Validation of a Z-Matrix Finite Element Program for Analyzing Electrostatic Fields

P. F. Barber
Department of
Electrical Engineering
United States Military Academy
West Point, New York 10996

T. S. Lauber
Department of
Electric Power Engineering
Rensselaer Polytechnic Institute
Troy, New York 12181

E. D. Patterson
Department of
Electrical Engineering
United States Military Academy
West Point, New York 10996

INTRODUCTION

Modern electrical equipment designs demand more compact, complex geometries and require more stringent analyses of the associated three-dimensional electrostatic fields. Analytical solutions of such fields are limited to relatively simple geometries. Numerical solutions using the finite element method are limited by available computer storage and processing time. Lauber posed an economical method for solving three-dimensional finite element electrostatic fields by forming the finite element system matrix using Brown's Z-matrix techniques. (Lauber, 1982) (Brown, 1975) This method was implemented by a FORTRAN computer program. (Barber and Lauber, 1986) (Barber, 1988) The purpose of this paper is to briefly describe the present version of the program called FEWZ (Finite Elements using a Window and the Z-matrix) and to show how it was validated.

THE FEWZ PROGRAM

The FEWZ program uses constant-stress tetrahedral finite elements to solve for potentials, interelectrode capacitances, and dielectric stresses in Laplacian electrostatic fields. A constant-stress tetrahedron has an assumed linear potential variation within it. Constant-stress tetrahedral elements were used in the formulation for several reasons. These simple, practical elements are well-known and have accrued a good amount of computing experience. Typical field geometries can be readily subdivided by a computer into tetrahedra. Zienkiewicz et al have shown how the finite element relationships for a system of these elements are obtained from the minimization of an appropriate functional. (Zienkiewicz et al, 1967) A close examination of the resulting finite element system matrix equation reveals that it is mathematically analogous to the admittance matrix (Y-matrix) equation for an electrical network. The solution of either requires the inversion of a sizeable coefficient matrix. The inverse of the electrical network Y-matrix is the impedance matrix or Z-matrix. Techniques for building and solving Z-matrices have been used for many years to solve large electrical network problems and are illustrated in the Appendix. (Brown, 1975) A major advantage of the Z-matrix method is that the potentials at a subset of nodes in a large problem can be obtained using modest amounts of computer resources.

The FEWZ program is used to solve the electrostatic finite element problem in the following manner. The three-dimensional problem space is divided into two regions, one region of primary interest (referred to as the "solution window" or simply the "window") where the solution of the field is desired, and another region of secondary interest (referred to as the "external region") from which effects on the primary region are desired but where solution details are not needed. The problem is solved in two passes analyzing the region first outside and then inside the window as shown in Figure 1. In the first pass, the Z-matrix representing the effects of the external region as observed at the window boundary nodes is assembled. Z-matrix axes representing nodes not in the window and not part of an electrode may be discarded as soon as all incident finite element connections are made; thus, the size of the Z-matrix can be reduced. The Z-matrix with axes representing nodes on the window surface and other axes representing electrodes is saved for subsequent use. In the second pass, the space within the solution window is modeled in the Z-matrix. When finished, the Z-matrix contains axes representing every dielectric node in the window and every electrode in the problem. Using this matrix and the specified electrode potentials, the solution is calculated. If design changes are desired within the window, only a second pass (reusing the results of the first pass) is required to obtain the new solution, giving significant economy of computer processing time.

Each FEWZ pass requires two modular processes. A tetrahedral model of the electrostatic field geometry is generated by the field geometry (FG) module. Using this tetrahedral model, the Z-matrix building (ZM) module assembles the Z-matrix and saves or solves it. As shown in Figure 2, the FEWZ program accomplishes this using several data files with the two modules. The user must prepare the first- and second-pass input files (IIN1 & IIN2) which describe the boundaries of the problem

