Predictions of Transient Eddy Current Fields Using Surface Impedances in Shell Structures

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Introduction

The surface impedance techniques have been discussed with moderate interest over the past fifteen years [1-6]. They are approached as a means to express sinusoidal fields by yielding a Neuman condition on the surface of conductors. They are generally applied when the skin depth is much less than the thickness characterizing conductor thicknesses within the problem. Some discussion has been given to their utilization near corners and slots [3]. They are not generally used as a means of approaching transient problems. The main focus of this work is in their utilization both in steady state and transient problems, primarily for shell type structures.

The first mention of the use of surface impedances with shell structures was given by E. M. Deeley [6]. Using Faraday's law, the idea is to relate the normal derivative of the magnetic scalar potential on a conducting surface to the values of the scalar potentials above and below that surface. The formulations are expressed generically using transfer relations in this paper. That is, the normal derivative of magnetic scalar potential is expressed as a combination of potentials on either side of a conducting shell structure.

The use of such expressions allows one to formulate an eddy current problem, modeling only the nonconducting regions. This produces a tremendous savings and reduction in complexity for modeling both two and three dimensional eddy current problems. This in itself produces a threefold reduction in the order of unknowns representing the problem. The formulation is applicable to finite element, finite difference, and boundary integral techniques. As shown below, it is useful to know a priori the nature of tangential field dependence on the interface of the conductor. For many problems, especially those in nondestructive evaluation and testing, this dependence is built in to the experimental set-up of the NDE procedure itself. When the characteristic problem skin depth is comparable to the thickness of the conducting structures in question, the rate of change of the magnetic field tangential to the conducting structure is negligible compared to its variation in the normal direction. In both of the above cases, the exact field is determined using the surface impedance approach without iterating at all. When no a priori knowledge of the interfacial field change is known, it is necessary to iterate the solution procedure once or twice. This is especially true in a transient problem when one is interested in the extended decay behavior of the field; in a transient field problem, the equivalent skin depth becomes infinite near the tail of the transient. Fortunately for most problems, the spatial rate of change in the field on the interface does not change drastically during the transient.

The technique fostered here is especially useful in nondestructive evaluation and testing (NDT and NDE). Consider, for instance, the problem of materials characterization for pipes. Among the parameters of interest are changes in thickness of the pipe, conductivity, and the pinpointing of flaws within the pipe. It is prohibitively expensive using volume discretization procedures to re-characterize a grid to reflect a thickness change in the material. Using the surface impedance technique, however, is relatively straightforward to make a slight change in the position of the interface; indeed, the interfacial thickness is an easily adjustable parameter built into the transfer relations. With the use of appropriate linear programming and optimization techniques, one can predict the most likely material change which best describes the search coil signature being registered.

Theoretical Approach

The surface impedance technique offers speed and flexibility at little cost in accuracy and it is applicable to both steady state and transient problems. In the transient problem, the Laplace transform of all field quantities must be taken, and the inverse transform applied at the end of the analysis. The inverse transform can be approached using Gauss-Laguerre quadrature formulas. The approach is the same for all geometries. For pedagogical reasons, we choose to outline the approach appropriate for Cartesian slabs. The problem is shown in the insert of Table 1 where we have a slab of conductivity σ and permeability μ, of thickness Δ. The field above and below the slab in the nonconducting regions can be represented as the gradient of a scalar. The question being addressed then is, "How is the normal derivative of the scalar potential at the upper and lower portions of the slab related to the value of the scalar potential itself on both surfaces?" The approach begins by writing Faraday's law and taking the curl of it.

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}(\mu)\nabla \times \vec{H} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} \tag{1}$$

Expanding this and noting that no charges exist in the problem of interest gives

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} \tag{2}$$

We next take the Laplace transform of the equation which yields

$$\nabla^2 \vec{e} + \mu\sigma\left(s\vec{e}(s) - \vec{E}(t=0)\right) = 0 \tag{3}$$

(where lower case "e" is used for the transformed quantity). For the purposes of this problem, we assume that the electric field in any direction can be represented as

$$e = \hat{e}(x)e^{j(k_y y + k_z z)} \tag{4}$$

where x is the normal component and y and z are the tangential components (see insert, Table I).

