On the Bounded Part of the Kernel in the Cylindrical Antenna Integral Equation

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Abstract

The kernel in the cylindrical antenna integral equation was partitioned by Schelkunoff into a complete elliptic integral and a bounded integral. This paper gives an exact expression for the bounded part.

1 Introduction

The cylindrical antenna integral equation for total axial current $I(z)$ on perfectly conducting tube of length '$2h$' and radius 'a' is [1]:

$$j\omega\left(\frac{d^2}{dz^2}+k^2\right)A_z(z)=k^2E_z^i(z)\,, \tag{1}$$

where

$$A_z(z)=\frac{\mu}{4\pi}\int_{-h}^{h}K(z-z')I(z')dz'\,. \tag{2}$$

The kernel in (2) is:

$$K(z-z')=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}d\phi'\,, \tag{3}$$

where

$$R=R(z-z',\phi')=\left[(z-z')^2+4a^2\sin^2\left(\frac{\phi'}{2}\right)\right]^{1/2}\,. \tag{4}$$

Here ϵ and μ characterize the medium [1], and k is the wavenumber.

For thin structures ($a\ll h$ and $a\ll\lambda$), $K(z-z')$ is usually substituted by a reduced kernel approximation

$$K\simeq K_r(z-z')=\frac{e^{-jkr}}{r}\,, \tag{5}$$

where

$$r=r(z-z')=\left[(z-z')^2+a^2\right]^{1/2}\,. \tag{6}$$

The kernel (3) possesses a logarithmic singularity (see for example [2]). This needs to be included while solving (1). Schelkunoff has partitioned the kernel into two parts.

The first part is a complete elliptic integral and the second part is a bounded integral. By expanding the elliptic integral, Pearson [3] has derived an expression which includes the logarithmic singularity. On the other hand, as the exact expression of the bounded integral is not available, it is usually evaluated numerically. Karwowski [4] has suggested a closed form approximation; an alternative to the numerical integration. In this presentation we obtain an exact expression for the bounded part. Certain recurrent relations are obtained to simplify its computation. The stability of the recurrence relations is analyzed and checked.

Before we conclude this section we shall briefly review other research accomplishments regarding the kernel (3). Wang [5] has obtained an 'exact' expression for the kernel (3) in terms of spherical Hankel functions. Werner [6] has presented an alternative 'exact' expression replacing the spherical Hankel functions by complex exponential functions. This has an advantage from the point of view of analytical and numerical evaluation of the kernel. Werner has presented 'extended' approximations [7] to the kernel, by truncating the series to a few terms. This is an alternative to the reduced approximation (5). Interestingly, the vector potential in (2) due to the singular part K_E has been worked by Werner, et al [8]. The results of this paper on the bounded part K_B can be useful when combined with the results of Pearson [3] or Werner, et al [8].

2 Kernel

Recall the Schelkunoff partition:

$$K(z-z') = K_E + K_B\,, \tag{7}$$

where

$$K_E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{R}d\phi' \tag{8}$$

and

$$K_B = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-e^{-jkR}}{R}d\phi'\,. \tag{9}$$

Expanding the first integral, an elliptic integral, Pearson has obtained the following expression, which we reproduce, with a correction:

$$K_E = -\frac{1}{\pi a}\ln|z-z'| + K_{EB}\,, \tag{10}$$

where

$$\begin{aligned} K_{EB} &= \frac{1-\beta}{\pi a}\ln|z-z'| + \frac{\beta}{\pi a}\cdot \\ &\quad \left\{\ln 8a - \ln\beta + \left(\frac{1}{2}\right)^2\left[\ln\left(\frac{4}{\beta_1}\right) - \left(\frac{2}{1\cdot 2}\right)\right]\beta_1^2 \right. \\ &\quad \left. + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left[\ln\left(\frac{4}{\beta_1}\right) - \left(\frac{2}{1\cdot 2}\right) - \left(\frac{2}{3\cdot 4}\right)\right]\beta_1^4\cdots\right\}. \end{aligned} \tag{11}$$

We have used the following notation

$$\beta = \frac{2a}{[4a^2+(z-z')^2]^{1/2}}\,, \quad \beta_1 = (1-\beta^2)^{1/2}\,. \tag{12}$$

The series is convergent for $|z-z'| \le 4a$. The leading term in K_{EB} can be shown to be of the form $(z-z')^2\ln|z-z'|$. Next, we shall obtain an exact expression for K_B.

