Efficient MCF Evaluation in a Turbulent Atmosphere over Large Structure Constant Interval

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***Abstract* ─** A fast and accurate method is derived and simulated to compute the mutual coherence function (MCF) of an electromagnetic beam wave propagating through atmospheric turbulence over a large interval of structure constant. This method is based on expanding the integral equation solution for one value of structure constant into its Pade' approximation to cover large fluctuation interval. The expansion is compared with numerical results, and a very good agreement was obtained. Such computations are important in practical fields as remote sensing, imaging systems, and optical communications.

***Index Terms* ─** Atmospheric scattering, atmospheric turbulence, coherence, mutual coherence function, Pade' approximation.

I. INTRODUCTION

Electromagnetic wave scattering problem by a turbulent atmosphere was solved several decades ago [1], and does not represent a challenging problem anymore. Solutions were over and done with the Rytov method or by perturbation theory for several moments such as the second order statistical moment known as mutual coherence function (MCF) [1, 2]. MCF within Rytov approximations were compared to that which resulted from the parabolic equation for the case of weak and strong fluctuations, including cases of plane, spherical and beam waves [3]. In recent years, the advent of advanced processors and modern applications in astronomy, remote sensing, free space optical communications, and imaging [4] renewed the interest in atmospheric scattering of beam electromagnetic waves. Consequently, finite element methods [5] and finite difference methods [6] were applied extensively to solve the scattering problem [7].

In all mentioned methods, field solutions were obtained for a single value of structure constant or at a single or double excitation frequencies, and none were obtained over an interval of structure constant. Several authors succeeded in obtaining the solution over wide range of frequency [8], and over a range of complex permittivity [9]. Nevertheless, those methods dealt with deterministic media. In this work, solution for MCF is found over a continuous interval of structure constant in a turbulent atmosphere. The solution method is based on computing the MCF integral at a single value of structure constant, expanding integral solution into its truncated power series, finding power series coefficients from which Pade' approximants [10] are calculated, then approximations are established.

Generally, solving Maxwell's equations or the wave equations for non-symmetric geometries or obstacles usually requires resorting to numerical techniques. Domain discretization in such techniques, in either finite difference methods or finite element methods, is a necessity. Such procedures consume computer time and memory, especially for large scale problems, which put limits on the electrical size of problems under study, specifically when the solution is required for several values of some parameter of interest. Pade' approximation offers a method of reducing computer CPU time and memory, while maintaining the high accuracy of the solution.

II. FORMULATION

Consider a beam electromagnetic wave propagating along the z direction in a randomly turbulent atmosphere. The wave length is assumed to be much smaller than eddies forming fluctuating permittivity field. Such wave satisfies the stochastic Helmholtz equation [1-3]:

$$\nabla_t^2 U(\boldsymbol{\rho}) + 2jk\frac{\partial U(\rho)}{\partial z} + k^2\tilde{\varepsilon}(\boldsymbol{\rho})U(\boldsymbol{\rho}) = 0, \qquad (1)$$

where $\tilde{\varepsilon}$ is the stochastic permittivity field, ∇_t^2 is the transverse scalar Laplacian, $\boldsymbol{\rho}$ is the transverse radial vector, j is the imaginary unit, $U(\boldsymbol{\rho})$ is the transverse electric field component, $k = 2\pi/\lambda$ is the free space wave number, and λ is the free space wave length. The important parameter of interest characterizing such waves, is the mutual coherence function MCF, defined [1, 2]:

$$\Gamma_2(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \langle U(\boldsymbol{\rho}_1, z)U^*(\boldsymbol{\rho}_2, z)\rangle, \qquad (2)$$

where $\langle\cdot\rangle$ represent ensemble average, $\boldsymbol{\rho}_1 = x_1\hat{x} + y_1\hat{y}$, $\boldsymbol{\rho}_2 = x_2\hat{x} + y_2\hat{y}$, and U^* indicates conjugation. Assuming

