On Second-Order Asymptotic Expansions at a Caustic

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Abstract: This paper presents a detailed analysis of second-order asymptotic expansions for the description of the geometrical optics field in the vicinity of a caustic. It is shown that the stationary phase solution commonly found in textbooks and papers is valid only at the caustic, but not away from the caustic, as is usually claimed. The exact stationary phase expansion is derived and is compared to steepest descent solutions presented elsewhere in the literature.

I. INTRODUCTION

It is well known that the geometrical optics (GO) field reflected from a surface is equivalent to the first-order stationary phase (SP) asymptotic expansion of the corresponding radiation integral. When the reflector is concave, the reflected field forms a caustic surface as depicted in Figure 1. Near the caustic, where the first-order expansion begins to fail, two dominant stationary (saddle) points contribute to the asymptotic solution of the integral. These stationary points are each associated with a geometrical optics type ray, the rays merging into one another at the caustic. At the caustic the first-order expansions have a singularity and tend to infinity. Several second-order SP and steepest descent (SD) expansions describing the fields due to these stationary points are available in the literature ([1-19], listed in order of publication, including some related topics). The SP expansion originally introduced by Kay and Keller [1] involves only a single stationary point and does not decompose into two distinct GO expressions away from the caustic. Others employing the SP method derived similar expressions [2, 6 eq. (37), 9], all of which are not valid in the vicinity of or further away from the caustic, but only at the caustic itself. The incorrect application of this second-order SP solution away from the caustic has led to the introduction of heuristic caustic correction factors by Albertsen *et. al.* [8], which would not have been necessary if the exact expansion had been used. The limitations of these
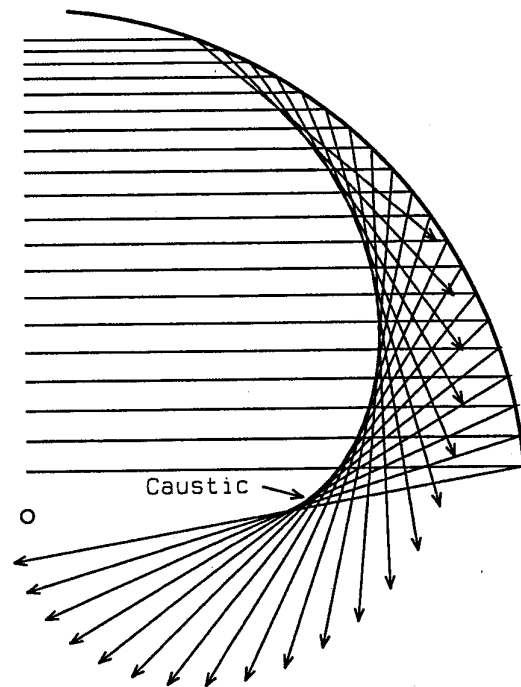

Figure 1. Reflected field of a concave reflector illuminated by a plane wave.

caustic correction factors have been pointed out by Meloling and Marhefka [19].

A Steepest Descent solution was introduced by Chester *et. al.* [3], which incorporated both stationary points and was valid at and away from the caustic. Their solution formed the basis of much of the work later published on the topic [4, 5, 7, 11, 14, 16, 19]. The new SP solution derived in this paper will be shown to yield results identical to these SD results. It is also possible to derive integral expressions for the fields in the vicinity of the caustic, as proposed by Ziolkowski and Deschamps [10], Hongo *et. al.* [12] and Hongo and Ji [13], but as a topic it falls beyond the scope of this paper. General reviews of high frequency techniques were presented by Arnold [11] and Bouche *et. al.* [17].

It will first be shown that the existing second-order SP solution is invalid away from the caustic. The exact second-order SP solution will then be derived and will be shown to be equivalent to the existing SD solutions. A detailed analysis of the scattering problem shown in Figure 1 will then be presented. This example will show clearly how the isolated rays contribute to the total solution. The paper is intended to present the new second-order SP solution, as well as to s how those who are not well familiar with asymptotic theory how to interpret the various asymptotic solutions found in the literature.

II. EXISTING SP EXPANSIONS IN THE VICINITY OF A CAUSTIC

Returning to two-dimensional scattering problem depicted in Figure 1 of which the paramater definitions are shown in Figure 2, the scattered electric field for the Tm_z polarization can be expressed in integral form as [20]

$$E_z = \frac{-k\eta}{4} \int_S J(t)\, H_0^{(2)}(k\, r(t))\, dt$$

$$\sim \frac{-k\eta}{4} \sqrt{\frac{2j}{\pi k}} \int_S \frac{J(t)}{\sqrt{r(t)}}\, e^{-jkr(t)}\, dt \qquad (1)$$

$$= \frac{-k\eta}{4} \sqrt{\frac{2j}{\pi k}}\, R \int_S \frac{J(\theta)}{\sqrt{r(\theta)}}\, e^{-jkr(\theta)}\, d\theta,$$

where J is the induced current on the circular reflector of radius R, r the distance from the integration point $t=R\theta$ on the equivalent reflector surface to the field evaluation point, and $H_0^{(2)}$ is a Hankel function of the second type. In the second line of (1) the large argument form of the Hankel function [21] was employed. The incident field in Figure 1 is assumed to be a plane wave. For this particular problem and similar problems in general, the scattered field integral can be expressed as

$$I = \int_S F(t)\, e^{jkq(t)}\, dt \quad , \qquad (2)$$

where in this case

$$F(t) = \frac{J(t)}{\sqrt{r(t)}}$$
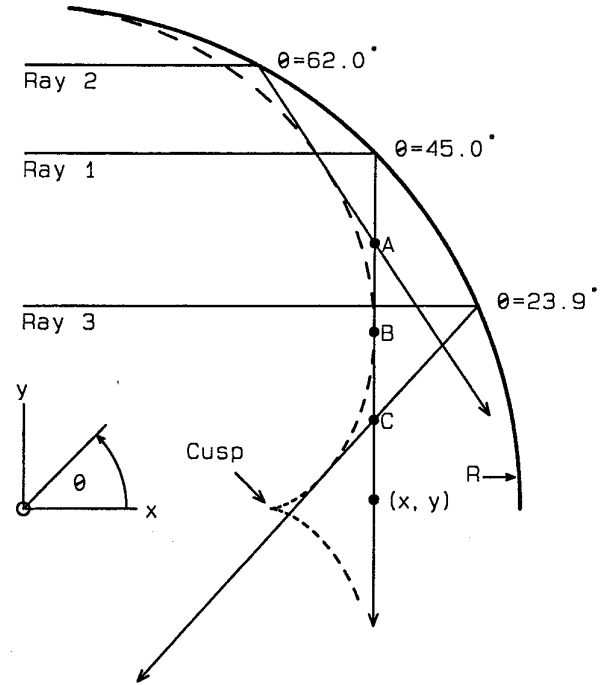
and $q(t) = -r(t)$.

Figure 2. Problem geometry and parameter definition.

If the reflector is large enough in terms of wavel ength, scattering from the reflector can be analysed by means of high frequency Geometrical Theory of Diffraction (GTD) [22] techniques, namely geometrical optics type reflected rays and diffraction from the edges of the reflector surface. This paper is not concerned with the diffracted fields, but only with the geometrical optics fields. For cases where the reflector is electrically large, also referred to as problems where k is large, the integral in (2) will have a stationary (or saddle) point for each geometrical optics ray. Such solutions are also referred to as the asymptotic solutions of the integral in (2). For any such stationary point, denoted by t_s and defined by $q'(t_s) = 0$, the first-order stationary phase solution can be expressed as [7, p. 387]

$$I \sim F(t_s) \sqrt{\frac{2\pi}{kq''(t_s)}}\, e^{jkq(t_s)}\, e^{j\frac{\pi}{4}}\, , \qquad (3)$$

where $q''(t_s)$ is the second derivative of q with respect to t, evaluated at the stationary point, and it was assumed that $q''(t_s) > 0$. When $q''(t_s) < 0$, a phase jump of -90° occurs, since from (3) we have

$\left(-1\right)^{-1/2} = e^{-j\pi/2}$. The asymptotic solution of the integral in (2) can then be expressed as

$$I \sim F(t_s)\sqrt{\frac{-2\pi}{kq''(t_s)}}\, e^{jkq(t_s)}\, e^{-j\frac{\pi}{4}} \quad . \tag{4}$$

With some effort it can be shown that (3) is in fact identical to the associated geometrical optics ray reflected at point t_s [23].