3 Expression for K_B

K_B can be written as

$$K_B = K_{Br} + K_{Bi} \tag{13}$$

where K_{Br} is the real part and K_{Bi} is the imaginary part

$$K_{Br} = \frac{-1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-\cos kR)}{R}d\phi' \tag{14}$$

and

$$K_{Bi} = \frac{-j}{2\pi}\int_{-\pi}^{\pi}\frac{\sin kR}{R}d\phi' . \tag{15}$$

Expanding $\sin kR$ in eq. (15) and setting $(z-z') = u$, we get the infinite series:

$$K_{Bi} = -\frac{j}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m k^{2m+1}A_{m-1}(u,a)}{(2m+1)!} \tag{16}$$

where

$$A_{m-1}(u,a) = \int_0^{\pi} R_1^{2m}d\mu \tag{17}$$

and

$$R_1 = \left[u^2 + 4a^2\sin^2\mu\right]^{1/2} . \tag{18}$$

We find from eqs. (17) and (18) that

$$A_{-1}(u,a) = \pi \tag{19}$$

and

$$A_0(u,a) = \pi\left(u^2+2a^2\right) . \tag{20}$$

In eq. (17), we rewrite $A_{m-1}(u,a)$ as:

$$\begin{aligned} A_{m-1}(u,a) &= \int_0^{\pi}\left(u^2+4a^2\sin^2\mu\right)^{m-1}\left(u^2+4a^2\sin^2\mu\right)d\mu \\ &= \left(u^2+2a^2\right)A_{m-2}(u,a) - 2a^2\int_0^{\pi}\left(u^2+4a^2\sin^2\mu\right)^{m-1}\cos 2\mu d\mu . \end{aligned}$$

Apply integration by parts to the integral. After certain algebraic manipulations, we get the recurrence relations:

$$\begin{aligned} [1+(m-1)]\,A_{m-1} &= \left(u^2+2a^2\right)[1+2(m-1)]\,A_{m-2} \\ &\quad -(m-1)u^2\left(u^2+4a^2\right)A_{m-3}\,; \quad m = 2,3,4,\cdots \end{aligned} \tag{21}$$

Using A_{-1}, A_0 from eqs. (19) and (20) we find A_{m-1}; $m = 2,3,4,\cdots$ and then determine K_{Bi} from eq. (16) up to a desired accuracy. The stability of the recurrence relations needs to be examined for a given radius 'a' and length '$2h$' of the wire. This has been done in the stability section. Now, on similar lines the real part, namely K_{Br}, becomes

$$K_{Br} = -\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m k^{2m+2}B_{m-1}(u,a)}{(2m+2)!} \tag{22}$$

where

$$B_{m-1}(u,a) = \int_0^{\pi} R_1^{2m+1} d\mu\,. \tag{23}$$

We discuss the computation of K_{Br} for two distinct cases of $u = z - z'$, namely for $u = 0$ and for $u \neq 0$.

From eq. (23) we find

$$B_{-1}(0,a) = 4a \tag{24}$$

and then obtain recurrence relation

$$B_{m-1}(0,a) = \left[\frac{2m}{(2m+1)}\right](2a)^2 B_{m-2}(0,a)\,. \tag{25}$$

Computed values of B_{m-1}, $m = 1, 2, 3, \cdots$ are substituted in eq. (22). It may be noted that the recurrence relation is stable for $a < 0.5$. This is a dimensionless number.

We obtain the following results using [9]

$$\begin{aligned} B_{-1}(u,a) &= 2u\int_0^{\pi/2}\left(1+q^2\sin\mu\right)^{1/2} d\mu \\ &= 2usE\left(\frac{q}{s}\right), \end{aligned} \tag{26}$$

where we have used the notation $q = (2a/u)$, $s = (1+q^2)^{1/2}$. Further,

$$B_0(u,a) = 2u^3\left[sE\left(\frac{q}{s}\right) + q^2 I(q)\right], \tag{27}$$

where

$$I(q) = \frac{1}{3}\left\{\left[\frac{(2q^2+1)}{q^2}\right] sE\left(\frac{q}{s}\right) - \left(\frac{s}{q^2}\right) F\left(\frac{q}{s}\right)\right\}. \tag{28}$$

E and F are elliptic integrals of second and first kind, respectively, and are evaluated to a desired accuracy using Landen's Transformation [10]. The recurrence relation now is:

$$\begin{aligned} B_{m-1}(u,a)\left[1+\left(\frac{2m-1}{2}\right)\right] &= \left(u^2+2a^2\right) B_{m-2}(u,a)\left[1+(2m-1)\right] \\ &\quad - \left[\left(\frac{2m-1}{2}\right) u^2\left(u^2+4a^2\right) B_{m-3}(u,a)\right] \\ &\quad m = 2, 3, 4, \cdots \end{aligned} \tag{29}$$

Using eqs. (26) and (27) we can obtain B_{m-1}, $m = 2, 3, 4, \cdots$ and then evaluate K_{Br}. The stability of eq. (29) is also included in the next section.