Note that this implies some sort of periodicity in the y and z directions. We can, in fact, for the purposes of NDE, choose a priori what k_y and k_z are without any loss of generality [6]. Expanding, Eq. (3) becomes

$$\frac{d^2\hat{e}}{dx^2} - k_y^2\hat{e} - k_z^2\hat{e} + \mu\sigma s\hat{e}(s) = +\mu\sigma\vec{E}(t = 0) \tag{5}$$

which after substitution is

$$\frac{d^2\hat{e}}{dx^2} - \gamma^2\hat{e} = \mu\sigma\vec{E}(t = 0) \tag{6}$$

where

$$\gamma = \sqrt{-\mu\sigma s + k_y^2 + k_z^2}$$

For the purposes of this paper, we shall focus on problems where the particular solution is zero at time t = 0. Ignoring its contribution does not effect the prediction of the characteristic response of the system. As shown in the first example to follow, the total field to any excitation is easily found by convolution once the characteristic step or impulse field response is known. The general solution of Eq. (6) is written in terms of a particular solution $\vec{e}_p$ and the value of electric field of the α and β surfaces as

$$\hat{e} = \hat{e}^{\alpha} \frac{\sin\gamma x}{\sin\gamma\Delta} + \hat{e}^{\beta} \frac{\sin\gamma(\Delta - x)}{\sin\gamma\Delta} + \vec{e}_p \tag{7}$$

The particular solution can be ignored in most realistic problems. It is definitely easier to account for $\vec{e}_p$ after the homogeneous solution is in hand; we will account for it when we are predicting the total solution in the last section of this paper. We now return to Faraday's law and represent it in terms of its equivalent Laplace transform as

$$\nabla \times \vec{E} = -\frac{\partial \mu \vec{H}}{\partial t} \tag{8a}$$

$$\nabla \times \hat{e} = -\mu\left(s\hat{h}(s) - \hat{H}(o)\right) \tag{8b}$$

The tangential components of this equation are

$$\frac{\partial e_y}{\partial x} - \cancelto{0}{\frac{\partial e_x}{\partial y}} = -\mu\left(sh_z(s) - H_z(o)\right) \tag{9}$$

$$-\frac{\partial e_z}{\partial x} + \cancelto{0}{\frac{\partial e_x}{\partial z}} = -\mu\left(sh_y(s) - H_y(o)\right) \tag{10}$$

The tangential derivatives of e_x must be zero on the interface as must e_x to insure current continuity. We next substitute Eq. (9) for the expression derived in (7) and write the results at both x = 0 and x = Δ to give

$$\gamma\left\{\hat{e}_y^{\alpha}\cot(\gamma\Delta) - \frac{\hat{e}_y^{\beta}}{\sin\gamma\Delta}\right\} = -\mu\left(s\hat{h}_z^{\alpha} - \hat{H}_z^{\alpha}(o)\right) \tag{11}$$

$$\gamma\left\{\frac{\hat{e}_y^{\alpha}}{\sin\gamma\Delta} - \hat{e}_y^{\beta}\cot(\gamma\Delta)\right\} = -\mu\left(s\hat{h}_z^{\beta} - \hat{H}_z^{\beta}(o)\right) \tag{12}$$

It is convenient to represent (9) and (10) in transfer relation form as

$$\begin{vmatrix} -\cot\gamma\Delta & \frac{1}{\sin\gamma\Delta} \\ \frac{-1}{\sin\gamma\Delta} & \cot\gamma\Delta \end{vmatrix} \begin{vmatrix} \hat{e}_y^{\alpha} \\ \hat{e}_y^{\beta} \end{vmatrix} = \frac{\mu}{\gamma} \begin{vmatrix} s\hat{h}_z^{\alpha} - \hat{H}_z^{\alpha}(o) \\ s\hat{h}_z^{\beta} - \hat{H}_z^{\beta}(o) \end{vmatrix} \tag{13}$$

Inverting this equation yields the result

$$\begin{vmatrix} \hat{e}^{\alpha}_y \\ \hat{e}^{\beta}_y \end{vmatrix} = \frac{\mu}{\gamma} \begin{vmatrix} \cot\gamma\Delta & -\frac{1}{\sin\gamma\Delta} \\ \frac{1}{\sin\gamma\Delta} & -\cot\gamma\Delta \end{vmatrix} \begin{vmatrix} s\hat{h}^{\alpha}_z - \hat{H}^{\alpha}_z(o) \\ s\hat{h}^{\beta}_z - \hat{H}^{\beta}_z(o) \end{vmatrix} \tag{14}$$

We now return to Faraday's equation

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{15}$$

the normal component of which is

$$\frac{\partial e_z}{\partial y} - \frac{\partial e_y}{\partial z} = -\mu\left(s\hat{h}_x - \hat{H}_x(o)\right) \tag{16}$$

