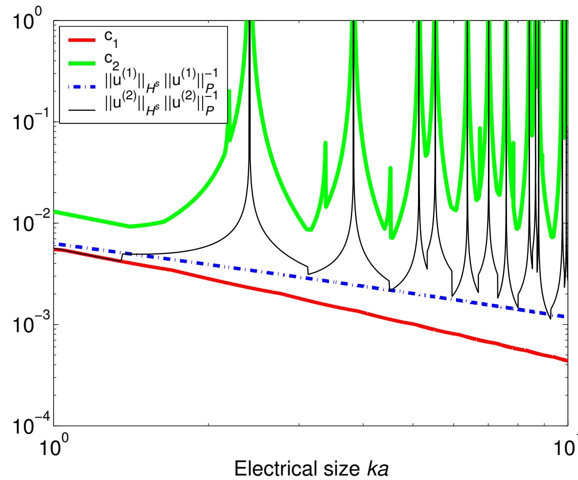

Fig. 1. Illustration of norm equivalence for two different test currents (TM polarization). The constants c_1 and c_2 always bound the ratio $\|\cdot\|_{\mathcal{H}^s}/\|\cdot\|_P$, which ratio is shown for the current induced by a plane wave ($u^{(1)}$) and a nearest-to-resonance, single-mode current $u^{(2)}$. Similar results are obtained for the TE polarization.

and

$$u_{nm}^{df} = \frac{\sqrt{\pi}}{k\eta} i^{-n-1} \frac{\sqrt{2n+1}}{[ka\, h_n^{(1)}(ka)]'} (\delta_{m,-1} + \delta_{m,1}). \tag{33}$$

Figure 2 verifies the bound (30) for this surface current. Note that the ratio $\|\boldsymbol{J}\|_{H_{div}^{1/2}} |P(\boldsymbol{J})|^{-1/2}$ is always greater than $c_1^{3\text{D}}$, as predicted by Eq. (30). We also see from Figs. (1) and (2) that away from resonances

$$\|u\|_P \approx \frac{1}{c_1}\|u\|_{\mathcal{H}^s}\ ,\ \sqrt{|P(\boldsymbol{J})|} \approx \frac{1}{c_1^{3\text{D}}}\|\boldsymbol{J}\|_{H_{div}^{1/2}}. \tag{34}$$

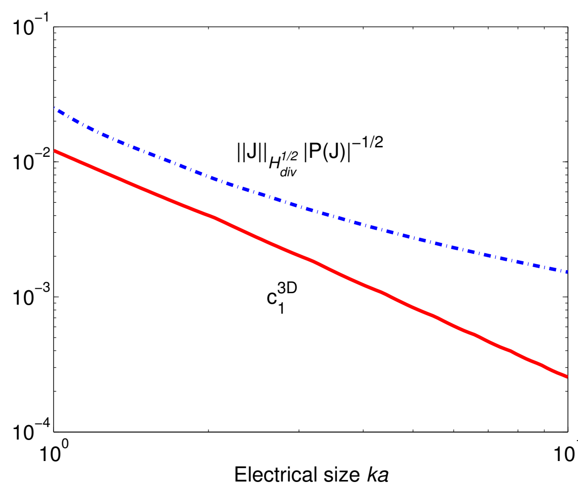

Fig. 2. Ratio of the Sobolev norm to the square root of the magnitude of the forward scattering amplitude for a plane wave induced current on a PEC sphere. The ratio is always bounded below by $c_1^{3\text{D}}$, as proved.

V. APPLICATION TO ERROR ANALYSIS

The equivalency statement (8) provides a physical interpretation for the Sobolev norm of the current solution error in the estimate (1). Suppose a moment method solution $\hat{u}$ to

$\mathcal{L}u = E^i$ on a PEC circular cylinder is generated, with current error defined by $\Delta u = u - \hat{u}$. The Sobolev norm of the error is approximately proportional within a range specified by the constants c_1 and c_2 to the total power radiated by the error current Δu if it were impressed on the scatterer contour $\mathcal{C}$. This transforms (1) into an error estimate for a physically meaningful quantity:

$$|P(\Delta u)| \le c_1^{-2} \|\Delta u\|^2_{\mathcal{H}^s} \le (C/c_1)^2\, h^{2r}, \tag{35}$$

which implies that the forward scattering amplitude or total supplied power associated with the current error Δu must decay at least as quickly as h^{2r}.