For the problem shown in Figure 1, reflected rays will traverse the caustic surface as indicated. Figure 2 depicts isolated rays traversing the caustic. At every point near the caustic, but not near the cusp of the caustic ($\theta = 0°$), the total field will be given by the sum of two distinct rays. In Figure 2 we keep Ray 1 fixed as the ray reflected at $\theta = 45°$. Progressing from the reflector towards the caustic along the path of Ray 1, Ray 1 will be crossed by a Ray 2, which has already traversed the caustic surface (evaluation point A). At the caustic, Ray 2 will have merged into Ray 1(point B). As we continue to progress past the caustic along the path of Ray 1, Ray 1 will now be crossed by a Ray 3, which is yet to traverse the caustic (evaluation point C). At the indicated cross-over points, which are close to the caustic surface, both rays contribute to the total solution. At evaluation points far from the caustic, in both directions, the stationary points become well separated and the total field (excluding diffracted fields) is given by the sum of two GO rays, namely

$$I \sim \begin{array}{l} F(t_{s1})\sqrt{\dfrac{-2\pi}{kq''(t_{s1})}}\, e^{jkq(t_{s1})}\, e^{-j\frac{\pi}{4}} \\[2em] + \; F(t_{s2})\sqrt{\dfrac{2\pi}{kq''(t_{s2})}}\, e^{jkq(t_{s2})}\, e^{j\frac{\pi}{4}} \; , \end{array} \tag{5}$$

where it was assumed that $q''(t_{s1}) < 0$ and $q''(t_{s2}) > 0$. Equation (5) is valid for those evaluation points in the region between the reflector and the caustic where the two stationary points are well separated. For evaluation points well beyond the caustic, such as the point (x,y) in Figure 2, the signs of the second derivatives will have changed and t_{s1} and t_{s2} in (5) must be interchanged.

As the caustic is approached, $t_{s1} \rightarrow t_{s2}$ where the

derivatives $q''(t_{s1}), q''(t_{s2}) \rightarrow 0$ and equations (3) to (5) become invalid. To overcome this problem, a second-order SP solution must be derived. The general procedure as adopted by [1, 9] is as follows. It is assumed that $t_{s1} = t_{s2} = t_s$, which allows a Taylor expansion of q to be made about the stationary point t_s, yielding

$$q(t) \approx \begin{array}{l} q(t_s) + q'(t_s)(t - t_s) \quad + \\[1em] \dfrac{q''(t_s)}{2}(t-t_s)^2 + \dfrac{q'''(t_s)}{6}(t-t_s)^3 \; . \end{array} \tag{6}$$

Only the first three derivatives were retained in (6) and by definition of a stationary point, $q'(t_s) = 0$. When (6) is substituted into (2), followed by the transformation of integration variable t to variable u (see [8] for details)

$$t - t_s = \left(\frac{2}{kq'''(t_s)}\right)^{\frac{1}{3}} u \; - \; \frac{q''(t_s)}{q'''(t_s)} \quad , \tag{7}$$

the asymptotic solution

$$I = 2\pi F(t_s)\left(\frac{2}{kq'''(t_s)}\right)^{\frac{1}{3}} e^{j\zeta} e^{jkq(t_s)} Ai(-\sigma) \tag{8}$$

is obtained, where Ai denotes the Airy function of the first kind [24],

$$\sigma = \left(\frac{k}{2}\right)^{\frac{2}{3}} \frac{[q''(t_s)]^2}{[q'''(t_s)]^{4/3}} \geq 0 \tag{9}$$

and

$$\zeta = \frac{2}{3}\sigma^{3/2} \quad . \tag{10}$$

At the caustic $q''(t_s) = 0$, yielding

$$I = 2\pi F(t_s)\left(\frac{2}{kq'''(t_s)}\right)^{\frac{1}{3}} e^{jkq(t_s)} Ai(0) \quad . \tag{11}$$

If (8) is supposed to represent the total field in the vicinity of the caustic, the contributions from the two distinct stationary points should become evident as one moves away from the caustic. This is clearly not the

case for (8), as it incorporates only one stationary point. If (8) is supposed to represent the contribution of a single stationary point only, and that a similar expression should be added for the contribution from the other stationary point, two problems arise. Firstly, at the caustic where $t_{s1} = t_{s2} = t_s$, the value of the field will be twice that given by (11), which is not correct. Secondly, when one moves away from the caustic, the two stationary points begin to separate and $\sigma > 0$. For widely separated stationary points, $\sigma \gg 0$ and one can employ the large argument form of the Airy function in (8). This is given by [8, 24]

$$Ai(-x) \sim \frac{1}{\sqrt{\pi}} \sigma^{-\frac{1}{4}} \sin(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \quad , \qquad (12)$$

an oscillatory function which does not decrease monotonically away from the caustic as is characteristic of the GO field given by (3). Equation (8) thus represents neither the total field nor the contribution of a single stationary point away from the caustic. Keller and Kay however expressly state that (8) above, presented in a different form in their paper, is the transition function between the field at the caustic and the first-order solution given by (3) above [1, paragraph following eq. (50)].

In an attempt to overcome this problem, (12) was written as

$$Ai(-\sigma) \sim \frac{1}{2\sqrt{\pi}} \sigma^{-\frac{1}{4}} (e^{j\frac{2}{3}\sigma^{\frac{3}{2}} - \frac{\pi}{4}} + e^{-j\frac{2}{3}\sigma^{\frac{3}{2}} + \frac{\pi}{4}}) \qquad (13)$$

by [8], which was then substituted back into (8). This substitution supposedly takes both rays into account, since (8) now "tends towards a sum of two terms, each of which is similar to" (3) above [8]. Since only one stationary point appears in (13), it was then suggested that each geometrical optics ray in (5) above be multiplied by a "caustic correction factor"

$$\frac{Ai(-\sigma)\sqrt{\pi}\,\sigma^{\frac{1}{4}}}{\sin(\frac{2}{3}\sigma^{\frac{3}{2}} + \frac{\pi}{4})} \quad . \qquad (14)$$

The very use of the term "correction factor" suggests that there is something wrong with the asymptotic expansion given in (8), which is not the case as long as

(8) is evaluated at the caustic only. As will become clear, this confusion stems purely from the fact that (8) is not an exact asymptotic expansion for the field at *and away from* a caustic.

