Assessment of ALEGRA Computation for Magnetostatic Configurations

Michael Grinfeld [1], John Niederhaus [2], and Andrew Porwitzky [1]

[1] The U.S. Army Research Laboratory
Aberdeen Proving Ground, MD 21005, USA

[2] Sandia National Laboratories
Albuquerque, NM 87185, USA

Abstract **─** A closed-form solution is described here for the equilibrium configurations of the magnetic field in a simple heterogeneous domain. This problem and its solution are used for rigorous assessment of the accuracy of the ALEGRA code in the quasistatic limit. By the equilibrium configuration we understand the static condition, or the stationary states without macroscopic current. The analysis includes quite a general class of 2D solutions for which a linear isotropic metallic matrix is placed inside a stationary magnetic field approaching a constant value $H_i°$ at infinity. The process of evolution of the magnetic fields inside and outside the inclusion and the parameters for which the quasi-static approach provides for self-consistent results is also explored. It is demonstrated that under spatial mesh refinement, ALEGRA converges to the analytic solution for the interior of the inclusion at the expected rate, for both body-fitted and regular rectangular meshes.

Index Terms **─** Electrostatics, exact solutions, magnetohydrodynamics, magnetostatics, verification and validation.

I. INTRODUCTION

Computational electromagnetics is a strong and continually growing area within modern applied electromagnetics. Computational electromagnetics tools are introducing new insights in many technical fields and contributing to the development of new measurement techniques. The soundness of these insights depends on rigorous assessment, or verification and validation (V&V), of these tools. Presented here is a simple problem in magnetostatics, and an analytic solution, which are useful for verification of computational electromagnetics capabilities.

Assessment is applied here to the computational electromagnetics code ALEGRA, which is currently being used successfully in development of laboratory electromagnetic measurement techniques. [1] ALEGRA is a multipurpose code handling a variety of mechanical and electromagnetic phenomena. This multiphysics capability is a key feature of ALEGRA. Here we assess the accuracy of the portion of the ALEGRA code used to compute transient magnetic diffusion, for a problem involving a permeable, conducting inclusion in a magnetized medium.

The project pursues two goals. First, we explore in the quasi-static approximation the process of evolution of the magnetic fields inside and outside the inclusion and the parameters for which the quasi-static approach in ALEGRA provides for self-consistent results. Second, we explore how reliable ALEGRA is in its static limit. By the static limit we understand the stationary states without macroscopic current. We choose quite a general class of 2D solutions for which a linear isotropic metallic matrix is placed inside a stationary magnetic field approaching a constant value $H_i°$ at infinity.

In this paper, we begin by reviewing the system of equations used to describe quasistatic magnetization and formulating our master system. Next, we analyze the equilibrium configuration of the magnetic field for an elliptical cylinder of linear isotropic material immersed in a 2D uniform magnetic field $H_i°$. The solution outside the ellipse is quite complex, however, inside it is remarkably simple. Therefore, this solution is very convenient for the verification purposes. We describe the ALEGRA code and the setup of the simulations, as well as the quasistatic evolution computed by the code. The quantitative verification analysis concludes the paper.

II. MHD MASTER SYSTEM

In the theoretical part of this project we follow the classical textbook of theoretical physics [1]. The analysis of quasi-statics is based on the following reduced Maxwell system:

$$z^{ijk}\nabla_j E_k = -\frac{1}{c}\frac{\partial B^i}{\partial t}, \qquad z^{ijk}\nabla_j H_k = \frac{4\pi}{c}J^i. \qquad (1)$$

These bulk partial differential equations should be augmented with the constitutive equations $J^i = \sigma^{ij}E_j$ (Ohm's law), the constitutive equation $B^i = B^i(H^k)$, boundary conditions $[B^i]_-^+ n_i = 0$, $[H^i]_-^+ \tau_i = 0$, the conditions at infinity, and appropriate initial conditions.

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Here, z^i and t are the spatial (Eulerian) Cartesian coordinates and time; E_i, H_i, and B_i are the electric and magnetic field and magnetic induction, respectively; J_i is the electric current density of free charges, c is the speed of light in vacuum, σ_{ij} is electrical conductivity. In the boundary conditions, n_i and τ_i are the normal and the tangent vectors to the discontinuity boundaries.