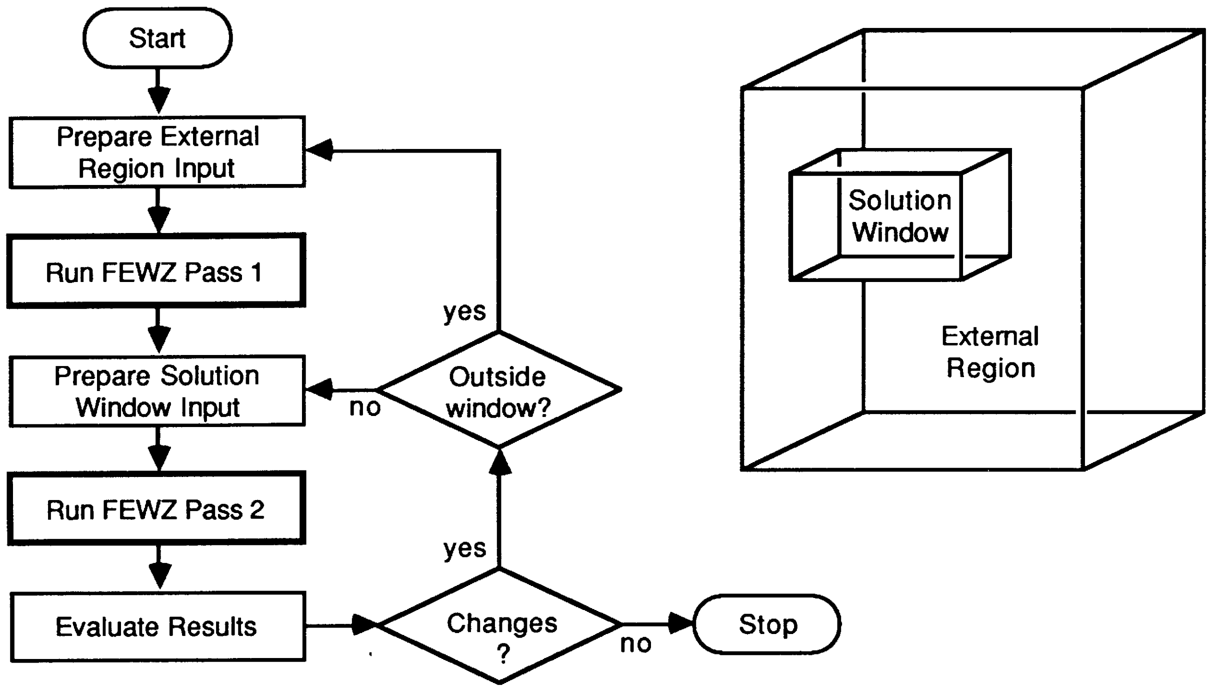


Figure 1 The Solution within a Window in Two Passes

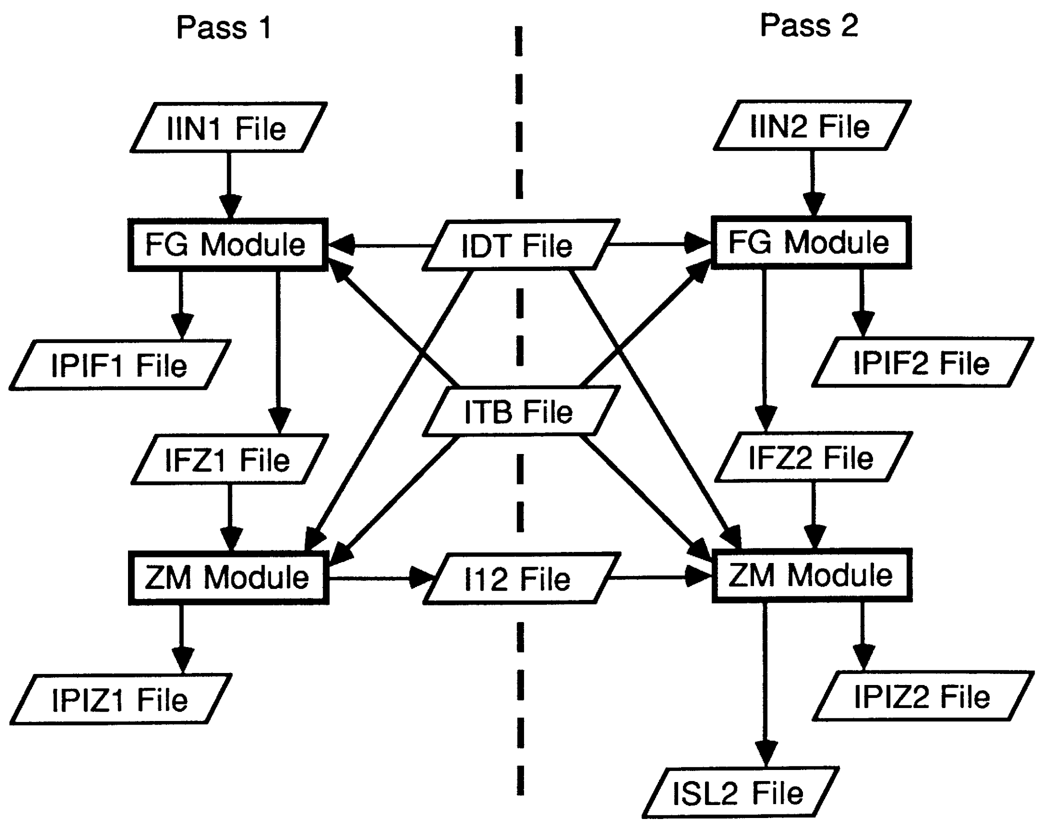


Figure 2 FEWZ Module and File Relationships

and the window, the desired regular grid, the electrode and dielectric bodies in the problem, and, in the second pass, the specified electrode potentials. Bodies are described by one or more geometric components. General polyhedra and/or convex polyhedra are used to describe dielectric bodies. Points, lines, polygons, general polyhedra and/or convex polyhedra are used to describe electrodes. The FG module develops the tetrahedral model for the user's problem using the well-known three-dimensional geometric relationships between points, lines, planes and tetrahedra. Various subprocedures contained within the FG module read the input file, set up the regular Cartesian grid, relocate a number of the regular grid points to conform to the electrode and dielectric body geometries, and automatically decompose the problem into a numerically suitable set of tetrahedra. The tetrahedra are numerically suitable if they provide an exclusive model for all space within the problem limits and if all tetrahedra have volumes greater than some minimum value (determined by the geometry of the problem). The tetrahedra are described by compact data structures which are stored in an interface file (IFZ1 or IFZ2) for subsequent use by the ZM module. The ZM module builds the Z-matrix for the space outside the window during the first pass and stores it in an interface file (I12). During the second pass, the ZM module completes the Z-matrix passed in the interface file (I12) and obtains the solution. The problem space is subdivided into columns and blocks as shown in Figure 3. Z-matrix building subprocedures contained within the ZM module evaluate the tetrahedra within each block of space, build Z-matrices representing individual blocks of space using component tetrahedron parameters, build Z-matrices representing columns of space by combining component block Z-matrices, and finally build the Z-matrix which represents the overall space by combining column Z-matrices. Columns that pass

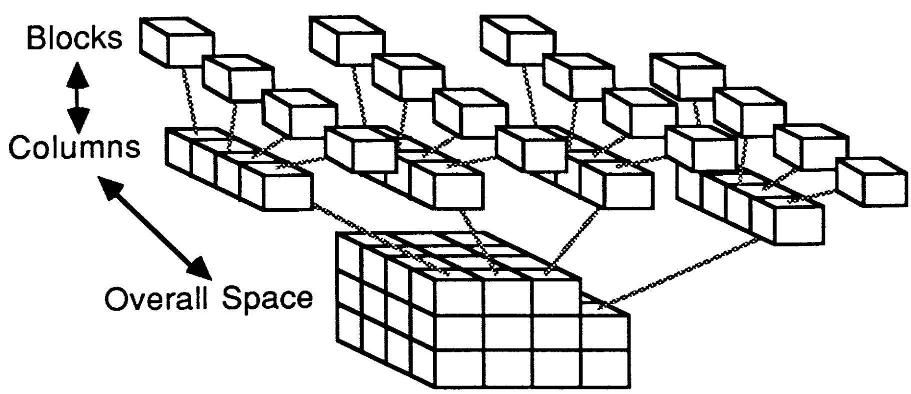


Figure 3 Space Subdivisions for Z-Matrix Assembly