To utilize (16), we must repeat the above process for the z component of the electric field. Upon doing this, we find

$$\begin{vmatrix} \hat{e}^{\alpha}_z \\ \hat{e}^{\beta}_z \end{vmatrix} = -\frac{\mu}{\gamma} \begin{vmatrix} \cot\gamma\Delta & -\frac{1}{\sin\gamma\Delta} \\ \frac{1}{\sin\gamma\Delta} & -\cot\gamma\Delta \end{vmatrix} \begin{vmatrix} s\hat{h}^{\alpha}_y - \hat{H}^{\alpha}_y(o) \\ s\hat{h}^{\beta}_y - \hat{H}^{\beta}_y(o) \end{vmatrix} \tag{17}$$

With a definition of a generic impedance matrix Z as

$$\underline{\underline{Z}} = \frac{1}{\gamma} \begin{vmatrix} \cot\gamma\Delta & -\frac{1}{\sin\gamma\Delta} \\ \frac{1}{\sin\gamma\Delta} & -\cot\gamma\Delta \end{vmatrix}, \tag{18}$$

we can write Eq. (16) in transfer relation form

$$\begin{vmatrix} Z_{11z} & Z_{12z} & Z_{11y} & Z_{12y} \\ Z_{21z} & Z_{22z} & Z_{21y} & Z_{22y} \end{vmatrix} \begin{vmatrix} \frac{\partial^2\hat{\psi}^{\alpha}}{\partial y^2} + \frac{1}{s}\frac{\partial \hat{H}^{\alpha}_y}{\partial y}(t=0) \\ \frac{\partial^2\hat{\psi}^{\beta}}{\partial y^2} + \frac{1}{s}\frac{\partial \hat{H}^{\beta}_y}{\partial y}(t=0) \\ \frac{\partial^2\hat{\psi}^{\alpha}}{\partial z^2} + \frac{1}{s}\frac{\partial \hat{H}^{\alpha}_z}{\partial z}(t=0) \\ \frac{\partial^2\hat{\psi}^{\beta}}{\partial z^2} + \frac{1}{s}\frac{\partial \hat{H}^{\beta}_z}{\partial z}(t=0) \end{vmatrix} = \begin{vmatrix} \frac{\partial\hat{\psi}^{\alpha}}{\partial x} + \frac{\hat{H}^{\alpha}_x(o)}{s} \\ \frac{\partial\hat{\psi}^{\beta}}{\partial x} + \frac{\hat{H}^{\beta}_x(o)}{s} \end{vmatrix} \tag{19}$$

where ψ^{α} and ψ^{β} are the magnetic scalar potentials on the α/β surface ($H = -\nabla\psi$).

The subscripts on the impedance refer to variations of the y or z components of the electric field, respectively. For most problems, this dependence is the same for both components. Note that we now have the result desired; that is, we have related the normal component of the scalar potential to its tangential derivatives on the upper and lower surfaces. We can, in fact, control the tangential derivative behavior by the nature of the excitation we choose in many NDE problems. Thus, this relationship, summarized in Table 1, gives us a very unique and time saving linkage which allows us to model only the scalar potential region. We have commonly chosen to represent that region using the integral formulation since it is ideal for handling unbounded space. The integral technique simply states that the scalar potential at any point in a region can be related to the normal derivative and the value itself on the surface interface which bounds the volume of interest. When the field point in question falls on the interface itself, the value of the potential is modified to be one-half ($\varepsilon = 1/2$)

$$\varepsilon\psi = \int \left(\frac{\partial\psi(r_q)}{\partial n_q} G(r_p,r_q) - \psi(r_q) \frac{\partial G(r_p,r_q)}{\partial n_q}\right) dS_q \tag{20}$$

The process can be repeated for a number of geometries. In each case, the procedure is as follows:

(1) Solve Eq. (1) for the region of interest.

(2) Use the tangential component of Faraday's law to relate $\vec{E}$ and $\vec{H}$ across the interface.

(3) Use the normal component of Faraday's law to relate the scalar normal derivative to its value on both sides of the shell.

For a cylindrical geometry, i.e., a cylindrical shell of inner radius β, outer radius α, the above procedure is followed to yield the set of transfer relations summarized in Table 2. For a spherical shell with only ϕ directed currents, the same procedure yields the transfer relations of Table 3.

