A Direct Solver Based on Rank-Structured Matrix for Large Arrays in Method of Moment

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Abstract ─ Rank-structured matrices such as $\mathcal{H}$-matrix, $\mathcal{H}^2$-matrix and hierarchically semi-separable (HSS) have be applied to solve integral equation problems in some engineering applications. In Method of Moment (MoM), the discretization of electric field integral equation (EFIE) usually leads to a dense matrix. However, by considering the low-rank properties of off-diagonal blocks, the rank-structured theory provides a novel sparse representation for the resulting matrix. In this paper, we propose a direct solver based on one-level rank-structured matrix to analysis the electromagnetic characteristics of large arrays. The memory requirements are compared to those of direct solver and advantages of the proposed method are validated by numerical examples.

Index Terms ─ Direct solver, large arrays, rank-structured matrix.

I. INTRODUCTION

Method of Moment (MoM) is a typical numerical method to obtain the electromagnetic characteristics based on the solution of surface integral equation (SIE) [1]. Unknowns and test functions by discretization generally result in dense impedance matrix. The memory requirement of direct solver is proportional to O(N^2), with N being the matrix size. In most engineering applications, the iterative solver is commonly used owing to its high efficiency. The iterative methods based on Krylov subspace such as GMRES, BiCGStab and others depend on fast matrix-vector products. A typical application is the fast multiple method (FMM) which speeds up the matrix-vector product from O(N^2) to O($N^{1.5}$) and its improvement multilevel fast multi-pole (MLFMA) algorithm which reduces the complexity to O($NlogN$) [2].

Another branch of methodology is the fast direct solver which focuses on reducing the scale of the impedance matrix. Compared to iterative method, direct method does well with multiple right hand sides, which means once an efficient factorization is obtained, all right hand sides can be solved with relatively low computational cost [3]. One type of mature method based on the physical and geometrical features of targets applies fewer high order synthetic basis functions to approximate the properties of targets. The characteristic basis function method (CBF) and the synthetic basis function method (SBF) are two typical representatives [4]. Another method is based on the low-rank property of the matrix itself. Matrix compression technique like Adaptive Cross Approximation (ACA) is applied to reduce the memory requirement [5].

Hierarchical Semi-separable (HSS) matrix is a typical theory in rank-structured matrices. The term 'semi-separable' originated in the theory that if an integral kernel is approximated by an outer sum, then the system could be with a number of operations essentially determined by the order of the approximation [6]. In the same period, Greengard and Rokhlin proposed the FMM which was limited to the solution of Green's function. To some extent, HSS can be thought as the algebraic counterpart of FMM [7]. In this paper, we develop a direct solver based on one-level rank-structure matrix to analyze the electromagnetic properties of large arrays. Specifically, we maintain the information of diagonal blocks and compress the off-diagonal blocks based on the low-rank properties. The size of the compressed matrix depends on the numerical rank and influences the accuracy of solution. Finally, the remaining matrix is solved directly after ULV factorization with modest memory consumption.

II. COMPRESSION ANALYSIS

Consider the electric field integral equation (EFIE):

$$\boldsymbol{n}\times ik\eta\iint_S \begin{pmatrix} \boldsymbol{J}(\boldsymbol{r}')g(\boldsymbol{r},\boldsymbol{r}') \\ +\frac{1}{k^2}\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')\nabla g(\boldsymbol{r},\boldsymbol{r}') \end{pmatrix} dS' = \boldsymbol{n}\times\boldsymbol{E}^{inc}, \quad (1)$$

in which $g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}$ represents Green's function, J is the induced surface current density and E^{inc} is the imposed electric field. Discretize J with a series of

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RWG basis functions and get the following linear system of equations:

$$ZI = V. \tag{2}$$

The impedance matrix Z, although dense, can be thought of "data-sparse" in HSS theory. The HSS representation is a hierarchical structure and based on a recursive row or column partitioning of the matrix. The resulting matrix can be approximated in form of the multiplication of the several low dimension matrices [6, 8].