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a beam with Gaussian amplitude distribution at the transmitting aperture $z = 0$, with a waist radius w_o and a phase front with radius of curvature R_o, an expression for MCF within the second order Rytov approximation was derived and evaluated for a beam wave propagating through a turbulent atmosphere [1-3]:

$$\Gamma_2(\boldsymbol{\rho}_c, \boldsymbol{\rho}_d, z) = \frac{w_o^2}{w^2} exp(g_1) exp(g_2),$$
$$g_1 = -\frac{k}{2}\left(\frac{w_o^2}{w^2}\right)(g_3 - j2g_4),$$
$$g_2 = -4.352 k C_n^2 \int_0^{L_z} \left(\gamma_I(z)\frac{L_z - z}{k}\right)^{5/6} {}_1\mathrm{F}_1\left(-\frac{5}{6}, 1; g_5\right) dz,$$
$$g_5 = -\frac{k|\gamma_R \boldsymbol{\rho}_d - j2\gamma_I \boldsymbol{\rho}_c|^2}{4\gamma_I(z)(L_z - z)},$$
$$g_3 = 2\alpha_1\left(\rho_c^2 + \frac{\rho_d^2}{4}\right),$$
$$g_4 = [\alpha_2 - (\alpha_1^2 + \alpha_2^2)L_z](\boldsymbol{\rho}_c \cdot \boldsymbol{\rho}_d),$$
$$w^2 = w_o^2[(1 - \alpha_2 L_z)^2 + \alpha_1^2 {L_z}^2],$$
$$\boldsymbol{\rho}_c = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2},$$
$$\boldsymbol{\rho}_d = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1,$$
$$\alpha = \alpha_1 + j\alpha_2 = \frac{2}{k w_o^2} + j\frac{1}{R_o},$$
$$\gamma(z, L_z) = \frac{1 + j\alpha z}{1 + j\alpha L_z} = \gamma_R - j\gamma_I, \quad (3)$$

where ${}_1F_1$ is the confluent hypergeometric fuction, C_n^2 is the refractive index structure constant, L_z is the distance from the output aperture. Expanding the unknown MCF in Eq. (3) into its power series about an arbitrary structure constant C_{no}^2, as:

$$\Gamma_2 = \sum_{i=0}^{\infty} a_i (C_n^2 - C_{no}^2)^i, \quad (4)$$

where

$$a_i = \frac{1}{i!}\frac{\partial \Gamma_2}{\partial C_n^2}\bigg|_{C_n^2 = C_{no}^2} = \frac{1}{i!}\frac{w_o^2}{w^2}\left(\frac{1}{C_n^2} g_2\right)^i exp(g_1) exp(g_2)\bigg|_{C_n^2 = C_{no}^2}. \quad (5)$$

$[L/M]$ Pade' approximants are obtained by truncating the power series at N, then matching to a rational:

$$\sum_{i=0}^{N} a_i (C_n^2 - C_{no}^2)^i \approx \frac{\sum_{l=0}^{L} p_l (C_n^2 - C_{no}^2)^l}{1 + \sum_{m=1}^{M} q_m (C_n^2 - C_{no}^2)^m}. \quad (6)$$

$N + 1$ equations reached from expansion of Eq. (6). The $q's$ are attained from the last M of these equations:

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} = -\begin{bmatrix} a_L & a_{L-1} & \cdots & a_{L-M+1} \\ a_{L+1} & a_L & \cdots & a_{L-M} \\ \vdots & \vdots & & \vdots \\ a_{L+M-1} & a_{L+M-2} & \cdots & a_L \end{bmatrix}^{-1} \begin{bmatrix} a_{L+1} \\ a_{L+2} \\ \vdots \\ a_{L+M} \end{bmatrix}, \quad (7)$$

whereas the $p's$ are found from the rest of equations:

$$\begin{bmatrix} a_o & 0 & \cdots & 0 \\ a_1 & a_o & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_L & a_{L-1} & \cdots & a_o \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_L \end{bmatrix} = \begin{bmatrix} p_o \\ p_1 \\ \vdots \\ p_L \end{bmatrix}. \quad (8)$$

III. NUMERICAL APPLICATIONS AND VALIDATION

With the derivation of asymptotic expansion coefficients accomplished, an investigation must now be made with regard to their applicability and validity. To demonstrate the efficiency of the technique, three simulations will be carried out on a 1.6 GHz personal computer. In all simulations, 41 structure constant values and 9 transverse space points are implemented for direct numerical solution. As a first check, power series expansion around $C_{n0}^2 = 10^{-16}$ with $[2/5]$ Pade' approximants used, and both expansions compared with direct numerical solution as shown in Fig. 1. The propagation parameters given by $w_0 = 0.05\, m$, $L_z = 2.5\, km$, $\lambda = 630\, nm$. It requires 145.9531s to obtain the solution with direct solution, though, it only takes 4.3281s for single point expansion. It may be inferred that the power series approximates well for small values of C_n^2, but drops sharply for larger values, while Pade' expansions shows very good approximation even deep in lager C_n^2 values region.

