CIRCLE-FIT SUMMATION ACCELERATION OF PERIODIC GREEN'S FUNCTIONS

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ABSTRACT. The circle-fit algorithm is shown to be an attractive alternative to the spectral domain form of the two-dimensional periodic Green's function when the radiated fields are evaluated in the plane of the array. As the name implies, the circle-fit algorithm predicts the summation limit by fitting circles to the converging spiral of spatial domain partial sums in the complex plane. Several numerical examples comparing the raw spatial, spectral, and circle-fit accelerated spatial sums demonstrate the algorithm's computational savings. While other series extrapolation methods are shown to be more efficient, the circle-fit algorithm has the advantage of providing insight into how the Green's function converges.

1. INTRODUCTION

When using the Method of Moments (MM) to solve electromagnetic scattering problems, infinite array geometries invariably lead to infinite summations. The semi-infinite arrays explored in our recent work are no exception [1,2]. In seeking a numerical solution, one is forced to truncate the summations in one way or another. The obvious choice is to terminate the summation after a given number of terms, but determining how many terms to include before truncation is not always a straightforward task. As is often the case, the obvious choice is not always the "best" choice. Many authors have proposed various methods for improving the convergence properties of these summations [3,4,5]. Michalski provides a useful summary of the most common techniques in his paper on numerical integration of Sommerfeld integral tails [6].

In the following, we describe a novel algorithm that takes advantage of the convergence characteristics of the infinite summation to accelerate its convergence. In particular, the spatial domain form of the periodic Green's function becomes an attractive alternative to the spectral domain form when the fields are evaluated close to the array plane. In this case, we demonstrate that the circle-fit algorithm successfully accelerates the convergence of the raw spatial summation, providing a computational advantage over its spectral domain counterpart.

2. PERIODIC GREEN'S FUNCTIONS

The spatial domain Green's function for a two-dimensional array of line sources with inner-element spacing, d, along the y-axis is given by

$$G(x,y)=\sum_{m=-\infty}^{\infty}\frac{-j}{4}H_o^{(2)}\left(k\sqrt{x^2+(y-md)^2}\right) \quad (1)$$

where $H_o^{(2)}$ is the Hankel function, zeroth-order, second kind and k is the wave number. (Note: The array's reference element is located at the origin.) Unfortunately, the spatial domain form given in Equation (1) converges very slowly due to the fractional power decay of the Hankel function's large argument behavior. A commonly used alternative to the spatial domain form is the spectral domain Green's function given by

$$G(x,y)=\sum_{m=-\infty}^{\infty}\frac{-j}{2kd}e^{-j\left(kx+2m\pi y/d\right)} \quad (2)$$

where

$$k=\begin{cases}\sqrt{k^2-(2m\pi/d)^2}, & k^2>(2m\pi/d)^2\\ -j\sqrt{(2m\pi/d)^2-k^2}, & k^2<(2m\pi/d)^2\end{cases}$$

This series converges very quickly due to the evanescence plane waves' exponential decay. However, when the observation point lies in the array plane, the decay is no longer exponential negating any advantage over the spatial domain form. This case is of practical interest in the application of the MM to periodic planar arrays. In the next section we describe an acceleration technique which, when applied to Equation (1), provides a computationally efficient alternative to the spectral domain sum.

3. CIRCLE-FIT ALGORITHM

The circle-fit acceleration algorithm applies primarily to oscillatory summations. The circle-fit algorithm is a modification of the spiral average method proposed by Skinner [7]. Both are based on the observation that, for a given test location, the consecutive spatial domain partial sums form a spiral about the convergence point in the complex plane. This phenomenon is due to the fact that the distances between each successive array element and the test point asymptotically approach the inner-element spacing. By estimating the center of the spiral, one obtains an approximation for the eventual convergence point.

The primary distinction between Skinner's spiral average algorithm and the circle-fit algorithm lies in the technique used to estimate the center of the spiral. Skinner determines the number of partial sums required to have a complete phase rotation in the complex plane. He then performs an arithmetic average of the partial sums to obtain his estimate. When the inner-element spacing does not lead to an integer number of phase rotations, Skinner must choose an error parameter and search for the correct number of partial sums required to obtain a phase rotation within an error percentage of an integer number of rotations.