III. DERIVATION OF EXACT SP EXPANSION IN THE VICINITY OF A CAUSTIC

In order to derive an exact second-order SP solution for the field at a caustic, the general approach adopted by Felsen and Marcuvitz [7] will be followed. The general form of exponential integrals is

$$I = \int_P F(z)e^{kQ(z)}\,dz \quad , \qquad (15)$$

where the integral is evaluated over a path P in the complex z-plane. As before, a Taylor expansion about the stationary point z_s and a substitution for $z - z_s$ similar to (7) yields

$$I \sim F(z_s) \left(\frac{2}{kQ'''(z_s)} \right)^{\frac{1}{3}} e^{\xi} e^{kQ(z_s)} \int_P e^{\frac{u^3}{3} - \delta u}\,du \quad , (16)$$

where

$$\delta = \left(\frac{k}{2} \right)^{\frac{2}{3}} \frac{[Q''(z_s)]^2}{[Q'''(z_s)]^{4/3}} \qquad (17)$$

and

$$\xi = \frac{2}{3} \delta^{3/2} \quad . \qquad (18)$$

For the sake of further analysis the sense of integration in (16) is reversed by performing the substitution $u = -z$. Furthermore, for the special case where $Q(z) = j\,q(z)$, (16) can be written as

$$I = I_p \sim F(z_s) \left(\frac{2}{kq'''(z_s)} \right)^{\frac{1}{3}} e^{\xi} e^{jkq(z_s)} e^{-j\frac{\pi}{6}}$$

$$\times \int_{P'} e^{\delta z - \frac{z^3}{3}}\,dz \quad , \qquad (19)$$

where the relevant parameters in the integral can now be expressed in terms of (9) and (10) as

$$\delta = \left(\frac{k}{2} \right)^{\frac{2}{3}} \frac{[q''(z_s)]^2}{[q'''(z_s)]^{4/3}} e^{j\frac{\pi}{3}} = \sigma e^{j\frac{\pi}{3}} \qquad (20)$$

and

$$\xi = \frac{2}{3}\delta^{3/2} = j\frac{2}{3}\sigma^{\frac{3}{2}} = j\zeta \quad .$$

$$(21)$$

As we have assumed $q''(z_s) > 0$, the integral is designated $I = I_p$. Defining the integral in (19) as

$$C(\delta) = \int_P e^{\delta z - \frac{z^3}{3}} dz \quad ,$$

$$(22)$$

and considering the possible integration paths for this integral as shown in Figure 3, the asymptotic solution of (22) is [7]

$$C(\delta) = \begin{cases} 2\pi j Ai(\delta) & \text{if } P = L_{32} & (23a) \\ \pi[Bi(\delta) - jAi(\delta)] & \text{if } P = L_{21} & (23b) \\ \pi[Bi(\delta) + jAi(\delta)] & \text{if } P = L_{31} & (23c) \end{cases}$$
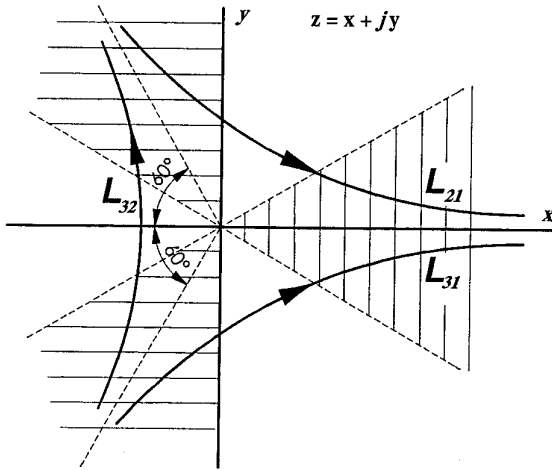
Figure 3. Integration paths for an Airy type integral of complex argument.

Since we need the integral to be in the form of (2) before we can derive a stationary phase path, following [7] we use the transformation $z = \Xi^{\frac{1}{3}} z'$ below to obtain

$$\int_{P'} e^{\delta z' - \frac{z'^3}{3}} dz' = \Xi^{\frac{1}{3}} \int_{P''} e^{\Xi(z - \frac{z^3}{3})} dz \quad ,$$

$$(24)$$

where

$$\Xi = \delta^{\frac{3}{2}} = j\Omega \quad , \quad \Omega = \sigma^{\frac{3}{2}} \quad .$$

$$(25)$$

The transformation has now allowed us to express (22) as

$$\int_{P'} e^{\delta z - \frac{z^3}{3}} dz = \Xi^{\frac{1}{3}} \int_{P''} e^{\Omega g(z)} dz \quad ,$$

$$(26)$$

where the function in the exponential term is

$$g(z) = j(z - \frac{z^3}{3}) \quad .$$

$$(27)$$

Equation (27) has stationary points where $g'(z_s) = 0$, which yields

$$z_s = \pm 1 \quad .$$

$$(28)$$

A stationary phase integration path requires that $\mathbb{R}e\{g(z)\} = \mathbb{R}e\{g(z_s)\}$, and with $z = x + jy$ we obtain the integration paths defined by

$$\mathbb{R}e\{g(z)\} = -y + x^2 y - \frac{y^3}{3}$$
$$= \mathbb{R}e\{g(z_s)\} = 0 \quad .$$

$$(29)$$

These integration paths are shown in Figure 4 and it is clear that an integration path corresponding to L_{32} in Figure 3 can be selected. For this choice the stationary phase solution of the integral in (22) is given by (23a) and the stationary phase solution of (19) for an isolated stationary point z_s becomes

$$I_p \sim 2\pi F(z_s) \left(\frac{2}{kq'''(z_s)}\right)^{\frac{1}{3}} e^{j\zeta} e^{jkq(z_s)} e^{j\frac{\pi}{3}}$$
$$\times Ai(\sigma e^{j\frac{\pi}{3}}) \quad .$$

$$(30)$$

When σ becomes large, we can use the large argument form of the Airy function of complex argument [24]

$$Ai(z) \sim \frac{1}{2\pi^{1/2} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} , \quad |\arg(z)| \leq \pi \quad (31)$$

to show that (30) reduces (3).

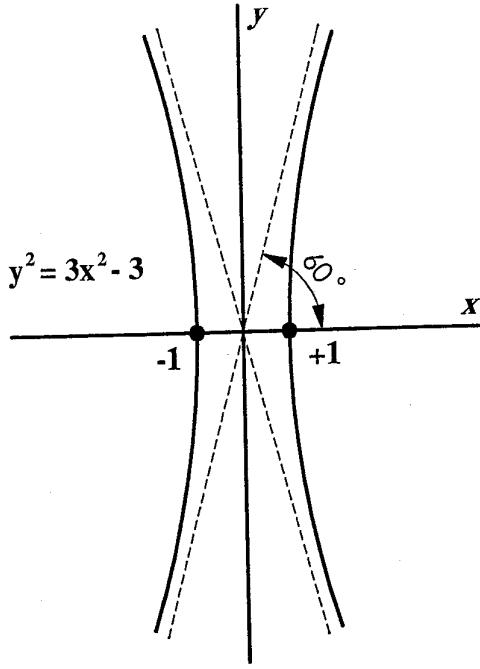When $q''(z_s) < 0$, we have from (25) and (20)

Figure 4. Stationary phase paths for an Airy type integral of complex argument.

$$\Xi_n = \delta^{\frac{3}{2}} = \frac{k}{2}\frac{[q''(z_s)]^3}{[q'''(z_s)]^2}e^{j\frac{\pi}{2}}$$

$$= \frac{k}{2}\frac{[|q''(z_s)|]^3}{[q'''(z_s)]^2}e^{j\frac{\pi}{2}}e^{j\pi} \qquad (32)$$

$$= -\Xi$$

and

$$\Xi_n^{\frac{1}{3}} = \delta^{\frac{3}{2}\cdot\frac{1}{3}} = \left(\frac{k}{2}\right)^{\frac{1}{3}}\frac{|q''(z_s)|}{[q'''(z_s)]^{\frac{2}{3}}}e^{j\frac{\pi}{6}}e^{j\frac{\pi}{3}}$$

$$= \Xi e^{j\frac{\pi}{3}} \quad . \qquad (33)$$

The negative sign in (32) will not result in different integration paths as defined by (29), but the transformation $z = \Xi_n^{\frac{1}{3}}z' = \Xi e^{j\frac{\pi}{3}}z'$ implies that the integration path must be rotated by 60° with respect to the case where $q''(z_s) > 0$. If we rotate the path through $z_s = +1$ in Figure 4 by 60° anti-clockwise, we can select an integration path corresponding to L_{21}, resulting in (22) becoming

$$C(\delta) = \pi[Bi(\sigma e^{j\frac{\pi}{3}}) - jAi(\sigma e^{j\frac{\pi}{3}})]$$

$$= 2\pi e^{-j\frac{\pi}{6}}Ai(\sigma e^{-j\frac{\pi}{3}}) \quad , \qquad (34)$$

where we have used the relationship

$$2Ai(ze^{\pm j\frac{2\pi}{3}}) = e^{\pm j\frac{\pi}{3}}[Ai(z) \mp jBi(z)] \quad . \qquad (35)$$

When (34) is substituted into (19) and recognising from (21) and (32) that ζ will also change in sign, we obtain

$$I_n \sim 2\pi F(z_s)\left(\frac{2}{kq'''(z_s)}\right)^{\frac{1}{3}}e^{-j\zeta}e^{jkq(z_s)}e^{-j\frac{\pi}{3}}$$

$$\times Ai(\sigma e^{-j\frac{\pi}{3}}) \quad . \qquad (36)$$

For σ becomes large, we can use (31) to show that (36) reduces to (4).