Considering a target consists of M blocks, the resulting matrix based on traditional MoM can be written as:

$$Z = \begin{pmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,M} \\ Z_{2,1} & Z_{2,2} & \cdots & Z_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{M,1} & Z_{M,2} & \cdots & Z_{M,M} \end{pmatrix}. \tag{3}$$

The sub-block $Z_{i,j}(i=1,2,\ldots,M)$ represents the interaction of RWGs on block i while $Z_{i,j}(i,j=1,2,\ldots,M, i\neq j)$ represents the mutual relationship between block i and block j. As mentioned above, the sub-block $Z_{i,j}$ has low-rank properties and can be written approximately as:

$$Z_{i,j}^{'} = UZ_{i,j}^{c} = UU^{H}Z_{i,j}. \tag{4}$$

The U matrix is called generator which is "tall and skinny" and $Z^c_{i,j}$ is "short and wide".

In this work, we apply one-level HSS matrix theory and use the singular value decomposition (SVD) to generate the compressed form. This work is based on two facts: the numerical rank of the iterative matrix is rather small for large array and thus the storage of off-diagonal blocks can be significantly reduced. Second, the computation complexity for compression of one level HSS form is rather small compared with multiple levels of HSS. For each column, apply SVD factorization to obtain the generator U. Here, we use numerical rank r to represent the size of U and r depends on the characteristics of the normalized singular values. The whole matrix can be compressed in this form:

$$Z^{c} = \begin{pmatrix} Z_{1,1} & U_1 Z^{c}_{1,2} & \cdots & U_1 Z^{c}_{1,M} \\ U_2 Z^{c}_{2,1} & Z_{2,2} & \cdots & U_2 Z^{c}_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ U_M Z^{c}_{M,1} & U_M Z^{c}_{M,2} & \cdots & Z_{M,M} \end{pmatrix}. \tag{5}$$

In the following step, we will show how to solve the equation by applying the ULV factorization method.

III. ULV FACTRORIZATION

ULV theory arose in an effort to stabilize the fast solver for matrices characterized by a hierarchical low numerical rank structure, where U and V are orthogonal matrices and L is a lower-triangular matrix [8]. And it belongs to backward stable algorithm.

We firstly introduce the main step in ULV factorization. The generator U has the special structure:

$$U = \Pi \begin{pmatrix} I \\ E^{r} \end{pmatrix}, \tag{6}$$

where matrix Π and E^r can be obtained in advance. Construct the transformation matrix:

$$\Omega = \begin{pmatrix} -E^{r} & I \\ I & 0 \end{pmatrix} \Pi^{-1}. \tag{7}$$

We can observe that $\Omega U = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Apply Ω to both off-diagonal and diagonal blocks in corresponding row blocks. Take the first row for example, we can get:

$$\Omega[Z_1 \; Z_2 \ldots Z_M] = [\Omega Z_1 \begin{pmatrix} 0 \\ Z_2^{c} \end{pmatrix} \ldots \begin{pmatrix} 0 \\ Z_M^{c} \end{pmatrix}], \tag{8}$$

then partition $W = \Omega Z_1$ into $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ where W_2 has r rows. We perform an LQ factorization: $W_1 = \begin{bmatrix} L & 0 \end{bmatrix} Q$. For 2×2 equation $ZI=b$, it can be transformed as:

$$\begin{aligned} &\begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} Z \begin{pmatrix} Q_1^{*} & 0 \\ 0 & Q_2^{*} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} L_1 & 0 \end{bmatrix} & 0 \\ W_{12}Q_1^{*} & Z^{c}{}_{1,2}Q_2^{*} \\ 0 & \begin{bmatrix} L_2 & 0 \end{bmatrix} \\ Z^{c}{}_{2,1}Q_1^{*} & W_{22}Q_2^{*} \end{pmatrix} \\ &= \begin{pmatrix} L_1 & 0 & 0 & 0 \\ W_{12}Q_{1;1}^{*} & W_{12}Q_{1;2}^{*} & Z_{1,2}^{c}Q_{2;1}^{*} & Z_{1,2}^{c}Q_{2;2}^{*} \\ 0 & 0 & L_2 & 0 \\ Z_{2,1}^{c}Q_{1;1}^{*} & Z_{2,1}^{c}Q_{1;2}^{*} & W_{22}Q_{2;1}^{*} & W_{22}Q_{2;2}^{*} \end{pmatrix}. \end{aligned} \tag{9}$$