through the window may be truncated into one or two smaller pieces. Other subprocedures in the ZM module maintain the data arrays, find the unknown node potentials, determine the interelectrode capacitances, and calculate the element stresses. The results are saved in a solution file (ISL2). The FEWZ solution satisfies Dirichlet boundary conditions because the potentials are explicitly specified on electrode problem boundaries. Although FEWZ does not force Neumann boundary conditions on dielectric problem boundaries, such conditions are approximately satisfied by the formulation. Intermediate processing information (progress reports, warnings, and error messages) generated by each module during each pass is saved in files (IPIF1, IPIZ1, IPIF2 and IPIZ2). Printing control codes regulate the amount of intermediate details provided; thus, process details can be traced at will. During both passes, each module refers to fixed data arrays read from the initialization files (IDT and ITB). These data arrays contain information used for tetrahedron decomposition and tetrahedron identification.

Validity, numerical accuracy, processing speed, computer storage requirements, and portability between computers were major considerations during development of the FEWZ program. The original FEWZ code was developed on an IBM3081D mainframe computer and subsequently moved with no changes to a PRIME 850 minicomputer. It was upgraded on the PRIME 850 and is now operating and being maintained in a UNIX environment on a MASSCOMP minicomputer. A UNIX shell script controls the processing by setting up the appropriate files in a problem subdirectory and executing the FEWZ modules. Documentation for the FEWZ program consists of the User's Manual, the Programmer's Manual and the FEWZ FORTRAN source code listing. The Programmer's Manual gives the theoretical basis and a detailed description of the data structures and processes for every module and subprocedure. The User's Manual gives a brief overview of the program, defines the human input requirements, describes the normal output products, and identifies abnormal conditions and potential user remedies. The FG and ZM modules contain about 1900 and 3200 lines of standard FORTRAN code respectively. About 75% of the lines are executable statements; the remainder are comment statements which help document the program. Most of the code uses single-precision floating-point arithmetic but one subroutine uses double-precision arithmetic to build the Z-matrix representing a block of space.