The quantity $\|\Delta u\|^2_P = |P(\Delta u)|$ in Eq. (35) is the forward scattering amplitude of the error current, which is not the error in the forward scattering amplitude as computed from the numerical current solution $\hat{u}$. In terms of the EFIE operator $\mathcal{L}$, we have for the forward scattering amplitude of the error current

$$\|\Delta u\|^2_P = |P(\Delta u)| = \frac{k\eta}{4} \left| \int_S (\mathcal{L}\Delta u)^* \Delta u \, ds \right|, \tag{36}$$

whereas the error in the computed scattering amplitude is

$$|\Delta S(\phi^i, \phi^s)| = |S - \hat{S}| = \frac{k\eta}{4} \left| \int_S E^{s*} \Delta u \, ds \right|. \tag{37}$$

To relate the Sobolev norm to a direct error quantity requires the scattering error to be put in a form containing two Δu terms. This may be done by defining an adjoint equation, $\mathcal{L}^a u^a = E^s$, where $\mathcal{L}^a$ is the adjoint of $\mathcal{L}$. Assuming that the adjoint equation is solved using the same procedure as the EFIE, but exchanging the roles of testing and basis functions, the following result is obtained [14]–[18]

$$|\Delta S(\phi^i, \phi^s)| = \frac{k\eta}{4} \left| \int_S (\mathcal{L}\Delta u)\, (\Delta u^a)^* \, ds \right| \tag{38}$$

where $\Delta u^a = u^a - \hat{u}^a$. Because $\mathcal{L}$ is not self-adjoint, Δu is not simply related to Δu^a for an arbitrary scattered field and Eq. (38) does not behave like an induced norm for Δu. However, in the backscatter direction $E^s = E^{i*}$, $u^a = u^*$, and assuming that $\hat{u}^a$ and $\hat{u}$ are expanded in the same basis (Galerkin testing), we also have $\hat{u}^a = \hat{u}^*$. This yields

$$|\Delta S(\phi^i, \phi^i)| = \frac{k\eta}{4} \left| \int_S (\mathcal{L}\Delta u)\, \Delta u \, ds \right|. \tag{39}$$

Note that $|\Delta S(\phi^i, \phi^i)|$ (39) differs from $\|\Delta u\|^2_P$ (36) only by a conjugate on the $\mathcal{L}\Delta u$ term. For a circular cylinder, ΔS in the backscattering direction is therefore similar to Eq. (7) and is given by

$$\Delta S(\phi^i, \phi^i) = -\sum_q \alpha_q^* (\Delta U_q)^2 \tag{40}$$

where we have used the fact that $\Delta U_{-q} = \Delta U_q$. The derivation of the lower constant c_1 applies equally well to the series (40) as it does to the series (7), therefore the inequality (35) is valid replacing $|P(\Delta u)|$ with $|\Delta S(\phi^i, \phi^i)|$, giving finally

$$|\Delta S(\phi^i, \phi^i)| \le c_1^{-2} \|\Delta u\|^2_{\mathcal{H}^s} \le (C/c_1)^2\, h^{2r}. \tag{41}$$

This is a new bound on the backscattering error, subject to the Galerkin testing condition. It shows that the error in the backscattering amplitude must decay at least as quickly as h^{2r}.

The curves in Fig. 3 were computed by generating a moment method solution for the EFIE and computing the coefficients ΔU_q numerically. A triangle (piecewise linear) expansion was chosen to avoid the Gibbs phenomenon associated with the Fourier coefficients of discontinuous functions. We see that the inequalities (35) and (41) are evident in the figure because both $|\Delta S(\phi^i, \phi^i)|$ and $\|u\|^2_P$ are both less than $1/c_1^2\, \|\Delta u\|^2_{\mathcal{H}^s}$. Further, we see that the error measures $\|\Delta u\|^2_{\mathcal{H}^s}$, $\|u\|^2_P$, and $|\Delta S|$ converge asymptotically at the same rate as the mesh is refined. While this is required of the first two error measures by the equivalency statement (8), $|\Delta S(\phi^i, \phi^i)|$ may actually converge faster than the Sobolev measure $\|\Delta u\|^2_{\mathcal{H}^s}$ without violating any inequality derived in this paper. We also note that all three error measures converge as h^5, which rate is proved for $|\Delta S|$ analytically in [12]. This is much faster than the $2r = 1$ rate predicted by the Sobolev bound (1), implying that these bounds are not tight. To achieve this convergence rate required a quadrature rule that combined lin-log Gaussian quadrature [19] with a Gauss-Legendre rule.