Confirmation of the Theory

The technique is tested in both cylindrical and spherical geometries. The hollow cylinder shown in Figure 1 is stressed by a homogeneous external field decaying with characteristic time of 40 ms. This is the so-called FELIX cylinder experiment and is discussed in [7]. We choose to treat the problem as a two dimensional one, examining the fields at the midsection plane of the cylinder. It is easily shown [8] that the scalar potential, both inside and outside the hollow cylinder, has a cosinusoidal dependence.

$$\psi = \hat{\psi}(r)\cos\theta \tag{21}$$

The defining integral equation for the problem is

$$\frac{\hat{\psi}}{2}(r_p) = \oint \left(\frac{\partial\hat{\psi}_q}{\partial n} G - \hat{\psi}_q \frac{\partial G}{\partial n}\right) ds_q \tag{22}$$

where

$$G = - \frac{\ln|\bar{r}_p - \bar{r}_q|}{2\pi} . \tag{23}$$

The term $\partial\hat{\psi}/\partial n$ in (22) is replaced everywhere by its equivalent surface impedance representation in Table 2. After making that substitution, the pair of equations, both on the inner surface and outer surface of the cylinder, are as follows

$$\left|\begin{array}{c|c} \frac{\cos(\theta_p)}{2} + \int^{r=\alpha} \frac{\partial G}{\partial n} \cos\theta_q + \int^{r=\alpha} \left(\frac{G\cos\theta_q F_z(\beta,\alpha)}{\alpha^2}\right) ds_q & \int \frac{G\cos\theta_q G_z(\alpha,\beta)}{\alpha\beta} ds_q \\ \int^{r=\beta} G\cos\theta_q \frac{G_z(\beta,\alpha)}{\alpha\beta} ds_q & \frac{\cos\theta_p}{2} + \int^{r=\beta} \frac{\partial G}{\partial n} \cos\theta_q ds_q + \int GF_z(\alpha,\beta) \cos\theta_q ds_q \end{array}\right| \left|\begin{array}{c} \hat{\psi}^{out}(r=\alpha) \\ \hat{\psi}^{in}(r=\beta) \end{array}\right|$$

$$= \left|\begin{array}{c} \int^{r=\alpha} Gds \left[\frac{F_z(\beta,\alpha)}{s\alpha} \frac{\partial H_\theta^\alpha(t=0)}{\partial\theta} + \frac{G_z(\alpha,\beta)}{s\alpha} \frac{\partial H_\theta^\beta(t=0)}{\partial\theta} - \frac{H_r^\alpha(t=0)}{s}\right] \\ \int^{r=\beta} Gds \left[\frac{G_z(\beta,\alpha)}{s\beta} \frac{\partial H_\theta^\alpha}{\partial\theta} + \frac{1}{s\beta} F_z(\alpha,\beta) \frac{\partial H_\theta^\beta(t=0)}{\partial\theta} - \frac{H_r^\beta(t=0)}{s}\right] \end{array}\right| \tag{24}$$

Observe that the impressed external field is represented as

$$H_r(t=0) = \frac{\partial H_\theta(t=0)}{\partial\theta} = J_o\cos\theta \tag{25}$$

The complete solution can be pursued by one of two routes. The first is to determine those values of the Laplace transform variable s for which the determinant of the LHS of Eq. (24) goes to zero. Then, use the residue theorem to numerically realize the inverse transform of the scalar potentials from Eq. (24). For this problem, it is a bit easier to simply seek those values of the Laplace transform variable s for which determinant goes to zero, and then to form a convolution of this step function response with the actual 40 ms decay field [8]. The total and induced predicted field following the second procedure is shown in Figure 2. This result differs from the exact response by less than 0.01%. The characteristic transient decay time for the exact two dimensional step response is 56.56, while that found from the surface impedance procedure is 56.58.

The second test problem is shown in Figure 3. Here, a hollow conducting sphere is placed in a uniform external magnetic field of 1 tesla which is turned on at time t = 0. The integral equations are again written on both the outer surface and inner surface of the sphere, substituting where appropriate for the normal derivative of the scalar potential. The result is

$$\begin{vmatrix} \frac{\cos\theta_p}{2} + \iint \frac{\partial G}{\partial n} \cos\theta_q ds_q + \frac{f_1(\alpha,\beta)}{DeT} \frac{2}{\alpha^2} \iint G\cos\theta_q ds_q & \frac{-g_1(\alpha,\beta)2}{DeT\ \alpha\beta} \iint G\cos\theta_q ds_q \\ \frac{g_1(\beta,\alpha)}{DeT} \frac{2}{\alpha\beta} \iint G\cos\theta_q ds_q & \frac{\cos\theta_p}{2} + \iint \frac{\partial G}{\partial n} \cos\theta_q ds_q - \frac{f_1(\beta,\alpha)}{\beta^2\ DeT} 2 \iint G\cos\theta_q ds_q \end{vmatrix} \begin{vmatrix} \hat{\psi}^{out}(r=b) \\ \hat{\psi}^{in}(r=a) \end{vmatrix} = 0 \quad (26)$$

where

$$G = \frac{1}{4\pi(\bar{r}_p - \bar{r}_q)}$$

The right-hand side of Eq. (26) has been set to zero since we are interested in the step function response; this is found by determining those values of the Laplace transform variable s for which the determinant of (26) is zero.