For each row i, right hand sides relate to L can be solved directly:

$$Ly = \Omega b. \tag{10}$$

Then, we need to update the right-hand side by eliminated the unknowns corresponding to L:

$$b' = b - W_{i;2}Q_{i;2}^{*}y_i - \sum_{j=1, j\neq i}^{M} Z_c Q_{j;2}^{*}. \tag{11}$$

The remaining unknowns can be merged together to be solved in a rather small scale. The whole process is illustrated in Fig. 1.

In the end, we discuss the memory requirement in this method, which consists of two main parts: one is the main remaining unknowns and the other is the backward matrix Q for each column. Consider a linear system of N unknowns, the memory requirement of conventional MoM is $O(N^2)$. If the proposed method is applied and the system is divided into M^2 sub-blocks, the total memory requirement is $O(N^2/M+r^2M^2)$ in which r is numerical rank and generally far less than

N. Moreover, we use the compression ratio, defined as Mr/N, to measure the low rank properties of the off-diagonal blocks.

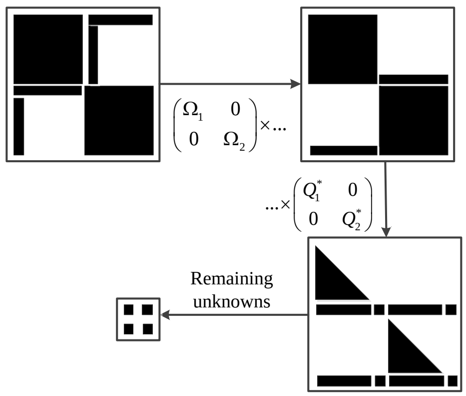

Fig. 1. Illustration of LUV factorization process.

IV. NUMERICAL RESULTS AND VALIDATION

To illustrate the validity and accuracy of the proposed method, we present several numerical examples based on EFIE. Firstly, consider a 3×3 PEC cylinder arrays as is shown in Fig. 2.

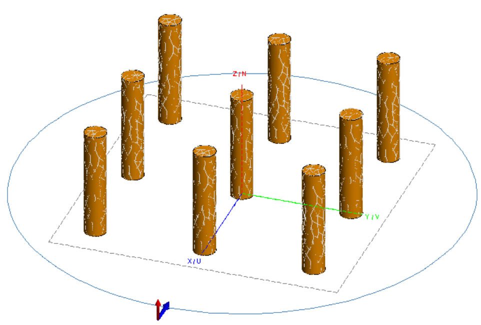

Fig. 2. Geometry of a 3×3 PEC cylinder arrays.

The excitation source is a +z polarized plane wave coming from x axis with frequency f=300 MHz. The array elements are geometrically isolated and thus the matrix will be divided into 81 sub-blocks. After discretization of the surface, 2628 triangular patches and 3942 RWG functions are defined. The size of the sub-block is 438. Before proceeding further, we first determine the numerical rank for off-diagonal blocks by setting the threshold ρ after SVD factorization. To be specific, the formulation $Err = \frac{\|I_r - I_{MoM}\|_F}{\|I_{MoM}\|_F}$ is used to illustrate the compression accuracy of the generator U, where "$\|\bullet\|_F$" denotes the Frobenius norm. In Table 1, we compare the current coefficient in this method with the one obtained by conventional MoM. Besides, we also show the changing tendency of current coefficients with the compression ratio.

Table 1: Relationship between compression ratio and *Err*

Threshold ρ	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Compression ratio	0.05	0.14	0.22	0.32
Err	$7.2e^{-3}$	$1.4e^{-3}$	$9.1e^{-5}$	$1.7e^{-5}$

It is obvious that the numerical rank of the off-diagonal blocks can be much smaller than the original dimension, which means the memory can be reduced significantly. From the solution results, the accuracy meets the demand. Bistatic RCS of the example are calculated using the proposed method with p=10^{-2}, and results of Feko and MoM are also given here for comparison in Fig. 3. The memory requirement of original matrix Z is 237.1 MB. In contrast, the whole memory cost in this method is 27.4 MB.