In contrast, the circle-fit algorithm does not require an integer number of phase rotations. The only requirement is that there is some phase rotation over the collection of partial sums. As the name implies, the algorithm simply attempts to fit a circle to the sequence of partial sums. The center of the circle provides an estimate to the infinite sum.

As an illustration, consider the two-dimensional Green's function for a semi-infinite wire array given by Equation (1). For large m values, one can use the asymptotic form of the Hankel function to express the summand as [8]

$$\begin{aligned} s_m &= H_o^{(2)}\left(k\sqrt{x^2+(y-md)^2}\right) \\ &\sim H_o^{(2)}\left(k|md|\right) \\ &\sim \sqrt{\frac{j2}{\pi k|md|}}\, e^{-jk|md|}, \quad m \to \infty \end{aligned} \tag{3}$$

Note that each summand contributes a component with a constant phase increment and monotonically decreasing amplitude.

Figure 1 depicts a typical sequence of partial sums. The sequence was generated using Equation (1) with the parameters: $d = 0.8\lambda$, $x = 0.0\lambda$, $y = 0.4\lambda$. The figure also contains a plot of the "best fit" circle to the last six partial sums. The center of this circle represents the current estimate of the infinite sum. To obtain a final estimate, one compares the current estimate with the estimate generated from the next six partial sums. When the difference between subsequent estimates reaches a specified tolerance, one calls the process converged. Note in this case the estimates are generated after every sixth partial sum. In general, a new estimate could be generated using any number of previous partial sums (greater than three) every time a new partial sum is calculated.

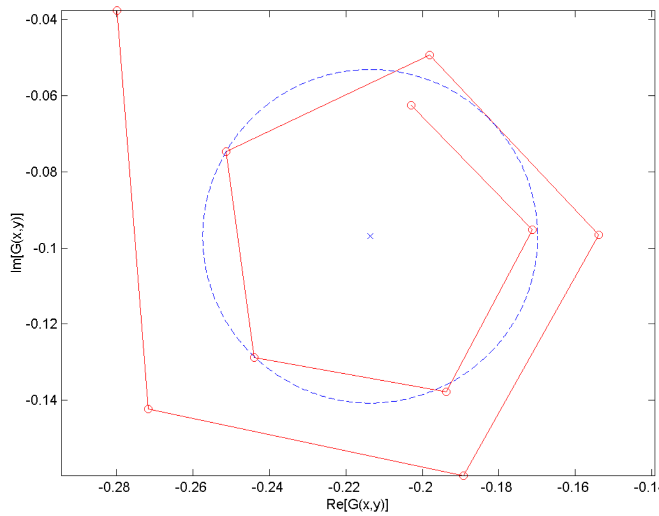

Figure 1 Circle-fit Algorithm Applied to Sequence of Partial Sums of the Spatial Domain Green's Function.

The method for fitting the circles comes from a paper by Moura and Kitney [9]. In it they describe how to determine the circle that minimizes the *square error* between itself and the set of data points. Simply stated, the problem amounts to finding a center point, (x_o,y_o), and a radius, r_o, such that the error defined by

$$E = \sum_{i=1}^{N}\left[(x_o - x_i)^2 + (y_o - y_i)^2 + r_o^{\,2}\right] \tag{4}$$

is minimized. This is accomplished by setting to zero the partial derivatives with respect to x_o, y_o, *and* r_o. The latter constraint gives a relationship between the data set, (x_i,y_i), the circle's center point, (x_o,y_o), and its radius, r_o

$$r_o^{\,2} = \frac{1}{N}\sum_{i=1}^{N}\left[(x_o - x_i)^2 + (y_o - y_i)^2\right] \tag{5}$$

From the remaining constraints, Moura and Kitney derive a matrix equation whose solution gives the circle's center, (x_o,y_o).