For purposes of comparison, we set up the exact field in terms of the ϕ component of the vector potential A. For this problem, it is appropriate to assign the vector potential in each of the three regions (outside, conducting, and interior region) as follows

$$A_\phi(r>b) = B_o r\sin\theta + \frac{C_1}{r^2}\sin\theta \quad (27)$$

$$A_\phi(a<r<b) = \{C_2 j_1(kr) + C_3 y_1(kr)\} \sin\theta \quad (28)$$

$$A_\phi(r<a) = C_4 r\sin\theta \quad (29)$$

The solution follows by requiring that the tangential component of H and the normal component of B be continuous across the outer and inner interfaces. The result is

$$\begin{vmatrix} 0 & j_1(ka) & -a & y_1(ka) \\ -\frac{1}{b^2} & j_1(kb) & 0 & y_1(kb) \\ 0 & kj_o(ka) - \frac{1}{a} j_1(ka) & -2 & ky_o(ka) - \frac{1}{a} y_1(ka) \\ \frac{1}{b^3} & kj_o(kb) - \frac{1}{b} j_1(kb) & 0 & ky_o(kb) - \frac{1}{b} y_1(kb) \end{vmatrix} \begin{vmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{vmatrix} = \begin{vmatrix} 0 \\ bB_o \\ 0 \\ B_o \end{vmatrix} \quad (30)$$

The characteristic decay time following the exact procedure is 19.86, whereas for the surface impedance procedure, it is 19.87. The predicted step function induced field is shown in Figure 4. These predicted field values differ from the exact values by no more than 0.01%.

Computing the Transient

In general, it is difficult to compute the exact field after calculating the general transient dependence, i.e., the eigenvalues $\lambda = -s$. In the development of the procedure, it was stated that the $\vec{E}$ field was to be ignored at $t = 0$. For most transient eddy current problems, it is the $\vec{B}$ field which remains fixed at $t = 0^+$, currents being induced instantaneously to preserve flux. By experience we have found that it is easiest to ignore <u>all</u> initial field values; both $\vec{E}$ and $\vec{H}$ at $t = 0$. The procedure is as follows:

(1) Set all initial fields to zero at $t = 0$.

(2) Compute the eigenvalues of the matrix defining the unknowns using either the integral, finite difference, or finite element approach. For the last example worked, this means finding those values of k for which the determinant of the matrix defined in (26) is zero. This is a nonlinear procedure and will result in several (actually an infinite number) eigenvalues S_1, S_2, S_3, ...

(3) Express the $\vec{H}$ or $\vec{B}$ field in terms of the interfacial values of the E field for all eigenvalues (eigenfunction expansion).

(4) Solve for the unknown interfacial E fields by writing the $\vec{H}$ or $\vec{B}$ field from Step (3) at several points in the conductor at $t = 0$ (the number of points being equal to twice the number of eigenvalues kept from Step (2)) (each eigenvalue involves one unknown for the upper surface and one for the lower).

(5) Write the $\vec{H}$ field both inside and outside the conductor from Faraday's law (an alternative is to relate interfacial values of ψ to $\vec{E}$ using the relations given in Tables 1 through 3).

Consider the example of the sphere. The $\vec{E}$ field internal to the sphere for the case of "N" eigenvalues can be written as

$$e_\phi(t) = \sin\theta \sum_{i=1}^{N} e_i^{\alpha}\left\{\frac{j_1(k_i r)y_1(k_i\beta) - j_1(k_i\beta)y_1(k_i r)}{j_1(k_i\alpha)y_1(k_i\beta) - j_1(k_i\beta)y_1(k_i\alpha)}\right\}e^{-\lambda_i t}$$

$$+ e_i^{\beta}\left\{\frac{j_1(k_i r)y_1(k_i\alpha) - j_1(k_i\alpha)y_1(k_i r)}{j_1(k_i\beta)y_1(k_i\alpha) - j_1(k_i\alpha)y_1(k_i\beta)}\right\}e^{-\lambda_i t} \quad (31)$$

The radial component of the induced H field is from Faraday's law

$$H_r = \frac{1}{\mu\lambda}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(e_\phi\sin\theta) = \frac{2\cos\theta}{r}\frac{e_\phi}{\mu\lambda} \quad (32)$$

This component of $\vec{H}$ must be $-H_o$ at $t = 0$, $\theta = 0$ for our test problem.