The asymptotic solution for the isolated stationary point $z_s = z_{s1}$ is thus given by (30) when $q''(z_{s1}) > 0$ and by (36) when $q''(z_{s1}) < 0$. For the second ray, defined by $z_s = z_{s2}$, we also apply either (30) or (36), depending on the sign of $q''(z_{s2})$. For evaluation points along the path of Ray 1, $q''(z_{s1}) < 0$ and $q''(z_{s2}) > 0$ before the caustic is traversed. Beyond the caustic $q''(z_{s1}) > 0$ and $q''(z_{s2}) < 0$. The total field is thus given by $I_{tot}(q_1'' < 0, q_2'' > 0) = I_n(z_{s1}) + I_p(z_{s2})$ for the case where the caustic is yet to be traversed, and by $I_{tot}(q_1'' > 0, q_2'' < 0) = I_p(z_{s1}) + I_n(z_{s2})$ beyond the caustic. At the caustic, having been approached from either side, $\sigma \to 0$ and the total field is given by $I = I_p + I_n$, namely

$$I_{tot} \sim 2\pi F(z_s)\left(\frac{2}{kq'''(z_s)}\right)^{\frac{1}{3}}e^{jkq(z_s)}e^{+j\frac{\pi}{3}}Ai(0)$$

$$+ 2\pi F(z_s)\left(\frac{2}{kq'''(z_s)}\right)^{\frac{1}{3}}e^{jkq(z_s)}e^{-j\frac{\pi}{3}}Ai(0) \qquad (37)$$

$$= 2\pi F(z_s)\left(\frac{2}{kq'''(z_s)}\right)^{\frac{1}{3}}e^{jkq(z_s)}Ai(0) \quad .$$

This is the same value as given by (11).

IV. SD ASYMPTOTIC EXPANSIONS FOR TWO NEARBY STATIONARY POINTS

A steepest descent asymptotic solution for the total field of (2) in the case of two nearby stationary (saddle) points is expressed by Felsen and Marcuvitz [7] as

$$I_{tot} \sim \pi k^{-\frac{1}{3}} [h_1 F(t_{s1}) + h_2 F(t_{s2})] e^{j a_o k}$$

$$\times Ai(-\sigma k^{\frac{2}{3}})$$

$$+ j \pi \sigma^{-\frac{1}{2}} k^{-\frac{2}{3}} [h_1 F(t_{s1}) - h_2 F(t_{s2})] e^{j a_o k}$$

$$\times Ai'(-\sigma k^{\frac{2}{3}}) ,$$

(38)

where

$$h_1 = \sqrt{\frac{-2 \sigma^{1/2}}{q''(t_{s1})}} , \quad h_2 = \sqrt{\frac{+2 \sigma^{1/2}}{q''(t_{s2})}} ,$$

(39)

$$a_o = \frac{1}{2} [q(t_{s1}) + q(t_{s2})] ,$$

(40)

and

$$\frac{2}{3} \sigma^{\frac{3}{2}} = \frac{1}{2} [q(t_{s1}) - q(t_{s2})] .$$

(41)

In (39) it is assumed that $q''(t_{s1}) < 0$ and $q''(t_{s2}) > 0$. The total field given by (38) is the sum of two rays, described respectively by

$$I_p \sim \frac{\pi}{2} k^{-\frac{1}{3}} [h_1 F(t_{s1}) + h_2 F(t_{s2})] e^{j a_o k}$$

$$\times [Ai(-\sigma k^{\frac{2}{3}}) + j Bi(-\sigma k^{\frac{2}{3}})]$$

$$+ j \frac{\pi}{2} \sigma^{-\frac{1}{2}} k^{-\frac{2}{3}} [h_1 F(t_{s1}) - h_2 F(t_{s2})] e^{j a_o k}$$

$$\times [Ai'(-\sigma k^{\frac{2}{3}}) + j Bi'(-\sigma k^{\frac{2}{3}})] ,$$

(42)

and

$$I_n \sim \frac{\pi}{2} k^{-\frac{1}{3}} [h_1 F(t_{s1}) + h_2 F(t_{s2})] e^{j a_o k}$$

$$\times [Ai(-\sigma k^{\frac{2}{3}}) - j Bi(-\sigma k^{\frac{2}{3}})]$$

$$+ j \frac{\pi}{2} \sigma^{-\frac{1}{2}} k^{-\frac{2}{3}} [h_1 F(t_{s1}) - h_2 F(t_{s2})] e^{j a_o k}$$

$$\times [Ai'(-\sigma k^{\frac{2}{3}}) - j Bi'(-\sigma k^{\frac{2}{3}})] .$$

(43)

It is easily verified that the sum of (42) and (43) yields (38).

At the caustic $t_{s1} = t_{s2}$, $\sigma = 0$, and h_1 and h_2 assume the limiting value [7]

$$|h_1| = |h_2| = \left[\frac{2}{q'''(t_s)} \right]^{\frac{1}{3}} .$$

(44)

Since the second term in (38) becomes zero, it is rudimentary to show that (38) reduces to (11). Equations (42) and (43) can likewise be shown to reduce to

$$I_p \sim 2 \pi F(t_s) \left(\frac{2}{k q'''(t_s)} \right)^{\frac{1}{3}} e^{j k q(t_s)} e^{+j \frac{\pi}{3}} Ai(0)$$

(45)

and

$$I_n \sim 2 \pi F(t_s) \left(\frac{2}{k q'''(t_s)} \right)^{\frac{1}{3}} e^{j k q(t_s)} e^{-j \frac{\pi}{3}} Ai(0)$$

(46)

respectively. To derive (45) and (46) the relationship

$$Ai(-\sigma) \pm j Bi(-\sigma) = 2 e^{\pm j \frac{\pi}{3}} Ai(\sigma e^{\pm j \frac{\pi}{3}})$$

(47)

was used. When the two stationary points are widely separated, (38) reduces to (5), as is shown in Appendix A. It is also shown that (42) and (43) reduce to (3) and (4), respectively.

It should now be clear that at the caustic and far away from the caustic, the SD expressions for the total field and the two separate rays are identical to the corresponding SP expressions derived in the previous section. A numerical example presented later will show that the SD and SP expansions yield identical results in the intermediate region as well.

V. NUMERICAL EXAMPLE OF THE SCATTERED FIELD NEAR A CAUSTIC

In order to demonstrate how the field behaves in the vicinity of the caustic and how the isolated ray contributions should be interpreted, we return to the problem depicted in Figures 1 and 2. The scattered field will be calculated by means of geometrical optics, integration of the physical optics current induced on the reflector surface, the newly derived SP expansions

and the SD expansions discussed in the previous section.