THE VALIDATION PROCESS

Validation occurred throughout the development of the FEWZ program. The finite element method is a well-known, valid method for solving field problems. Z-matrix techniques are also well-known and widely used to solve large electrical network problems. Careful development and some comparisons with various analytical solutions indicate that the fundamental theory of the method is valid. For example, since the tetrahedral elements were assumed to have a linear potential variation within them, the solution of fields known to have a linear potential variation should be exact except for computer roundoff error. FEWZ gives an exact solution to at least four significant figures for the electric field between two parallel-plate electrodes with a relatively coarse 4x4x8 grid.

The FEWZ program is organized and coded to enhance validation. Modules and subroutines were designed to have a definitive theoretical function or process operating on a distinct set of inputs to obtain a verifiable set of outputs such that the code could be easily documented and tested. The files generated by each module in each pass give intermediate information in a human-readable form. The interface files are thoroughly labeled to identify the printed information. Printing control codes implemented in the FEWZ program give added capability to print extra intermediate results. These codes give nine levels of increasing detail in three different categories. Comment statements identify the major decisions and steps in each process and provide links to the FEWZ documentation. Recursive checks were made between FEWZ documentation, code, and performance to verify the correct performance of each subroutine and finally each module.

In addition to the parallel-plate electrode geometry, the FEWZ program was applied to other electrostatic field geometries. One such geometry was used by Zienkiewicz et al to demonstrate the validity of their constant-stress tetrahedral finite element formulation. (Zienkiewicz et al, 1967) They considered a uniform dielectric cube with one face at a fixed potential of 1000 volts and all other faces at zero potential. Considering symmetry, only the quarter section shown in Figure 4

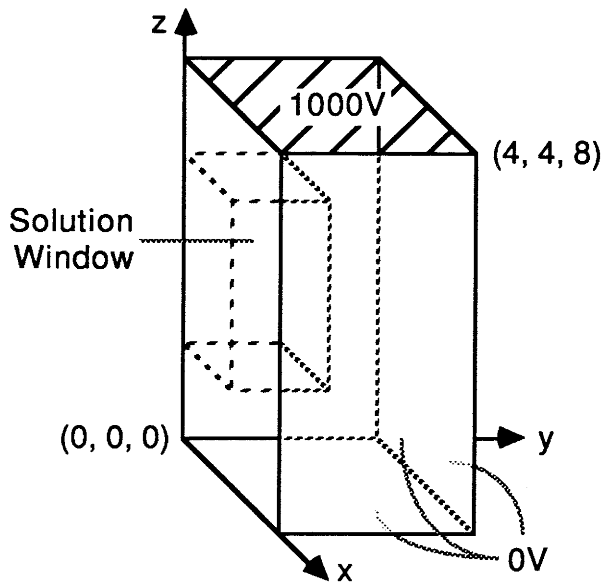


Figure 4 Dielectric Cube Geometry

needed to be analyzed. The general analytical solution for a potential in this geometry is given by

$$\phi = \frac{16\,V}{\pi^2} \sum_{\substack{m=\\1,3,5,\ldots}}^{\infty} \sum_{\substack{n=\\1,3,5,\ldots}}^{\infty} \frac{\sinh[k\,z]}{\sinh[k\,c]} \frac{\sin\left[\frac{\pi}{2}\frac{m}{a}(x+a)\right]}{m} \frac{\sin\left[\frac{\pi}{2}\frac{n}{b}(y+b)\right]}{n}$$

$$\text{where} \quad k = \frac{\pi}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

For the specific conditions shown in Figure 4 (ie, V = 1000, a = b = 4 and c = 8), the potential can be calculated using

$$\phi = \frac{16000}{\pi^2} \sum_{\substack{m=\\1,3,5,\ldots}}^{\infty} \sum_{\substack{n=\\1,3,5,\ldots}}^{\infty} \frac{\sinh[k\,z]}{\sinh[k\,8]} \frac{\sin\left[\frac{\pi}{8} m\,(x+4)\right]}{m} \frac{\sin\left[\frac{\pi}{8} n\,(y+4)\right]}{n}$$

$$\text{where} \quad k = \frac{\pi}{8}\sqrt{m^2 + n^2}$$

Zienkiewicz et al obtained four numerical solutions shown in Table I. Their finite element solutions (under the columns labeled Z FE) and finite difference solutions (under the columns labeled Z FD) may be compared with the analytical solution (under the column labeled ANSL). The numerical solutions were obtained using grid spacings of 2 (L/4) and then 1 (L/8). The FEWZ program was applied to the same geometry with grid spacings of 2 (L/4), 1 (L/8) and 0.5 (L/16). The potentials