For the sphere, the first three eigenvalues (λ_i) were computed in Step (2) to be 19.06, 666, 2551. Keeping the first two eigenvalues only means that there are 4 unknowns in (31). We choose 4 arbitrary points internal to the shell and solve for e_1^{α}, e_1^{β}, e_2^{α}, e_2^{β} using (31) and (32). The predicted induced field was found to be off by 0.885% using only the first eigenvalue and by 0.533% using the first two eigenvalues.

Conclusion

A procedure has been outlined for solving shell type structures using a surface impedance technique. If some a priori knowledge of the tangential character of the field exists, no iteration is necessary. The technique is applicable to integral, finite element, and finite difference procedures. Extremely accurate solutions have been obtained for two test problems using matrices of only size 2 x 2. Among the more outstanding questions yet to be addressed in the procedure are the following:

(1) How can the method be generalized to account for corners of various internal angles?

(2) How can the procedure be generalized for problems where the internal electric field at time $t = 0$ is not equal to zero as assumed in this work?

(3) Can the generalized inverse transform be performed efficiently using a Gauss-Laguerre procedure?

Certainly, for the simpler transient and sinusoidal problems involving shell-like structures, the procedure seems to be both efficient and useful.

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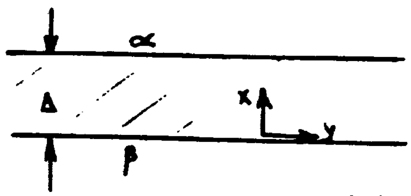
Table 1 Planar Slab



Solve $\nabla \times \nabla \times \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t}$

conductivity σ

permeability μ

Laplace transform $(\vec{E}) \rightarrow \vec{e}$

Assume $\vec{e} = \tilde{e}(x) e^{jk_y y} e^{jk_z z}$

$$\gamma = \sqrt{-\mu\sigma s + k_y^2 + k_z^2}$$

$$\begin{bmatrix} \tilde{e}^{\alpha} \\ \tilde{e}^{\beta} \end{bmatrix} = \frac{\mu}{\gamma} \begin{bmatrix} \cot \gamma\Delta & -\frac{1}{\sin \gamma\Delta} \\ \frac{1}{\sin \gamma\Delta} & -\cot \gamma\Delta \end{bmatrix} \begin{bmatrix} s\tilde{h}_z^{\alpha} & \tilde{H}_z^{\alpha}(0) \\ s\tilde{h}_z^{\beta} & -\tilde{H}_z^{\beta}(0) \end{bmatrix}$$

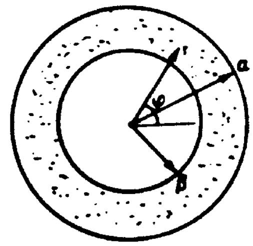
$$\begin{bmatrix} Z_{11z} & Z_{12z} & Z_{11y} & Z_{12y} \\ Z_{21z} & Z_{22z} & Z_{21y} & Z_{22y} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \tilde{\psi}^{\alpha}}{\partial y^2} + \frac{1}{s} \frac{\partial \tilde{H}_y^{\alpha}(t=0)}{\partial y} \\ \frac{\partial^2 \tilde{\psi}^{\beta}}{\partial y^2} + \frac{1}{s} \frac{\partial \tilde{H}_y^{\beta}(t=0)}{\partial y} \\ \frac{\partial^2 \tilde{\psi}^{\alpha}}{\partial z^2} + \frac{1}{s} \frac{\partial \tilde{H}_z^{\alpha}(t=0)}{\partial z} \\ \frac{\partial^2 \tilde{\psi}^{\alpha}}{\partial z^2} + \frac{1}{s} \frac{\partial \tilde{H}_z^{\beta}(t=0)}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{\psi}^{\alpha}}{\partial x} + \frac{\tilde{H}_x^{\alpha}(t=0)}{s} \\ \frac{\partial \tilde{\psi}^{\beta}}{\partial x} + \frac{\tilde{H}_x^{\beta}(t=0)}{s} \end{bmatrix}$$

where

$$\underline{\underline{Z}}_u = \frac{1}{\gamma} \begin{bmatrix} \cot \gamma\Delta & -\frac{1}{\sin \gamma\Delta} \\ \frac{1}{\sin \gamma\Delta} & -\cot \gamma\Delta \end{bmatrix},$$

the spatial wavelengths k_y, k_z are applied to the u (z and y component of the electric field); ($\underline{\underline{Z}}_y = \underline{\underline{Z}}_z$ for many problems)