The two-dimensional circular scatterer is assumed to have a radius $R = 200\lambda$ and is illuminated by a plane wave, which is TM$_z$ polarised. The incident electric and magnetic fields at the reflector surface are given by

$$\boldsymbol{E}^{i} = E^{i}\hat{z} = e^{-jkR\cos\theta}\,\hat{z} \qquad (48)$$

and

$$\boldsymbol{H}^{i} = H^{i}\hat{y} = -\frac{1}{\eta}e^{-jkR\cos\theta}\,\hat{y} \quad, \qquad (49)$$

where η is the free space wave impedance. With $E^{i} = 1$, the GO scattered electric field is given by

$$\boldsymbol{E}^{s} = E^{s}\hat{z} = -e^{-jkR\cos\theta}\sqrt{\frac{\rho}{\rho+l_{2}}}\,e^{-jkl_{2}}\,\hat{z} \qquad (50)$$

where l_{2} is the distance from the reflection point along the ray path and the radius of curvature ρ is given by

$$\frac{1}{\rho} = \frac{1}{l_{1}} - \frac{2}{R\cos\theta^{i}} = -\frac{2}{R\cos\theta^{i}} \quad . \qquad (51)$$

In (51) it was furthermore assumed that $l_{1} = \infty$ since the incident field is a plane wave. For a circular reflector, $\theta^{i} = \theta$.

The physical optics current J is given by

$$\begin{aligned}\boldsymbol{J} = J_{z}\hat{z} = 2\hat{n}\times\boldsymbol{H}^{i} &= \frac{2}{\eta}e^{-jkR\cos\theta}n_{x}H_{y}\hat{z} \\ &= \frac{2}{\eta}e^{-jkR\cos\theta}\cos\theta\,\hat{z} \quad, \end{aligned} \qquad (52)$$

where we have utilised the normal vector relationship $\hat{n} = n_{x}\hat{x} + n_{y}\hat{y} = -\cos\theta\,\hat{x} - \sin\theta\,\hat{y}$. In subsequent integral calculations (1) was evaluated by means of Gaussian integration, with 5 integration points/ wavelength used for the 0°-90° circular integration sector (about 1570 integration points).

With (52) substituted into (1), we have from (2)

$$F(\theta) = -\frac{k}{2}\sqrt{\frac{2}{\pi k l_{2}}}\,e^{j\frac{\pi}{4}}\cos\theta \quad, \qquad (53)$$

where for an observation point (x,y) shown in Figure 2,

$$l_{2} = \sqrt{(x - R\cos\theta)^{2} + (y - R\sin\theta)^{2}} \qquad (54)$$

and

$$q(\theta) = -(l_{2} + R\cos\theta) \quad . \qquad (55)$$

The derivatives of $q(\theta)$ are given in Appendix B. The first- and second-order expansions discussed above give a phase jump of $\pm45°$ across the caustic. Since the GO field encounters a phase jump of $+90°$ across the caustic, all SP and SD solutions were multiplied by a factor $e^{j\frac{\pi}{4}}$ to ensure that the GO and SP/SD phase terms are identical.

With the appropriate functions as defined above, the various field expressions were evaluated along the path of Ray 1 in Figure 2, which is vertically downward $(\theta_{s} = 45°)$. The evaluation points on this path progressed from the reflector surface to beyond the caustic surface. Ray 2 crosses Ray 1 at a distance of $\rho_{o}/2$ before the caustic, while Ray 3 crosses Ray 1 at a distance of $\rho_{o}/2$ past the caustic point of Ray 1, where $\rho_{o} = 0.5\,R\cos(\pi/4)$.
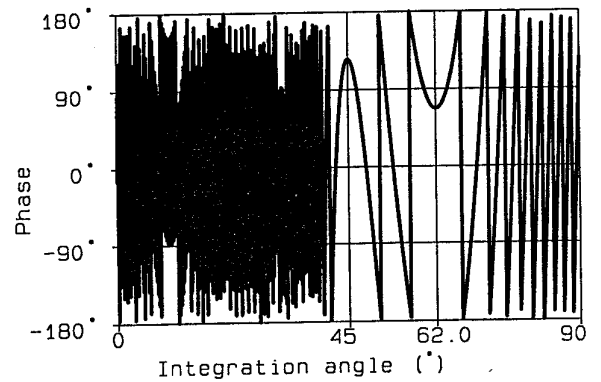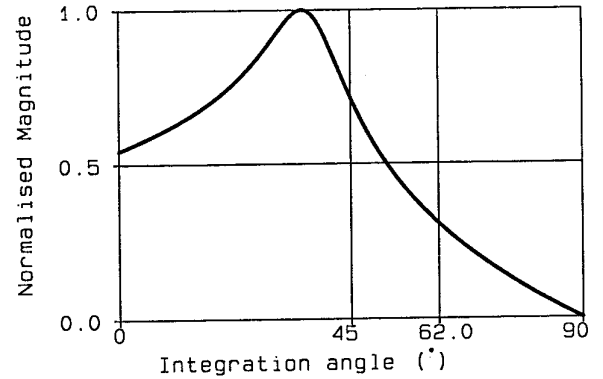

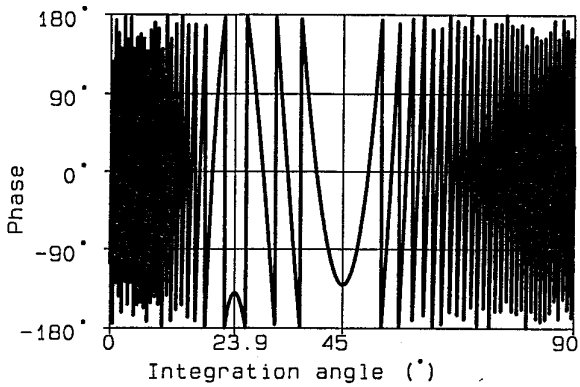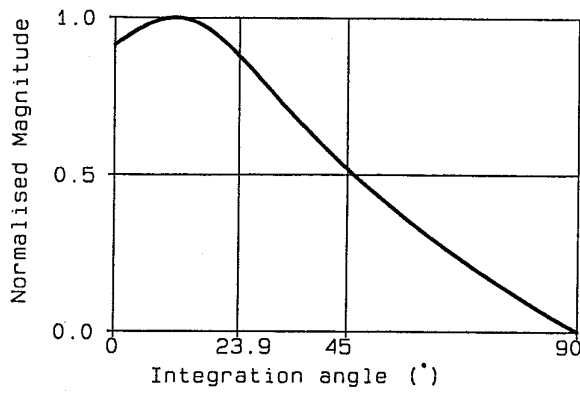
Figure 5. Integrand for Ray 2 crossing Ray 1.

Figure 6. Integrand for Ray 3 crossing Ray 1.

Figure 7. Integrand for Rays 1 and 2 having merged
at the caustic.