Table I Dielectric Cube Potentials

Coord x, y, z	FEWZ L/4	FEWZ L/8	FEWZ L/16	ANSL	Z FE L/8	Z FE L/4	Z FD L/8	Z FD L/4
0,0,6	439.4	454.2	457.0	458.1	454	444	450	436
2,0,6	367.9	370.8	372.3	372.9	371	365	369	362
0,2,6	367.9	370.8	372.3	372.9	371	365	369	362
2,2,6	304.1	306.1	307.0	307.2	307	303	306	304
0,0,4	164.6	166.4	166.6	166.7	167	166	166	167
2,0,4	124.4	123.3	122.9	122.7	124	125	124	128
0,2,4	124.4	123.3	122.9	122.7	123	125	124	128
2,2,4	88.96	91.52	90.96	90.64	92	94	93	98
0,0,2	51.44	51.46	51.16	51.02	52	53	52	55
2,0,2	36.01	36.90	36.64	36.50	37	38	37	40
0,2,2	36.01	36.90	36.64	36.50	37	38	37	40
2,2,2	26.83	26.45	26.25	26.13	27	28	27	30

obtained by FEWZ are also shown in Table I (under columns labeled FEWZ) and compare quite favorably with the numerical results obtained by Zienkiewicz et al and the analytical solution. Table I also shows how the error is reduced by successively finer grid spacings. A grid spacing of 2 gave a maximum error less than 4.1% while grid spacings of 1 and 0.5 gave maximum errors less than 1.3% and 0.5% respectively. The program is not presently dimensioned to handle finer grids such as L/32 or L/64 so stability problems arising from finer grids remain undiscovered. It can be concluded from these results that the FEWZ program provided a reasonably accurate solution for potentials in the dielectric cube.

Three situations involving a coaxial electrode geometry were also analyzed using the FEWZ program. This geometry was chosen because it has a well-known analytical solution and it exercised other capabilities of the FEWZ program. The radii of the inner and outer electrodes were 8 cm and 16 cm respectively. The length of the electrodes considered was 65 cm. A quarter section of the coaxial electrode geometry was modeled as shown in Figure 5. Since the problem boundary coincides with the ends, no fringing was present at the ends. The inner electrode was represented by a convex polyhedron while the outer electrode was modeled by a general polyhedron. The 0.5 cm x 1 cm x 1 cm conductive particle which was included in one of the FEWZ computations was located with its centroid at (x, y, z) = (0.25 cm, 8.5 cm, 7.5 cm) approximately 12 cm from the inner electrode centerline; it was represented by a convex polyhedron. A 10 x 18 x 18 regular grid with the window located as shown in Figure 6 was used in the solution.