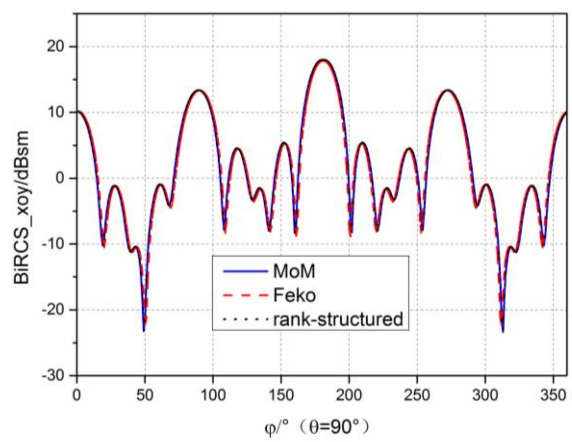

Fig. 3. Bistatic RCS of the cylinder arrays.

In fact, the rank-structured method, as purely algebraic method, is also adaptive to a single target, although the strength in memory reduction is not apparent. The second example is a PEC sphere with radius of 1 m. The excitation source is a +z polarized plane wave coming from x axis with frequency f=420 MHz A total of 7272 RWGs are defined on sphere surface. To use rank-structured method, the RWGs on the space is equally divided into 8 sub-blocks as is shown in Fig. 4. The RWGs on the connection zone are randomly distributed to sub-blocks, which has little influence on the final consequence.

Here, we just show the relationship between numerical rank and *Err* in Fig. 5. Usually, RCS results can achieve satisfying accuracy if the current coefficient error is no more than 1%.

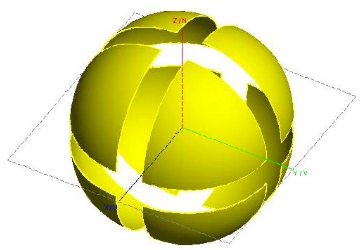

Fig. 4. Geometry of a PEC sphere after separation.

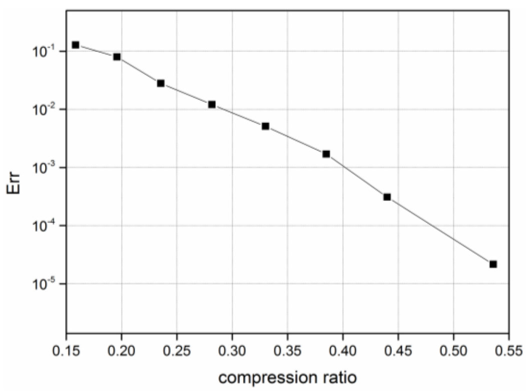

Fig. 5. Relationship between the compression ratio and *Err*.

We observe that to satisfy the accuracy of 1%, the size of reduced matrix should be about 30% of the origin matrix. In contrast, for large array in first example, the ratio of reduction can be 5% to meet the same accuracy. The reason lies in the fact that as all the groups are touching for a single target, the low-rank property is not apparent in just one-level structure. To achieve a better performance, a hierarchical and nested structure should be applied such as $\mathcal{H}^2$-matrix or HSS representation, which is our future research topic.

V. CONCLUSION

The paper proposes a memory-reduced direct method to deal with the electromagnetic problem of large scale arrays based on rank-structured theory. The method takes advantage of the low-rank properties of the impedance matrix and factorizes the matrix in form of multiplication of several low dimension matrices. The approach is perfectly suitable to the MoM and we validate this idea by two kinds of numerical results. In the end, we recommend randomized sampling algorithm [9] to generate the HSS matrix. The main advantage of this approach is that the original matrix does not need to be explicitly formed and only requires some selected elements and fast matrix-vector produce routine.

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