Figures 5 to 7 show the magnitude (normalised) and phase (not normalised) of the integrand of (2) for the cases where the two stationary points are separated from each other and when the two rays have merged into each other at the caustic. In Figure 5, Ray 2 has a stationary point $\theta_s = 62.0°$ and has already traversed the caustic surface (point A). The first derivative of q is clearly zero at the two stationary points (the definition of a stationary point), and it is also evident that $q''(\theta_{s1}) < 0$ and $q''(\theta_{s2}) > 0$. Figure 6 represents the case where Ray 1 is crossed by Ray 3 (point C). Ray 1 has already traversed the caustic, but Ray 3 ($\theta_s = 23.9°$) is yet to do so. As expected, the signs of the second derivatives have reversed and we now have $q''(\theta_{s1}) > 0$ and $q''(\theta_{s3}) < 0$. Figure 7 shows the integrand magnitude and phase for the case where the two rays have merged into one, with the evaluation point at the caustic of Ray 1 in Figure 2 (point B). The phase of the integrand displays an inflection point at the caustic and one can by inspection see that $q''(\theta_{s1}) = q''(\theta_{s2}) = 0$.
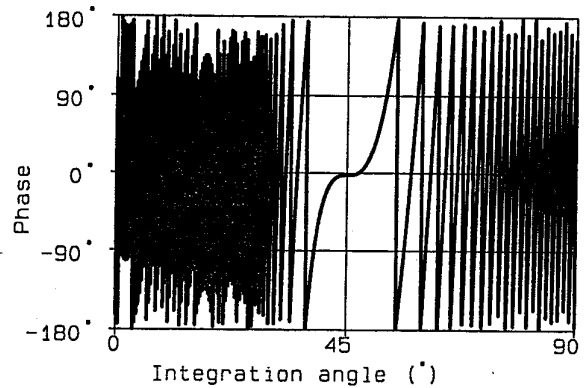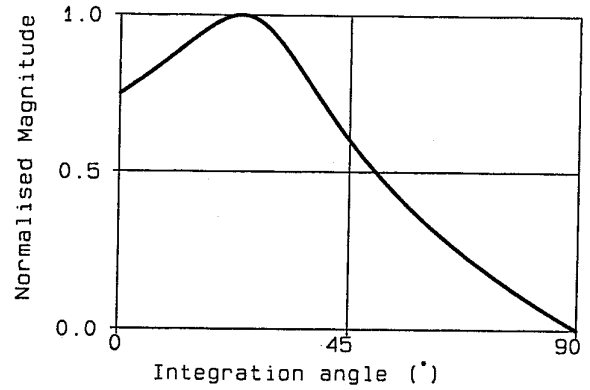
The scattered electric field along the path of Ray 1 was next calculated by means of the techniques mentioned above. The ray path progresse d from the surface of the reflector ($l_2 = 0$) to $l_2 = 1.5\rho_o$. Figure 8 shows the scattered field component of Ray 1 only, as calculated by means of geometrical optics (GO), the new stationary phase expansions deriv ed above (SP) and the steepest descent expansions given by [7] (SD).

The phase of the scattered field (ϕ) was normalised with respect to the GO linear phase (excluding the phase jump which occurs when the caustic is traversed), and the normalised phase is thus given by the expression $\phi_{nor} = \phi + k(R\cos(\pi/4) + l_2) - \pi$, the phase term π accounting for the negative reflection coefficient in (50) above. The magnitude was normalised with respec t to the value of the electric field at the caustic, designated by Ec.

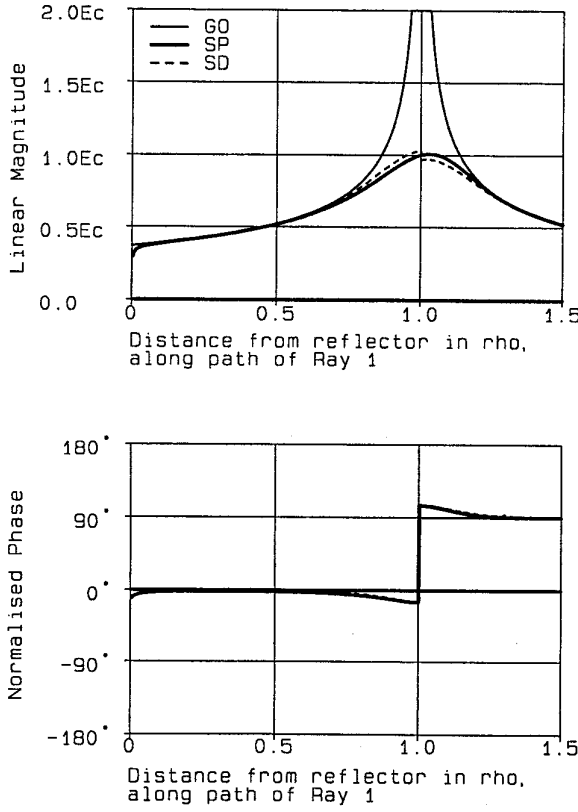Returning to Figure 8, Ray 1 seems to be discontin uous

Figure 8. Partial field given by Ray 1.

across the caustic, as the phase jumps by more than 90°. It should be kept in mind that $q''(\theta_{sI}) < 0$ as long as Ray 1 has not traversed the caustic, and $q''(\theta_{sI}) > 0$ when it has already done so. One must then use (36) in the former case and (30) in the latter. At every evaluation point along the path of Ray 1, one should simply choose the appropriate asymptotic solution, which in turn depends on the sign of $q''(\theta_s)$. The SP solution is not accurate right against the surface of the reflector, since we have used the large argument form of the Hankel function to derive (2), which is not correct when $l_2 \to 0$.

Calculation by means of the SD expansions is considerably more involved. It should be clear that (42) and (43) represent the case where Ray 1 is yet to traverse the caustic, with $q''(\theta_{sI}) = q_1'' < 0$ and $q''(\theta_{s2}) = q_2'' > 0$. When Ray 1 has traversed the caustic, the signs of these derivatives change and (42) and (43) are no longer valid. We can rewrite (42) and (43) in shorthand notation for the case where

$q_1'' < 0$, $q_2'' > 0$, $l_2 < \rho_0$ as

$$
\begin{aligned}
I_{2p} = \; & C_1 [h_1 F_1 + h_2 F_2] [Ai + jBi] \\
& + C_2 [h_1 F_1 - h_2 F_2] [Ai' + jBi'] \\
\sim \; & F_2 \sqrt{\frac{2\pi}{kq_2''}} \, e^{jkq_2} \, e^{+j\frac{\pi}{4}} \quad \text{for } \sigma \gg 0
\end{aligned}
\tag{56}
$$

and

$$
\begin{aligned}
I_{1n} = \; & C_1 [h_1 F_1 + h_2 F_2] [Ai - jBi] \\
& + C_2 [h_1 F_1 - h_2 F_2] [Ai' - jBi'] \\
\sim \; & F_1 \sqrt{\frac{-2\pi}{kq_1''}} \, e^{jkq_1} \, e^{-j\frac{\pi}{4}} \quad \text{for } \sigma \gg 0 \; .
\end{aligned}
\tag{57}
$$

The corresponding expressions for the case where $q_1'' > 0$, $q_2'' < 0$, $l_2 > \rho_0$ are

$$
\begin{aligned}
I_{2n} = \; & C_1 [h_1 F_1 + h_2 F_2] [Ai - jBi] \\
& - C_2 [h_1 F_1 - h_2 F_2] [Ai' - jBi'] \\
\sim \; & F_2 \sqrt{\frac{-2\pi}{kq_2''}} \, e^{jkq_2} \, e^{-j\frac{\pi}{4}} \quad \text{for } \sigma \gg 0
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
I_{1p} = \; & C_1 [h_1 F_1 + h_2 F_2] [Ai + jBi] \\
& - C_2 [h_1 F_1 - h_2 F_2] [Ai' + jBi'] \\
\sim \; & F_1 \sqrt{\frac{2\pi}{kq_1''}} \, e^{jkq_1} \, e^{+j\frac{\pi}{4}} \quad \text{for } \sigma \gg 0 \; .
\end{aligned}
\tag{59}
$$

It should be noted that the SD expansions do not quite give the correct values as the caustic is approached. This is because of various terms (h_1, h_2, σ) in the SD expansions tend to either zero or infinity as the caustic is approached. These infinite/zero numbers are added and subtracted to give the total contribution of Ray 1, with an obvious numerical round-off error giving rise t o the discrepancy. The field value jumps from just more than to just less than the actual value as the caustic is traversed. We know from (45) and (46) that the SD

solutions for Ray 1 and Ray 2 give the correct value at the caustic. The phase curves are identical.