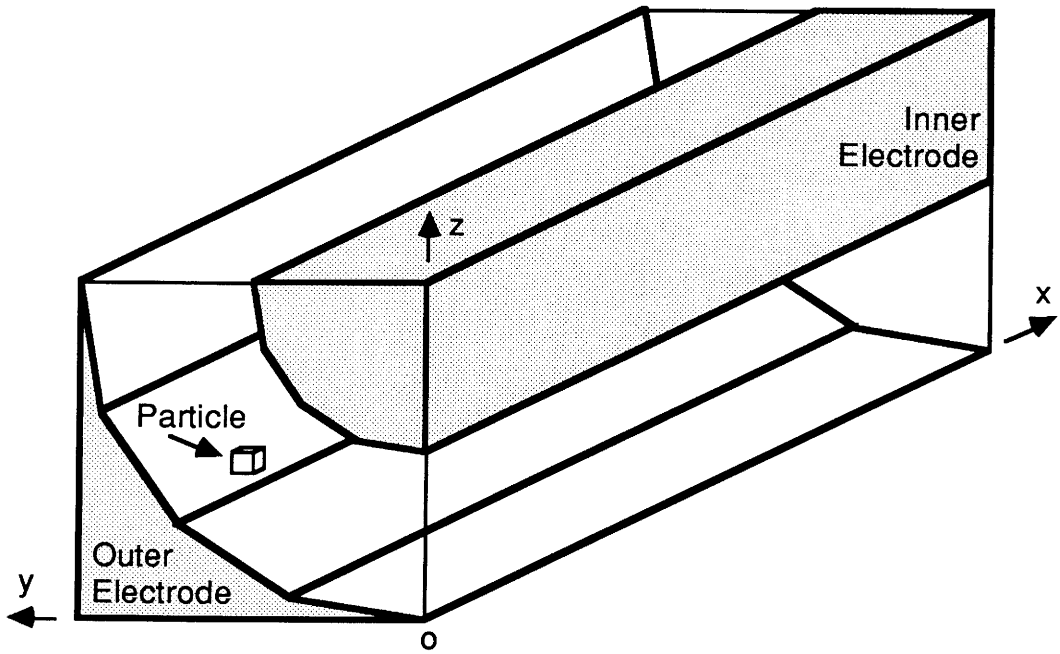


Figure 5 Coaxial Electrode Model

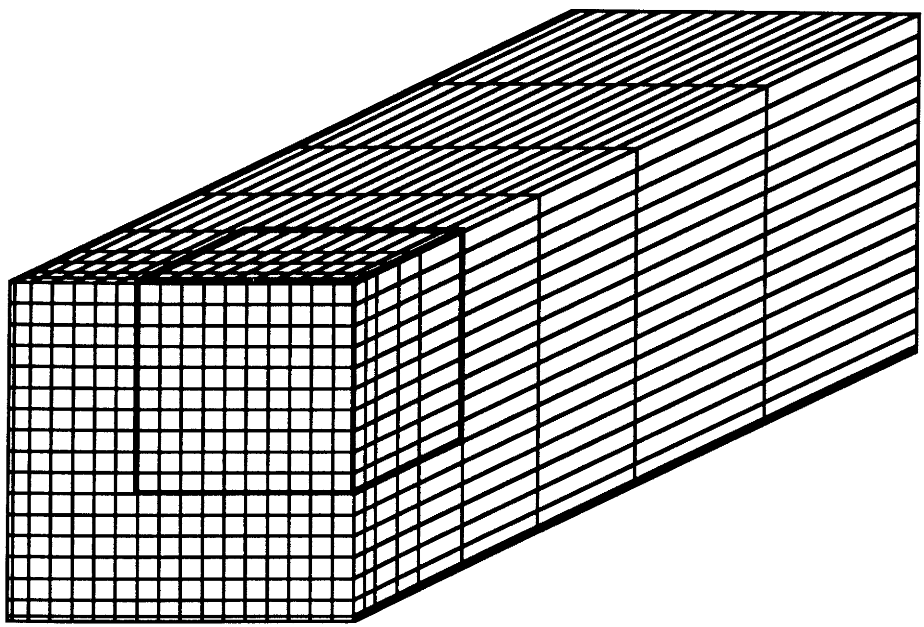


Figure 6 Regular Finite Element Grid and Solution Window

For the first test, the FEWZ program was used to solve for potentials, capacitance and stresses for the coaxial electrode geometry without the particle. The applied voltages on the inner and outer electrodes were 1 and 0 volts respectively. The potentials at a representative set of nodes are listed in Table II under the column labeled FEWZ Base Case. They may be compared with the analytical values listed under the column labeled Analytical Base Case. The potentials calculated by

Table II Selected Coaxial Electrode Potentials

Coord x	y	z	Analytical Base Case	FEWZ Base Case	FEWZ OtherV	FEWZ Particle
0	0	6	0.6781	0.6700	16.70	0.6698
0.5	0	6	0.6781	0.6700	16.70	0.6698
1	0	6	0.6781	0.6697	16.70	0.6696
2	0	6	0.6781	0.6693	16.69	0.6692
3	0	6	0.6781	0.6692	16.69	0.6691
5	0	6	0.6781	0.6691	16.69	0.6691
0	10	6	0.1781	0.1738	11.74	0.1869
0	0	7	0.8301	0.8243	18.24	0.8242
0	9	7	0.3301	0.3252	13.25	0.4080*
0	10	7	0.2501	0.2450	12.45	0.2725
0	8	8	0.5000	0.4956	14.96	0.4080*
0	10	8	0.3212	0.3160	13.16	0.3218
0	7	9	0.6926	0.6894	16.89	0.6752
0	10	9	0.3904	0.3852	13.85	0.3853
0	10	10	0.4563	0.4512	14.51	0.4500
0	10	11	0.5171	0.5118	15.12	0.5107
0	10	12	0.5710	0.5654	15.65	0.5647
0	10	13	0.6159	0.6098	16.10	0.6094
0	10	14	0.6498	0.6416	16.42	0.6413
0	10	15	0.6709	0.6618	16.62	0.6616
0	9	16	0.8301	0.8243	18.24	0.8242
0	10	16	0.6781	0.6700	16.70	0.6698

* These two potentials are located on corners of the particle.

FEWZ were within 2.5% of the analytical values. The FEWZ program calculated a capacitance factor (C/ε) of 147.91 for this geometry. The analytical value of this factor for the quarter section is 147.30. The FEWZ result is within 0.5% of the analytical result. The maximum dielectric stress of 0.2316 V/cm reported by the FEWZ program occurred in a tetrahedron adjacent to the inner electrode. The analytical solution shows that a maximum stress of 0.18033 V/cm occurs at the surface of the inner electrode. This indicates an error of about 28%. It should be noted that the dielectric stresses are found by considering partial derivatives of the potential variations. Derivatives of data having a particular error are subject to having greater error. A better indication of the maximum dielectric stress can be obtained by considering the average stress in the block containing the most stressed tetrahedron. In this case, the average block stress was 0.1882 volts/cm which is within 5% of the analytical result. These results indicate that the FEWZ program produces a valid solution.