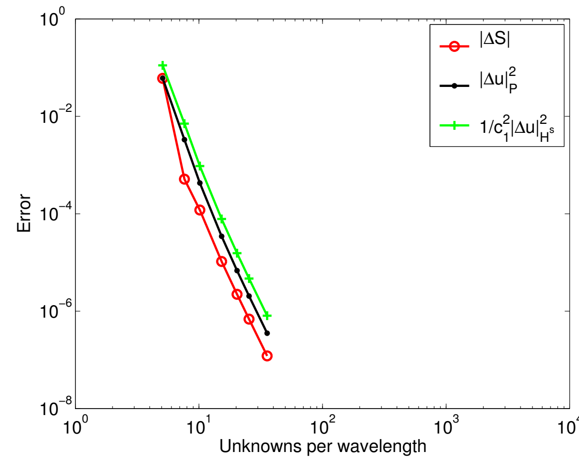

Fig. 3. Three different error measures for a moment method, TM polarized current solution, $ka = \pi/4$. The horizontal axis is λ/h. The backscattering amplitude error $|\Delta S|$ (circles) is computed from an MoM solution for $\mathcal{L}u = E^i$ with triangle expansion functions and Galerkin testing. The forward scattering amplitude of the current error, $\|\Delta u\|_P$ (dots) is equivalent in the rigorous sense to the Sobolev measure $\|\Delta u\|_{\mathcal{H}^s}$ (pluses). These error measures are related by Eqs. (35) and (41). For this particular value of ka, we have $c_1 \approx 0.006$.

We have given numerical examples of moment method error measures for scattering from a circular cylinder. Since the variational expression (38) applies also in three dimensions, we have

$$|\Delta S| \le (c_1^{\rm 3D})^{-2} \|\Delta \boldsymbol{J}\|^2_{H^{1/2}_{div}}. \tag{42}$$

Here, ΔS is the error in the backscattering amplitude for moment method solutions to scattering from a sphere. (To the authors' knowledge, there are no bounds analogous to Eq. (1) for 3D PEC scattering problems, although [2], [4] give Sobolev-type bounds for 3D scalar problems.) To give a numerical example similar to Fig. 3, computing the Sobolev measure $\|\Delta \boldsymbol{J}\|^2_{H^{1/2}_{div}}$ would require computing the inner products of $\Delta \boldsymbol{J}$

with Legendre polynomials and complex exponentials. Since this is tedious, we omit it. The impracticality of computing the Sobolev norm of error currents is one reason a physically meaningful alternative to the Sobolev norm is desirable.

VI. CONCLUSIONS

We have related the abstract Sobolev norm of an arbitrary current on an infinite, PEC circular cylinder to the forward scattering amplitude associated with that current. A slightly weaker result was derived for 3D currents on a PEC sphere. This equivalency was used to show that a small error current measured in the Sobolev norm implies that the error current radiates little energy. Therefore, Sobolev error estimates prove that moment method solutions converge in the sense that the energy radiated by the current error vanishes as the mesh is refined.

Further, a direct relationship was derived between the Sobolev norm of the current error to the error in the computed backscattering amplitude solution. This provides a link between error estimates in the Sobolev literature to physical quantities in practical CEM simulations. We conjecture that these observations hold for more general scatterers.

APPENDIX
INEQUALITY FOR COMPUTING c_2

Let a, b, c be real numbers with $|b| \leq M|a|$ and $M \geq 1$. Then we have the inequality

$$|a + i(b + c)| \geq \frac{|a| + |c|}{3M}. \tag{43}$$

Proof: Assume that $|b| > |c|$. Then we have

$$|a + i(b + c)| \geq |a| = \frac{|a|}{2} + \frac{|a|}{2} \tag{44a}$$

$$\geq \frac{|a|}{2} + \frac{|b|}{2M} \tag{44b}$$

$$\geq \frac{|a|}{2} + \frac{|c|}{2M} \tag{44c}$$

$$\geq \frac{|a| + |c|}{3M}. \tag{44d}$$

The second step (44b) follows from the given $|b| \leq M|a|$ and (44c) from the case statement $|b| > |c|$. The fourth line (44d) follows from the given $M \geq 1$. Now assume instead that $|b| \leq |c|$. It can be shown that

$$\sqrt{2}|a + i(b + c)| \geq |a| + |b + c| \geq |a| + |c| - |b|, \tag{45}$$

where we have used the triangle inequality and $|b| \leq |c|$. Claiming that

$$|a| + |c| - |b| \geq \frac{|a| + |c|}{2M}, \tag{46}$$

inequality (43) immediately follows. We can prove claim (46) by contradiction. Suppose that

$$|a| + |c| - |b| < \frac{|a| + |c|}{2M} \tag{47}$$

is true. Then the following inequalities are implied

$$(2M - 1)(|a| + |c|) < 2M|b| \tag{48a}$$

$$(2M - 1)(|b|/M + |c|) < 2M|b| \tag{48b}$$

$$|c| < \left(\frac{2M}{2M - 1} - \frac{1}{M}\right)|b|. \tag{48c}$$

For $M \geq 1$, the expression $2M/(2M - 1) - 1/M$ is less than one. This implies that $|c| < |b|$, a contradiction to the case statement $|b| \leq |c|$. Thus, the assumption (47) must be false and Eq. (46) must hold, completing the proof.

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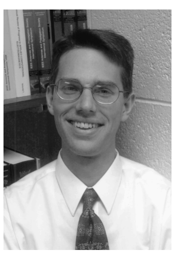

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