Table 2 Cylindrical Shell



$$\nabla^2 \vec{e} = \mu\sigma\left(s\vec{e}(s) - \vec{E}(t=0)\right)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(re_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 e_\theta}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial e_r}{\partial\theta} + \frac{\partial^2 e_\theta}{\partial z^2} = \mu\sigma\left(se_\theta - E_\theta(0)\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial e_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 e_z}{\partial\theta^2} + \frac{\partial^2 e_z}{\partial z^2} = \mu\sigma\left(se_z - E_z(0)\right)$$

Assume $\vec{e} = \tilde{e}(r)e^{j(m\theta+kz)}$

$$\frac{\partial^2 e_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial e_\theta}{\partial r} - \left(k^2 + \frac{m^2+1}{r^2} + \mu\sigma s\right)e_\theta = -\mu\sigma E_\theta(0)$$

$$\frac{\partial^2 e_z}{\partial r^2} + \frac{1}{r}\frac{\partial e_z}{\partial r} - \left(k^2 + \frac{m^2}{r^2} + \mu\sigma s\right)e_z = -\mu\sigma E_z(0)$$

conductivity σ
permeability μ

$\gamma = \sqrt{\mu\sigma s + k^2}$

$n = \sqrt{m^2+1}$

$$\begin{bmatrix} f_n(\beta,\alpha) + \frac{1}{\alpha} & g_n(\alpha,\beta) \\ g_n(\beta,\alpha) & f_n(\alpha,\beta) + \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{e}^\alpha_\theta \\ \tilde{e}^\beta_\theta \end{bmatrix} = -\mu \begin{bmatrix} s\tilde{h}^\alpha_z - \tilde{H}_z(0) \\ s\tilde{h}^\beta_z - \tilde{H}^\beta_z(0) \end{bmatrix}$$

$$\begin{bmatrix} f_m(\beta,\alpha) & g_m(\alpha,\beta) \\ g_m(\beta,\alpha) & f_m(\alpha,\beta) \end{bmatrix} \begin{vmatrix} \tilde{e}^\alpha_z \\ \tilde{e}^\beta_z \end{vmatrix} = \mu \begin{vmatrix} s\tilde{h}^\alpha_\theta - \tilde{H}^\alpha_\theta(0) \\ s\tilde{h}^\beta_\theta - \tilde{H}^\beta_\theta(0) \end{vmatrix}$$

$$f_\ell(x,y) = \gamma\frac{\left(Y_\ell(\gamma y)J'_\ell(\gamma x) - J_\ell(\gamma y)Y'_\ell(\gamma x)\right)}{J_\ell(\gamma x)Y_\ell(\gamma y) - Y_\ell(\gamma y)J_\ell(\gamma x)} \; ;$$

$$g_\ell(x,y) = \frac{2}{\pi x\left[J_\ell(\gamma x)Y_\ell(\gamma y) - J_\ell(\gamma y)Y_\ell(\gamma x)\right]} \; .$$

$$\begin{bmatrix} \frac{F^z_m(\beta,\alpha)}{\alpha} & \frac{G^z_m(\alpha,\beta)}{\alpha} & F^\theta_n(\beta,\alpha) & G^\theta_n(\alpha,\beta) \\ \frac{G^z_m(\beta,\alpha)}{\beta} & \frac{F^z_m(\alpha,\beta)}{\beta} & G^\theta_n(\beta,\alpha) & F^\theta_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}\frac{\partial^2\tilde{\psi}^\alpha}{\partial\theta^2} + \frac{1}{s}\frac{\partial\tilde{H}^\alpha_\theta(t=0)}{\partial\theta} \\ \frac{1}{\beta}\frac{\partial^2\tilde{\psi}^\beta}{\partial\theta^2} + \frac{1}{s}\frac{\partial\tilde{H}^\beta_\theta(t=0)}{\partial\theta} \\ \frac{\partial^2\tilde{\psi}^\alpha}{\partial z^2} + \frac{1}{s}\frac{\partial\tilde{H}^\alpha_z(0)}{\partial z} \\ \frac{\partial^2\tilde{\psi}^\beta}{\partial z^2} + \frac{1}{s}\frac{\partial\tilde{H}^\beta_z(0)}{\partial z} \end{bmatrix} = -\begin{bmatrix} \frac{\partial\tilde{\psi}^\alpha}{\partial r} + \frac{\tilde{H}^\alpha_r(t=0)}{s} \\ \frac{\partial\tilde{\psi}^\beta}{\partial r} + \frac{\tilde{H}^\beta_r(t=0)}{s} \end{bmatrix}$$