Other notation is the following. The metrics co- and contra-variant tensors z_{ij}, z^{ij} of the Eulerian coordinate system are used for lowering and raising ("juggling") the indexes, and for defining the covariant differentiation ∇_i with respect to the coordinates z^i; z_{ijk} is the so-called covariant Levi-Civita skew-symmetric tensor. Using tensor notation permits one to present all the equations in the universal covariant (i.e., coordinate-independent) form.

The ALEGRA code uses the vector potential A_i. The vectors A_i and H_i are interconnected by the covariant differential relation $H^i = z^{ijk}\nabla_j A_k$. With a known spatial distribution of the vector potential $A(z,t)$ one can immediately and explicitly recover the magnetic field H_i.

III. ELLIPTIC PLATE

There are few exact 2D and 3D solutions of the MHD master system. For the static equilibrium configuration a closed form solution can be obtained for an elliptic inclusion in an infinite isotropic matrix, in particular, in vacuum [2], [1], [3], [4]. This solution is described below and used in our project for verification purposes.

Consider an ellipse with the semi-axes a and b coinciding with the Cartesian axes z^1 and z^2. We assume that the elliptical domain is filled with a linear isotropic substance with magnetic permeability μ. We then assume that the ellipse is immersed in the unbounded space in which there is a uniform magnetic field $H^{i\circ}$. If there is an elliptical inclusion, the otherwise uniform field $H^i = H^{i\circ}$ will change. The changes are particularly strong inside the ellipse and in its vicinity. At infinity, the newly generated field H^i approaches its original value $H^{i\circ}$.

This problem was analyzed by many outstanding mathematicians and physicists working on it since Newton's times. First, it was focused on various problems of gravitation and cosmology. The exact solution of this problem has the following form.

a) Outside the ellipse:

$$H^1_{out} = \left(\alpha \frac{\partial^2\Theta}{\partial z^1 \partial z^1} + 1\right) H^{1\circ} + \beta \frac{\partial^2\Theta}{\partial z^1 \partial z^2},$$
$$H^2_{out} = \alpha \frac{\partial^2\Theta}{\partial z^2 \partial z^1} H^{1\circ} + \beta \left(\frac{\partial^2\Theta}{\partial z^2 \partial z^2} + 1\right) H^{2\circ}. \quad (2)$$

b) Inside the ellipse:

$$H^1_{ins} = \frac{a+b}{a+\mu b} H^{1\circ}, \quad H^2_{ins} = \frac{a+b}{\mu a + b} H^{2\circ}, \quad (3)$$

where $2\pi\alpha = (\mu - 1)(a+b)/(a+\mu b)$, and $2\pi\beta = (\mu - 1)(a+b)/(\mu a + b)$. The logarithmic potential of the ellipse $\Theta(z)$ is given by the relationship:

$$\Theta(z) = \int_{\omega_{ell}} d\omega^* d \ln|z - z^*|. \quad (4)$$

Analysis for related magnetic diffusion problems appears throughout the literature. Knoepfel [5] considered linear and nonlinear magnetic diffusion for simple geometries and non-permeable materials. Woodson and Melcher [6] analyzed permeable materials, but only for a slab geometry. Brauer [7] considered slabs and cylinders with linear and nonlinear permeability and finite-element modeling. Here we consider linear permeability with an elliptical geometry.

IV. NUMERICAL MODEL

The "transient magnetics" module of the ALEGRA MHD code [8], [9] (henceforward ALEGRA) computes solutions to the reduced Maxwell system of Equation 1 in quasi-static fashion. It is assumed that the medium is stationary, with variable electrical conductivity σ and fixed magnetic permeability μ. The system is recast in terms of the vector potential A_i and transformed to SI units, and appropriate constitutive relationships are incorporated. These include Ohm's law $J^i = \sigma E^i$, and a simple linear relationship between the magnetic field and the magnetic induction, $B^i = \mu H^i$. An implicit linear solver is used to solve the system with an unstructured finite-element discretization, and evolve the solution forward in time.

ALEGRA is equipped to handle a much broader class of problems, including deforming media, mechanical and electromagnetic forces, and adiabatic and Ohmic sources of heating. These are encompassed within ALEGRA's broader magneto-hydrodynamics (MHD) capability. For the present work, only the transient magnetics module is considered. The scope of the simulations here is restricted to the two-dimensional system described in Section III, treated in ALEGRA as an initial boundary value problem whose final state should be equilibrium.