Figure 9 shows the GO, SP and SD results for the contribution of Ray 2 as the evaluation point p rogresses along the path of Ray 1. An algorithm was developed to find the appropriate stationary point of Ray 2 for each of these evaluation points. The phase of the Ray 2 results was normalised by the same factor as used in Figure 8. Near the caustic Ray 2 tends to Ray 1 and experiences a similar phase variation, except that the phase jump is now -90°. When the results of Figures 8 and 9 are overlaid, one can see that the total field will be continuous in magnitude and phase.
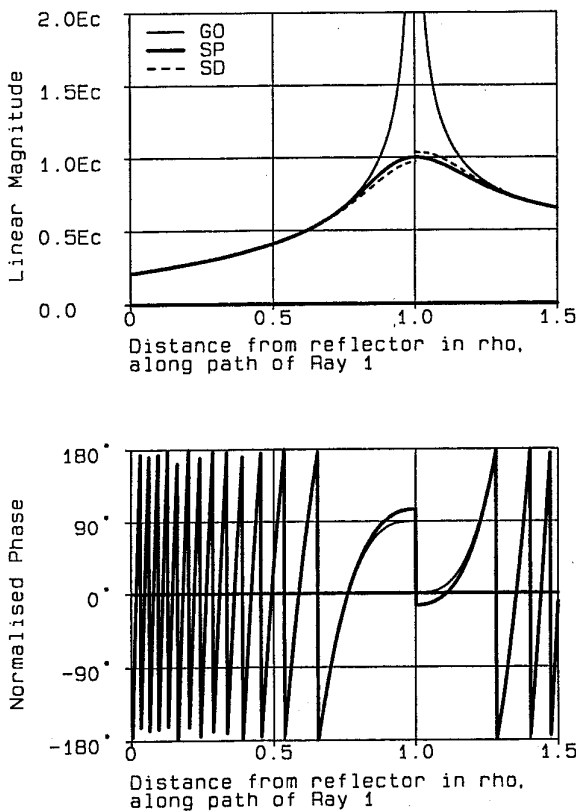

Figure 9. Partial field given by Ray 2.

It should be stressed that the expansions for isolated stationary points given by (30), (36) and (56) to (58) can yield an absolute phase va riation of maximum 15°, as is evident from Figures 8 and 9.

The total field as calculated by geometrical optics, the SP and SD expansions and integration of the physical optics (PO) current is shown in Fi gure 10. There is no

discernable difference between the results of any of the latter three methods. The physical optics integral solution should actually be regarded as the bench -mark, as it is based on the least number of approximations.

Of particular interest is the phase of the total field in th e vicinity of the caustic. The phase of the total field was normalised by the same factor used in Figures 8 and 9. Had there been no caustic effect present, the normalise d phase at $1.0 \rho_o$ would have been zero (the field would
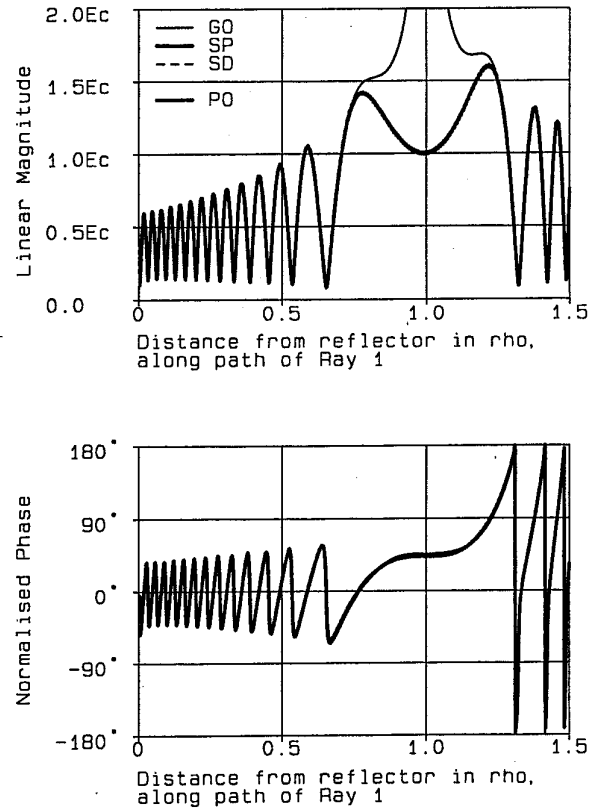

Figure 10. Total field taking both stationary points into account.

have been purely GO in nature). However, with the caustic present, the phase at the caustic is +45° with respect to the linear phase of Ray 1. In fact, if one considers the normalised phase in the region where the GO and SP magnitude curves begin to separate (the caustic region), it is clear that there is a gradual phase jump from 0° to +90° across the caustic region. This is in perfect agreement with the phase jump predicted by the first-order SP (and thus the GO) solutions of the

scattered field integral.

VI. CONCLUSION

Of the various published forms of asymptotic expansions for the field in the vicinity of a causti c, only the steepest descent expansions were found to be correct. The corresponding second-order stationary phase expansion which is commonly found in the literature and textbooks was found to be valid only at the caustic itself, but incorrect away from the caustic. In this paper the exact second-order stationary phase expansions were derived. These expansions yield identical results to the steepest descent expansions, but are considerably less complex to evaluate than the latter. This is due mainly because the new stationary phase expansions were derived in terms of the contribution of a single stationary point only, whilst the steepest descent expansions were derived in terms of two nearby saddle points. It is instructive to see that two seemingly different sets of expansions yield the same results. This is of course to be expected if both are correct.

The purpose of the paper was not only to present the exact stationary phase expansions, but also to serve as a tutorial for those who are not experienced in the fie ld. The importance of the numerical example presented here lies in the clear exposition of how the various expansions should be used, and in that it shows how th e phase of the total field experiences a gradual phase jump of 90° as the caustic is trav ersed. The expansions for the isolated stationary points do not yield a continuous phase jump across the caustic by themselves , only when they are added to obtain the total field. The second-order solutions discussed in this paper are not valid in the region of a caustic cusp, where higher orde r derivatives for the various functions are required [1].

It is hoped that this paper will contribute to a better understanding of asymptotic solutions at a caustic.

REFERENCES

[1]   I. Kay and J. B. Keller, "Asymptotic Evaluation of the Field at a Caustic", *J. Appl. Phys.*, Vol. 25, No. 7, pp. 876-883, July 1954.

[2]   I. Kay, "Fields in the neighborhood of a caustic", *IRE Trans. Antennas and Propagat.*, pp. 255-260, 1959.

[3]   C. Chester, B. Friedman and F. Ursell, "An Extension of the Method of Steepest Descents", *Proc. Cambridge Phil. Soc.*, Cambridge, London, U.K., Vol. 53,. Pp599-611, 1957.

[4]   D. Ludwig, "Uniform Asympt otic Expansions at a Caustic", *Comm. Pure Appl. Math.*, vol. XIX, pp. 215-220, 1966.

[5]   N. Bleistein, "Uniform asymptotic expansions of integrals with many nearby stationary point s and algebraic singularities", *J. Math. Mech.*, vol. 17, no. 6, pp. 533-559, 1967.

[6]   G. Hyde, "Studies of the focal region of a spherical reflector: stationary phase evaluation", *IEEE Trans. Antennas and Propagat.*, vol. AP-16, pp. 646-656, 1968.

[7]   L.B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, Englewood Cliffs, N.J.: Prentice-Hall, 1973, pp. 370-416.

[8]   N.C. Albertsen, P. Balling and N.E. Jensen, "Caustics and Caustic Corrections to the Field Diffracted by a Curved Edge", *IEEE Trans. Antennas Propagat.*, vol. AP-25, pp. 297-303, 1977.

[9]   G.L. James, *Geometrical theory of diffraction for electromagnetic waves*, London: IEE (Peter Peregrinus), 1976, 1980, p.30.

[10]   R.W. Ziolkowski and G.A Deschamps, "Asymptotic evaluation of high frequency fields near a caustic, an introduction to Maslov's method," *Radio Sci.*, vol. 19, no. 4,. pp. 1001-1025, Jul.-Aug., 1984.