Another test was made to verify that the finite element model gave correct results under other electrode voltage specifications. The same coaxial electrode geometry was used but the inner and outer electrodes were specified at potentials of 20 volts and 10 volts respectively (a dilation and then translation from the base case potentials in the first test). The resulting potentials are listed in Table II under the column labeled FEWZ OtherV and can be compared with the analytical and FEWZ base case solutions to verify the correct processing of other potential specifications. For example, the potential at (0,0,6) was 0.6700V from the FEWZ Base Case and (10(0.6700) + 10 =) 16.70V in this test. As expected, the capacitance and dielectric stress results were also consistent with the base case.

The third test used the same coaxial electrode geometry as in the first test but included the conductive particle without a specified voltage. Although an analytical solution is not available for this geometry, the FEWZ potentials listed in Table II under the column labeled FEWZ Particle can be compared with the results of the first test to identify expected variations and/or unexpected discrepancies. The two points on the particle shown in Table II had potentials of 0.3252 and 0.4956 volts in the first test; the potential on the floating particle was intermediate at 0.4080 volts. The interelectrode capacitance factors (C/ε) were calculated by the FEWZ program. Figure 7 shows the FEWZ values and an equivalent which can be compared with the result of the first test. The equivalent capacitance factor between the inner and outer electrodes is slightly larger when the particle is present. These results are consistent with what was expected and, thus, verify the processing of multiple electrodes and floating conductors.

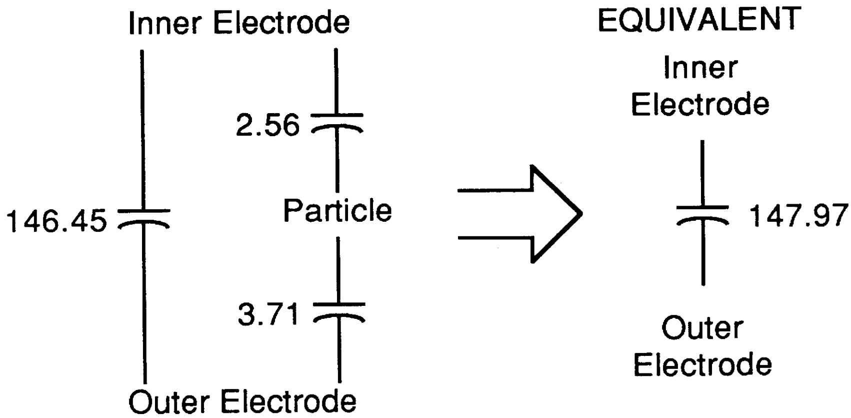


Figure 7 Interelectrode Capacitance Factors from FEWZ

RESULTS

The FEWZ program obtained good numerical solutions for these test cases with reasonable economy of computing resources. The coaxial electrode problem was solved by the FEWZ program on both a PRIME 850 minicomputer and a MASSCOMP minicomputer. Typical CPU times are shown in Table III below. The three coaxial electrode test cases posed differences within the window but used the same geometry outside of the window. Thus, to solve the three cases required execution of one Pass 1 process and three Pass 2 processes for a total of 111 CPU minutes on the PRIME 850 or 38 CPU minutes on the MASSCOMP. The Z-matrix required a maximum of 360 axes during the processing.

An earlier version of FEWZ was successfully used to analyze critical electric fields around the base insulator of a high voltage, low frequency antenna. (Barber and Lauber, 1986) (Barber, 1988) Some details of this practical problem are provided as follows. A 15 x 21 x 14 grid was used to model a tower electrode, a pedestal electrode and supporting insulators. A 6 x 6 x 4 window was selected in a region of known corona activity under the tower electrode rainshield corona ring. Potentials and stresses were obtained on an IBM3081D mainframe computer using about 30.5 minutes of CPU time. The Z-matrix required a maximum of 452 axes.

Table III Typical FEWZ CPU Times on the Coaxial Electrode Problem

	PRIME 850	MASSCOMP
Pass 1	36 min	14 min
Pass 2	25 min	8 min

CONCLUSIONS

The FEWZ program obtains a finite element solution of electrostatic fields using a solution window and Z-matrix techniques. The solution includes potentials, interelectrode capacitances and dielectric stresses. The program consists of two FORTRAN modules. The FG module analyzes the field geometry and creates the tetrahedron model. The ZM module builds the Z-matrix which is either saved for subsequent passes or solved.