This problem is shown schematically in Fig. 1. The simulations assume geometric parameters a = 1.8 cm and b = 0.56 cm, which imply an ellipse with an eccentricity of 0.831 and an aspect ratio of 3.24. A fixed magnetic tangential field $H^\circ = (0, 1/\mu_0)$ Ampere/m is imposed on the left and right boundaries, with zero tangential field on the top and bottom. For the ellipse, a magnetic permeability of $3\mu_0$, and a constant isotropic electrical conductivity of 10^7 S/m are used. The exterior has the permeability of free space, and an electrical conductivity of 10^{-6} S/m. The magnetic field is initially zero everywhere except on the domain boundary. At equilibrium, the exact solution given in Equation 3 predicts a uniform magnetic induction $B_1 = 0$ in the horizontal direction and $B_2 = 1.18657$ Tesla in the vertical direction, in the ellipse interior.

Two quadrilateral finite-element mesh configurations are used here in the attempt to capture this solution: (1) a simple regular rectangular mesh with multimaterial elements, and (2) a more complex irregular mesh fitted to the ellipse surface. In the former case, ALEGRA uses volume-averaged values of σ and μ in multimaterial elements. The meshes are chosen to provide roughly the same number of elements inside the ellipse for the two cases, to facilitate comparison.

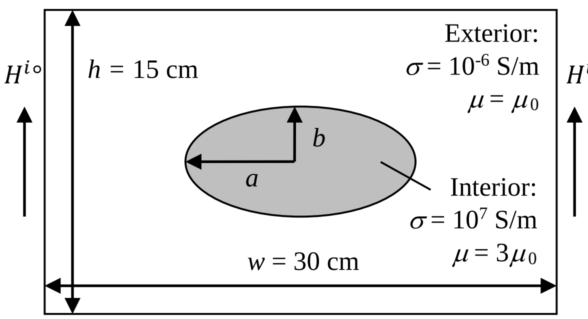

Fig. 1. Schematic layout (not to scale) for problem studied here showing elliptical inclusion.

V. TIME EVOLUTION

The equilibrium solution is obtained in ALEGRA in quasi-static fashion, by a series of timesteps capturing the time evolution. The time required to reach equilibrium can be approximated with a scaling relationship that arises from the 1D magnetic diffusion Equation [3], giving a diffusion time $\tau = \mu\sigma l^2$, where l is a characteristic length scale.

Using the shortest ellipse dimension (b = 0.56 cm), we obtain $\tau = 1.2$ ms as the time for one e-fold increase of the interior field magnitude. Therefore, the simulations are run out to a termination time of 0.01 seconds, in order to capture eight e-foldings, and ensure that a fully diffused field can be captured. The timestep size is fixed in this study to dt = 2 µs, so that 5000 timesteps are modeled.

The time evolution computed by ALEGRA is shown in Fig. 2. Here we see, as expected, that the magnetic field imposed at the boundary gradually diffuses into the elliptical inclusion, leading ultimately to a nearly uniform interior field. The exterior field is highly nonuniform, with the largest gradients appearing near the poles along the major axis of the ellipse.