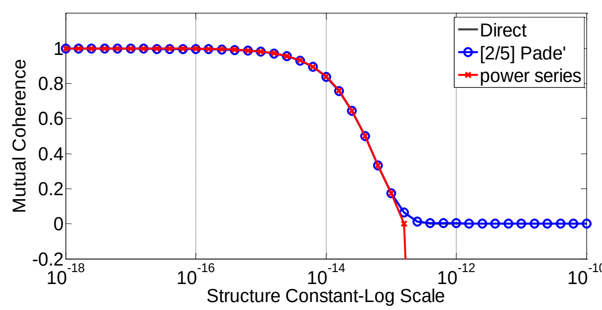

Fig. 1. Normalized mutual coherence function $\Gamma_2(0,0,L_z)$ versus C_n^2.

In Fig. 2, simulation parameters are $C_{n0}^2 = 10^{-16}$ $w_0 = 0.005\, m$, $L_z = 2.5\, km$, $\lambda = 630\, nm$. To study the effect of numerator and denominator degrees on quality of Pade' expansions, three different values are taken, namely, $[L/M] = [3/3]$, $[L/M] = [2/4]$ and $[L/M] = [4/2]$. Simulations show that when the denominator degree is larger than numerator degree, Pade' expansions approximated the solution very well. In the contrary, other orders did not for larger values of C_n^2, the approximation skyrockets when numerator degree is larger, and drops steeply when numerator and denominator degrees are equal. The difference in time between the three cases is a fraction of a second and can be neglected. However, it should be noted that as the number of power series coefficients is increased to larger integers, their values become very prohibitive and Pade' matrix becomes close to singular. This is expected since their values increase as powers of wave number.

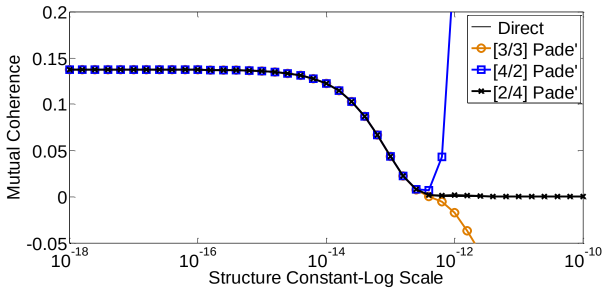

Fig. 2. Normalized mutual coherence function $\Gamma_2(\rho_c = 0.1, 0, L_z)/\Gamma_2(0,0,L_z)$ versus structure constant C_n^2 for different Pade' orders.

Turning into another case, and shifting the expansion point toward larger fluctuations, namely, $C_{n0}^2 = 10^{-14}$, as shown in Fig. 3. An obvious observation is that power series does not approximate in larger fluctuations region, and has an almost constant error in smaller fluctuations region, which is expected, since the solution has almost constant slopes for smaller fluctuations. Amazingly enough, Pade' expansions agrees very well with direct solution for both small and large fluctuations. Simulation parameters given by $w_0 = 0.05\ m$, $L_z = 2.5\ km$, $\lambda = 630\ nm$, $[L/M] = [2/5]$.

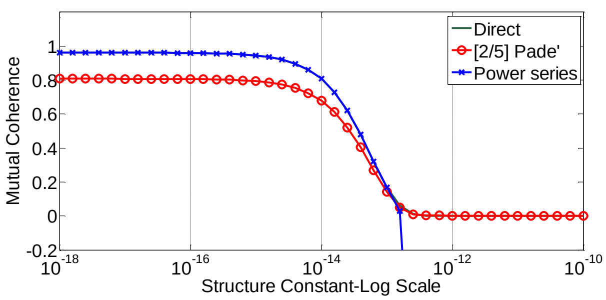

Fig. 3. Mutual coherence function $\Gamma_2(0,0,L_z)$ versus structure constant C_n^2 for a different expansion point $C_{n0}^2 = 10^{-14}$.

IV. CONCLUSION

Numerical methods offer a strong technique to compute the mutual coherence function of electromagnetic beam waves scattered by atmosphere. The efficiency is extremely increased when combined with Pade' approximation. It is shown that direct solution takes about 32 times the required time for Pade' expansion in presented simulations. In this research, it is also shown that a rational expansion with denominator degree larger than numerator gives much more accurate solution. Finally, rational asymptotes work very fine for small fluctuations region as well as for larger fluctuations.

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