The FEWZ program was validated for rectilinear and cylindrical geometries using analytical solutions and output from other programs. It gives good numerical results within reasonable computing times. The FEWZ program has been tested and operated on three different computers.

REFERENCES

Barber, P. F. Three-Dimensional Electrostatic Finite Element Solutions Using Z-Matrix Methods. PhD Dissertation, Rensselaer Polytechnic Institute, Troy, New York, May 1988.

Barber, P. F., and T. S. Lauber. "An Alternative Laplacian Electrostatic Field Finite Element Formulation." Conference Record of the 1986 IEEE International Symposium on Electrical Insulation, pp. 99-103, June 1986.

Brown, H. E. Solution of Large Networks by Matrix Methods. New York: Wiley, 1975. (2nd Ed., 1985.)

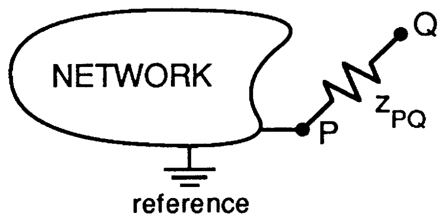
Lauber, T. S. "Calculation of Three-Dimensional Electric Fields." Final Report on DOE Contract DE-AS05-80RA50159, May 1982.

Zienkiewicz, O. C., A. K. Bahrani and P. L. Arlett. "Solution of Three-Dimensional Field Problems by the Finite Element Method." The Engineer, 27 October 1967, pp. 547-550.

APPENDIX
Z-MATRIX TECHNIQUES

Using Brown's techniques, the Z-matrix representing an electrical network may be assembled directly. The assembly starts from a reference node. A branch leading to a new node is incorporated into the Z-matrix using the radial branch addition process illustrated in Figure A.1. Other branches leading to other new nodes are processed in the same manner. Branches which connect between nodes already included in the Z-matrix are incorporated using the loop-closing branch addition process illustrated in Figure A.2. The Z-matrix at any stage represents the network

Radial Branch Addition
of branch impedance z_{PQ}
from existent node P
to new node Q



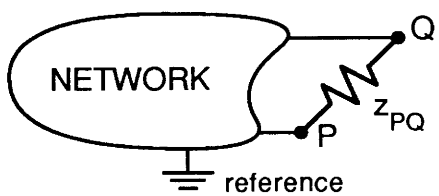
- Add an axis for node Q to Z-matrix

  where $Z(Q,Q) = Z_O(P,P) + z_{PQ}$

  and, for all $i \neq Q$, $Z(i,Q) = Z(Q,i) = Z_O(i,P)$
- When P is the reference node, let $Z_O(P,P) = Z_O(i,P) = 0$

Figure A.1

Loop-Closing Branch Addition
of branch impedance z_{PQ}
from existent node P
to existent node Q



- Add a "loop" axis L to Z-matrix

  where $Z(L,L) = Z_O(P,P) + Z_O(Q,Q) - 2\,Z_O(P,Q) + z_{PQ}$

  and, for all $i \neq L$, $Z(i,L) = Z(L,i) = Z_O(i,P) - Z_O(i,Q)$
- When P is the reference node, let $Z_O(P,P) = Z_O(i,P) = 0$
- When Q is the reference node, let $Z_O(Q,Q) = Z_O(i,Q) = 0$
- Finally, eliminate the "loop" axis using Kron reduction where, for all $i \neq L$ and $j \neq L$, $Z'(i,j) = Z(i,j) - [Z(i,L)\,Z(L,j) / Z(L,L)]$

Figure A.2

included up to that point. It is a full, diagonally symmetric matrix. These assembly techniques can be extended using submatrices to incorporate whole subnetworks at a time. A carefully crafted strategy for forming and assembling submatrices can significantly reduce both program complexity and computing time. In FEWZ, submatrices representing blocks of space are combined to assemble submatrices representing columns of space. Column submatrices are then combined to assemble the Z-matrix representing the overall problem space. After all branches have been processed, the final Z-matrix may be used in a matrix equation to solve for node potentials as illustrated in Figure A.3. Since there are few potentials specified in a typical problem, the matrix inversion and matrix multiplications required in the solution are usually modest.

Node Potential Solution from the Z-Matrix

- Partition the Z-matrix according to specified and unknown node potentials

$$\begin{bmatrix} Z_{uu} & Z_{us} \\ Z_{su} & Z_{ss} \end{bmatrix} \begin{bmatrix} 0 \\ Q \end{bmatrix} = \begin{bmatrix} \phi_u \\ \phi_s \end{bmatrix}$$

- Solve for unknown node potentials

$$\left[\phi_u\right] = \left[Z_{us}\right]\left[Z_{ss}\right]^{-1}\left[\phi_s\right]$$

Figure A.3