4 Stability

The characteristic equation [11] corresponding to eq. (21) is:

$$(1+n)x^2 - \alpha(1+2n)x + n\gamma = 0\,; \quad n = 1, 2, \cdots \tag{30}$$

where $\alpha = (u^2 + 2a^2)$, $\gamma = u^2\,(u^2 + 4a^2)$. Rewrite eq. (30) as

$$\frac{x(x-\alpha)}{\alpha x - \gamma} = \frac{n}{(1+n)}. \tag{31}$$

The above equation is useful to seek the stability condition, namely $|x| < 1$. For a given radius 'a' and length of wire '$2h$', this condition shall depend upon λ.

Denoting

$$\frac{n}{(1+n)} = p \tag{32}$$

The solution of (31) is

$$x = \frac{\alpha(1+p) \pm [\alpha^2(1-p)^2 + 16pa^4]^{1/2}}{2}. \tag{33}$$

The right hand side is a monotonic increasing function for given values of p and a, enabling direct evaluation of the maximum of x. For recurrence relations corresponding to the real part given in eq. (29), the analysis is repeated on similar lines with

$$p = \frac{(2n+1)}{(2n+2)}; \quad n = 1, 2, 3, \cdots, \tag{34}$$

yet the stability criteria is not altered. Thus, for any wire of known radius 'a' and length '$2h$', using eq. (33) we can easily establish the condition of stability on λ before computing the recurrence relations and determining K_B.

5 Examples

As a first case we have a dipole of half-wave length in length and whose radius is $(\lambda/8)$. From eq. (21), we note that the value of p varies from 1/2 to a limit of 1. Then, the maximum of x in eq. (33) is $5\lambda^2/16$, which implies that the condition for stability is $\lambda < 4/\sqrt{5}$, that is approximately 1.7888. This is not a strong condition as only a limiting value could be used. In the first example we set the radius of the wire to be 0.003 and λ to be 1.0. This is a case of thin wire and Karwowski's results are accurate. We have given in Table 1 for various values of $u = z - z'$, the (real, imaginary) values of K_B from (16) and (22). These values are compared with those of Karwowski [4].

Next, we consider a cylindrical wire of a wavelength in length and radius $(\lambda/4)$. The maximum value of x is $(5\lambda^2/4)$. Thus, for $\lambda < 2/\sqrt{5}$ (approximately 0.8944), the recurrence relations in eqs. (21) and (29) are stable. The condition in eq. (25) that $a < 1/2$ is satisfied in this case also. In the second example we have $a = 0.22$ and $\lambda = 0.88$. The corresponding results are given in Table 2. Here, while setting $\lambda = 0.90$, the recurrence relation was verified to be unstable, which agrees with our estimate. The values in the tables are correct up to four decimal places and appropriately rounded up to three decimal places. Only 12 terms in the series were used to obtain this accuracy. The difference between the sets of the values is due to the approximation introduced by Karwowski in replacing $(1/R)$ by $(\sin\phi/R)$ in (9) and thus apparently diminishing the contribution near the ends of the interval.

Table 1: Comparison between this work and that of Karwowski [4] for a thin wire half-wave dipole.

$a = 0.003$ and $\lambda = 1.0$		
u	Present	Karwowski
0.0	(−0.075, −6.282)	(−0.075, −6.282)
0.1	(−1.885, −5.877)	(−1.911, −5.877)
0.2	(−3.433, −4.755)	(−3.455, −4.755)
0.3	(−4.347, −3.170)	(−4.364, −3.170)
0.4	(−4.513, −1.469)	(−4.552, −1.469)
0.5	(−3.997, +0.226)	(−4.000, +0.226)

Table 2: Comparison between this work and that of Karwowski [4] for a thick wavelength long wire.

$a = 0.22$ and $\lambda = 0.88$		
u	Present	Karwowski
0.000	(−4.114, −3.063)	(−4.341, −2.894)
0.176	(−4.500, −2.001)	(−4.584, −1.852)
0.352	(−4.150, +0.150)	(−4.152, +0.250)
0.528	(−2.577, +1.406)	(−2.215, +1.451)
0.704	(−0.884, +0.903)	(−0.431, +0.909)
0.880	(−0.323, −0.371)	(−0.096, −0.380)

6 Conclusion

This paper makes an effort to evaluate the kernel by the definition of Schelkunoff [1]. An exact expression for the bounded part of the kernel K_B is given. This result can be used along with that of Pearson [3] for solving cylindrical antenna integral equations. Examples are considered where K_B is evaluated by the present method and is compared with that of Karwowski [4].

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