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} x_o \\ y_o \end{bmatrix} \tag{6}$$

where

$$G_{11} = 8\sum_{i=1}^{N} x_i^2 - \frac{8}{N}\left(\sum_{i=1}^{N} x_i\right)^2$$

$$G_{12(21)} = 8\sum_{i=1}^{N} x_i y_i - \frac{8}{N}\sum_{j=1}^{N} x_j \sum_{k=1}^{N} x_k$$

$$G_{22} = 8\sum_{i=1}^{N} y_i^2 - \frac{8}{N}\left(\sum_{i=1}^{N} y_i\right)^2$$

$$C_1 = -\frac{4}{N}\sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} x_j - \frac{4}{N}\sum_{i=1}^{N} y_i^2 \sum_{k=1}^{N} x_k + 4\sum_{i=1}^{N} y_i^2 x_i + 4\sum_{i=1}^{N} x_i^3$$

$$C_2 = -\frac{4}{N}\sum_{i=1}^{N} y_i^2 \sum_{j=1}^{N} y_j - \frac{4}{N}\sum_{i=1}^{N} x_i^2 \sum_{k=1}^{N} y_k + 4\sum_{i=1}^{N} x_i^2 y_i + 4\sum_{i=1}^{N} y_i^3$$

Significantly, this method is a *direct* method. (*i.e.*, the circle's defining parameters are calculated from the data points in one step.) There is no iterative process to refine an initial guess, a fact crucial in maintaining the circle-fit algorithm's computational efficiency.

4. NUMERICAL RESULTS

In this section, we examine the circle-fit algorithm's convergence characteristics. Taking the lead from Michalski [6], we employ the commonly used ε algorithm as a benchmark to evaluate the circle-fit algorithm's computational efficiency. Figure 2 clearly demonstrates this efficiency by comparing the convergence characteristics of the acceleration algorithm to the unaccelerated sums of Equations (1) and (2). The parameters here are the same as those used to generate Figure 1. Convergence is determined by calculating the relative error between each algorithm's estimate and a "truth" value obtained by setting the convergence tolerance to machine accuracy for either algorithm. The relative error is then defined by

$$\varepsilon = \left| \frac{\text{estimate} - \text{truth}}{\text{truth}} \right| \qquad (7)$$

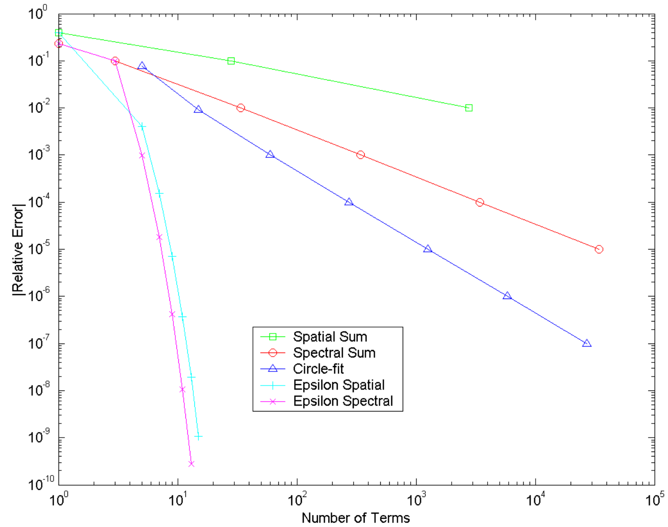

Figure 2 Relative Error Magnitude versus the Number of Terms for the Unaccelerated, Circle-fit, and Benchmark ε Accelerated Sums for (*x,y*) = (*0.0λ, 0.4λ*), *d* = *0.8λ*.

Table I contains the CPU times obtained on an 800MHz Pentium II laptop for the same array geometry. Clearly, the overhead in fitting the estimating circles to the partial sums is more than compensated by the enhanced convergence. (Note: "NC" indicates the particular summation calculation did not converge.)

Table I Green's Function Calculation Times for Two-Dimensional Array with (*x,y*) = (*0.0λ, 0.4λ*) and *d* = *0.8λ*.