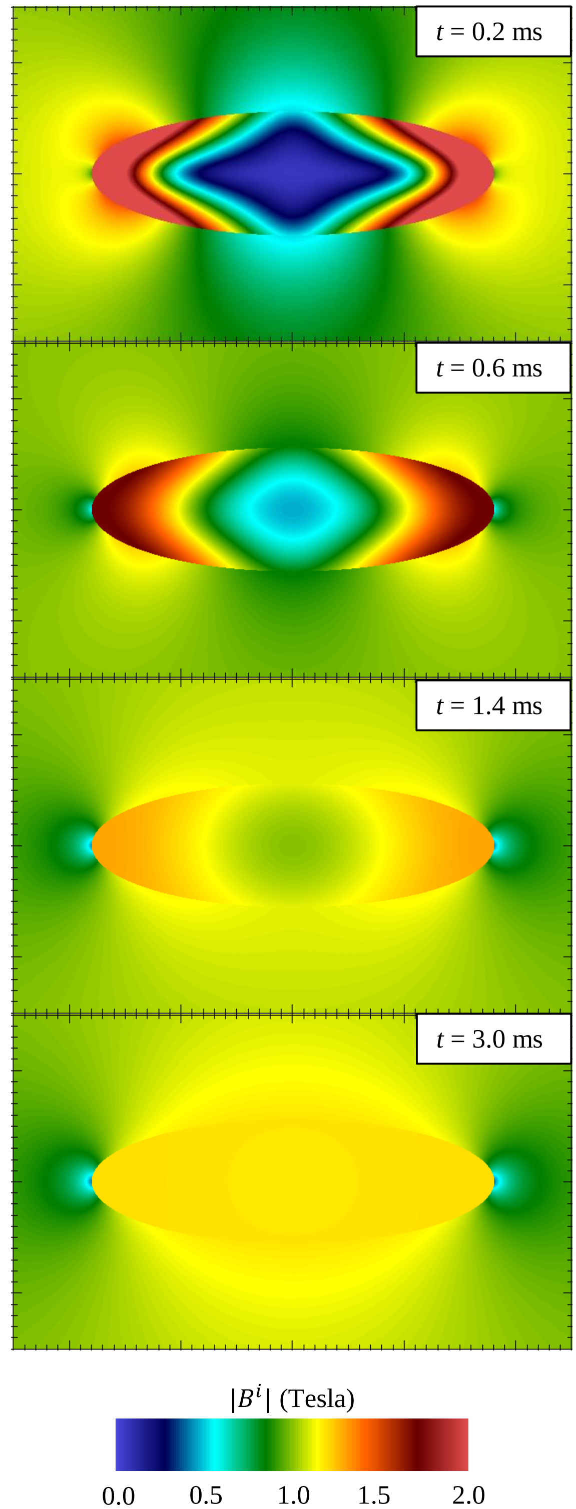

Fig. 2. ALEGRA simulation evolution (zoomed-in): magnetic induction magnitude for a body-fitted mesh with N = 1280 elements around the ellipse perimeter.

VI. SOLUTION VERIFICATION

The interior solution at t = 10 ms is computed using this setup for six levels of mesh refinement, in order to verify convergence of the solution to the analytic result in Equation 3. The error at each level is measured by summing the square of the local deviation of B_2 from the exact solution inside the ellipse, and normalizing by the tally of elements in the ellipse. Taking the square root yields a root-mean-square (RMS) error metric, and normalizing again by the analytic value of B_2 from Equation 3 yields a "fractional error." ALEGRA's transient magnetics methods are formally second-order accurate. However, verification here is done using the magnetic induction B (of most interest to ALEGRA users), rather than the native vector potential A. Therefore, we expect to see first-order convergence of the solution error with respect to the mesh interval.

Applying this technique for body-fitted and regular meshes with approximately 500 to approximately 500,000 elements inside the ellipse, we arrive at the convergence results shown in Fig. 3. We see that ALEGRA computes highly accurate solutions – to one part in 10^5 on the finest meshes. We find smooth, monotonic convergence at a rate of 0.9 for body-fitted meshes, and 1.3 for regular meshes.

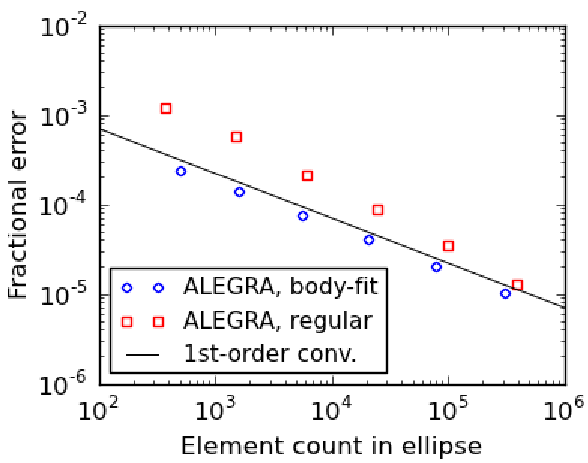

Fig. 3. Convergence of fractional error with respect to analytic solution, showing first order convergence.

VII. CONCLUSIONS

The verification analysis shows that the equilibrium state represented in the analytic solution can be reached with very good accuracy via computation with ALEGRA. The computed solution converges smoothly under mesh refinement at the expected rate, and shows a slight advantage for simple regular meshes over complex body-fitted meshes. This outcome suggests that analysts using tools like ALEGRA for more complex geometries need not avoid using regular meshes, which are much less costly to generate than body-fitted meshes. The closed-form solution provided here makes for convenient verification analysis, and the outcome invites empirical validation by means of laboratory measurement for a similar configuration.

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