[11]   J.M. Arnold, "Geometrical theories of wave propagation: a contemporary review", *IEE Proc.*, vol. 133, Pt. J, pp. 165-188, 1986.

[12]   K. Hongo, Y. Ji and E. Nakajima, "High-frequency expression for the field in the caustic region of a reflector using Maslov's method", *Radio Sci.*, vol. 6, pp. 911-919, Dec. 1986.

[13]   K. Hongo and Y. Ji, "Study of the field around the focal region of a sphe rical reflector antenna", *IEEE Trans. Antenna and Propagat.*, vol. AP-36, pp. 592-598, 1988.

[14] P.H. Pathak and M-C. Liang, "On a uniform asymptotic solution valid across smooth caustics of rays reflected by smoothly indented boundaries", *IEEE Trans. Antennas and Propagat.*, vol. AP-38, pp. 1192-1203, 1990.

[15] E. D. Constantinides and R.J. Marhefka, "Efficient and accurate computation of the incomplete Airy functions", *Radio Sci.*, vol. 28, no. 4, pp. 441-457, July-August 1993.

[16] E. D. Constantinides and R.J. Marhefka, "A UGO/EUTD solution for the scattering and diffrcation from cubic polynomial strips", *IEEE Trans. Antennas and Propagat.*, vol. AP-41, pp. 1088-1098, 1993.

[17] D.P. Bouche, F.A. Molinet and R. Mittra, "Asymptotic and hybrid techniques for electromagnetic scattering," *Proc. IEEE*, vol. 81, no. 12, pp. 1658-1684, 1993.

[18] E. D. Constantinides and R.J. Marhefka, "An extended UTD analysis for RCS computations involving higher order curved surfaces", *Annales Des Telecommunications*, vol. 50, no. 5-6, pp. 474-486, May-June 1995.

[19] J. H. Meloling and R.J. Marhefka, "A caustic corrected UTD solution for the fields radiated by a source on a flat plate with a curved edge", *IEEE Trans. Antennas and Propagat.*, vol. AP-45, pp. 1839-1849, 1997.

[20] R.F. Harrington, *Field Computation by Moment Methods*, Kriegler Publishing Company, pp. 41-61, 1982.

[21] R.F. Harrington, *Time-Harmonic Electromagnetic Fields*, McGraw-Hill, p. 463, 1961.

[22] J.B. Keller, "Geometrical Theory of Diffraction", *J. Opt. Soc. Amer.*, vol. 52, pp. 116-130, 1962.

[23] A.J. Booysen, "On the equivalence of geometrical-optics reflected fields and the stationary-phase solution of the associated radiation integrals", *IEEE Antennas and Propagation Magazine*, Vol. 44, Issue 5, pp. 120-123, Oct. 2002.

[24] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, 1965.

APPENDIX A

Equation (38) can be shown to reduce to (5) as follows. For widely separated stationary points, σ becomes large and we can use the large argument form of the Airy functions in (38). With (12) and [24]

$$Ai'(-x) \sim -\frac{1}{\sqrt{\pi}} \sigma^{\frac{1}{4}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right) \tag{60}$$

substituted into (38), we obtain

$$I \sim \sqrt{\frac{\pi}{k}} \, [F_1 \sqrt{\frac{-2}{q_1}} + F_2 \sqrt{\frac{2}{q_2}}] e^{ja_o k} \sin\alpha$$

$$- j\sqrt{\frac{\pi}{k}} \, [F_1 \sqrt{\frac{-2}{q_1}} - F_2 \sqrt{\frac{2}{q_2}}] e^{ja_o k} \cos\alpha \,, \tag{61}$$

where the shorthand notation $F_1 = F(t_{sI})$ and $q_1 = q''(t_{sI})$ was introduced for the sake of simplicity and

$$\alpha = \frac{2}{3}\sigma^{\frac{3}{2}} + \frac{\pi}{4} \quad . \tag{62}$$

Rearranging (61), we obtain

$$I \sim F_1 \sqrt{\frac{-2\pi}{kq_1}} (\sin\alpha - j\cos\alpha) \, e^{ja_o k}$$

$$+ F_2 \sqrt{\frac{+2\pi}{kq_2}} (\sin\alpha + j\cos\alpha) \, e^{ja_o k} \quad . \tag{63}$$

Equation (63) can be written as

$$I \sim F_1 \sqrt{\frac{-2\pi}{kq_1}} e^{-j\frac{\pi}{2}} e^{j\alpha} e^{ja_o k}$$

$$+ F_2 \sqrt{\frac{+2\pi}{kq_2}} e^{j\frac{\pi}{2}} e^{-j\alpha} e^{ja_o k} \tag{64}$$

which with the aid of (40) and (41) reduces to

$$I \sim F_1 \sqrt{\frac{-2\pi}{kq_1}} e^{-j\frac{\pi}{4}} e^{jkq_1} + F_2 \sqrt{\frac{2\pi}{kq_2}} e^{j\frac{\pi}{4}} e^{jkq_2} \quad . \tag{65}$$

Equation (65) is identical to (5).

The isolated rays given by (42) and (43) can in similar fashion be shown to reduce to the first and second terms in (5), respectively. Only (42) will be discussed. The large argument forms of the Airy functions of the second type are [24]

$$Bi(-x) \sim \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}} \cos(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \qquad (66)$$

and

$$Bi'(-x) \sim \frac{1}{\sqrt{\pi}} \sigma^{\frac{1}{4}} \sin(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \qquad . \qquad (67)$$

With (12), (60), (66) and (67) substituted into (42), we obtain

$$I \sim \sqrt{\frac{\pi}{k}} \ [F_1 \sqrt{\frac{-2}{q_1}} + F_2 \sqrt{\frac{2}{q_2}}]) e^{ja_o k}$$
$$\times [+\sin\alpha + j\cos\alpha]$$
$$+ j\sqrt{\frac{\pi}{k}} \ [F_1 \sqrt{\frac{-2}{q_1}} - F_2 \sqrt{\frac{2}{q_2}}]) e^{ja_o k} \qquad (68)$$
$$\times [-\cos\alpha + j\sin\alpha] \ ,$$

which upon further simplification yields (3).

APPENDIX B

Differentiation of

$$q(\theta) = -\sqrt{(R\cos\theta - x)^2 + (R\sin\theta - y)^2} - R\cos\theta \qquad (69)$$

with respect to θ yields

$$q'(\theta) = -[(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{1}{2}} \qquad (70)$$
$$\times (xR\sin\theta - yR\cos\theta) + R\sin\theta.$$

Further differentiation yields

$$q''(\theta) = [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{3}{2}}$$
$$\times (xR\sin\theta - yR\cos\theta)^2$$
$$- [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{1}{2}} \qquad (71)$$
$$\times (xR\cos\theta + yR\sin\theta) + R\cos\theta$$

and differentiation once more

$$q'''(\theta) = -\frac{\partial A}{\partial \theta} - \frac{\partial B}{\partial \theta} - R\sin\theta \qquad (72)$$

where

$$\frac{\partial A}{\partial \theta} = 3 [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{5}{2}}$$
$$\times (xR\sin\theta - yR\cos\theta)^3 \qquad (73)$$
$$- 2 [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{3}{2}}$$
$$\times (xR\sin\theta - yR\cos\theta)(xR\cos\theta + yR\sin\theta)$$

and

$$\frac{\partial B}{\partial \theta} = - [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{3}{2}}$$
$$\times (xR\sin\theta - yR\cos\theta)$$
$$\times (xR\cos\theta + yR\sin\theta) \qquad (74)$$
$$+ [(R\cos\theta - x)^2 + (R\sin\theta - y)^2]^{-\frac{1}{2}}$$
$$\times (-xR\sin\theta + yR\cos\theta) \ .$$

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