Relative Error	*Spatial (Terms/ CPU sec)*	*Spectral (Terms/ CPU sec)*	*Circle-fit (Terms/ CPU sec)*
1.0e-1	28 / 0.01	3 / 0.00	5 / 0.51
1.0e-2	2764 / 1.47	34 / 0.00	15 / 0.51
1.0e-3	NC	342 / 0.03	60 / 0.53
1.0e-4	NC	3417 / 0.31	275 / 0.67
1.0e-5	NC	34165/ 3.13	1265 / 1.27
1.0e-6	NC	NC	5850 / 4.09
1.0e-7	NC	NC	27170 / 17.5

Figures 3 and 4 show the error magnitude comparisons for different array spacing and observation locations (in the array plane). In all cases, the circle-fit algorithm converges faster than the spectral domain Green's Function. However, it is apparent the algorithm does not out-perform the ε algorithm.

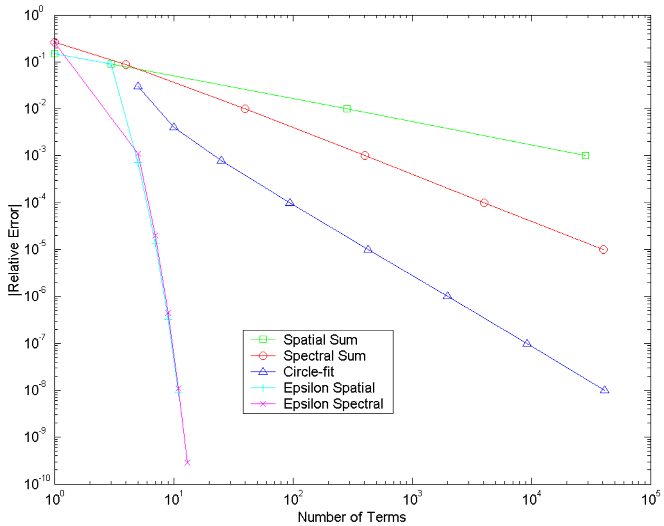

Figure 3 Relative Error Magnitude versus the Number of Terms for (*x,y*) = (*0.0λ,0.3λ*), *d = 0.6λ.*

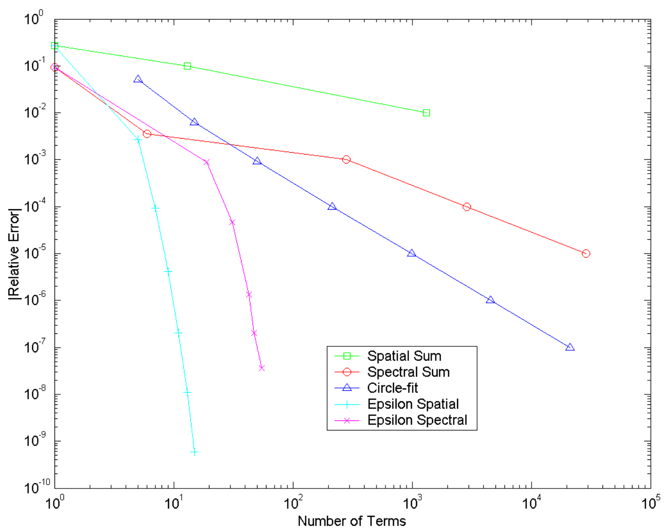

Figure 4 Relative Error Magnitude versus the Number of Terms for (*x,y*) = (*0.0λ,0.1λ*), *d = 1.2λ.*

5. CONCLUSION

We have investigated an alternative to the spectral domain transformation of the spatial domain Green's function for two-dimensional periodic arrays. Based on the decaying spiral behavior of the spatial domain partial sums in the complex plane, we approximate the convergence point as the center of a circle fit to the last few partial sums. The use of the circle-fit algorithm is shown to accelerate the convergence of the spatial domain Green's function, providing a computational advantage over the spectral domain when the fields are observed in the array plane. While the technique does not compete favorably with the ε algorithm, we point out the circle-fit algorithm does provide insight into how the periodic Green's function converges, something the more efficient algorithms cannot do. In addition, the circle-fit algorithm may provide a benefit when employed in combination with a more efficient algorithm. We are continuing our investigations of this technique to optimize the number of partials sums used given a particular array geometry.

6. REFERENCES

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