where

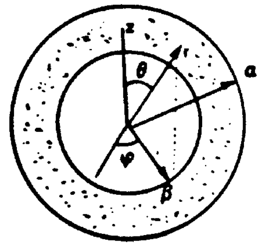
$$F_m(x,y) = \frac{1}{\gamma}\frac{\left(J'_m(\gamma x)Y_m(\gamma y) - Y'_m(\gamma x)J_m(\gamma y)\right)}{\left(J'_m(\gamma x)Y'_m(\gamma y) - J'_m(\gamma y)Y'_m(\gamma x)\right)} \; ; \; G_m(x,y) = \frac{1}{\pi\gamma(\gamma x)}\frac{2}{\left(J'_m(\gamma y)Y'_m(\gamma x) - J'_m(\gamma x)Y'_m(\gamma y)\right)}$$

$$F^\theta_n(x,y) = \frac{f_n(y,x) + \frac{1}{x}}{DeT} \; ; \; G^\theta_n(x,y) = \frac{g_n(y,x)}{DeT}$$

$$DeT = \left(f_n(\beta,\alpha) + \frac{1}{\alpha}\right)\left(f_n(\alpha,\beta) + \frac{1}{\beta}\right) - g_n(\beta,\alpha)g_n(\alpha,\beta) \; ; \; n = \sqrt{m^2+1}$$

Table 3 Spherical Shell

ϕ directed current, e_ϕ only



$$\nabla^2\vec{e} - \mu\sigma\left(s\vec{e}(s) - \vec{E}(0)\right) = 0$$

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial e}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial e}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 e}{\partial\phi^2} - \frac{e}{r^2\sin^2\theta} - \mu\sigma\left(se - E(0)\right) = 0$$

Assume $e_\phi = \tilde{e}_\phi(r)e^{jm\phi}\Phi(\theta)$

$$\Phi = P_n^{\sqrt{m^2+1}}(\cos\theta)$$

$$\begin{bmatrix} f_n(\beta,\alpha) & g_n(\alpha,\beta) \\ g_n(\beta,\alpha) & f_n(\alpha,\beta) \end{bmatrix}\begin{bmatrix} \tilde{e}^\alpha_\phi \\ \tilde{e}^\beta_\phi \end{bmatrix} = \mu\begin{bmatrix} s\tilde{h}^\alpha_\theta - \tilde{H}^\alpha_\theta(t=0) \\ s\tilde{h}^\beta_\theta - \tilde{H}^\beta_\theta(t=0) \end{bmatrix}$$

$$f_n(x,y) = k\,\frac{\{j_n'(ky)y_n(kx) - j_n(kx)y_n'(ky)\}}{\{j_n(ky)y_n(kx) - j_n(kx)y_n(ky)\}} + \frac{1}{y}\ ;$$

$$g_n(x,y) = \frac{-1}{kx^2}\,\frac{1}{\left(j_n(ky)y_n(kx) - j_n(kx)y_n(ky)\right)}$$

$$k = \sqrt{-\mu\sigma s}$$

$$\frac{1}{DeT}\begin{bmatrix} \frac{f_n(\alpha,\beta)}{\alpha} & \frac{\cot\theta f_n(\alpha,\beta)}{\alpha} & \frac{-g_n(\alpha,\beta)}{\alpha} & \frac{-\cot\theta g_n(\alpha,\beta)}{\alpha} \\ \frac{-g_n(\beta,\alpha)}{\beta} & \frac{-\cot\theta g_n(\beta,\alpha)}{\beta} & \frac{f_n(\beta,\alpha)}{\beta} & \frac{\cot\theta f_n(\beta,\alpha)}{\beta} \end{bmatrix}\begin{bmatrix} \frac{s}{\alpha}\frac{\partial^2\tilde{\psi}^\alpha}{\partial\theta^2} + \frac{\partial\tilde{H}^\alpha_\theta(t=0)}{\partial\theta} \\ \frac{s}{\alpha}\frac{\partial\tilde{\psi}^\alpha}{\partial\theta} + \tilde{H}^\alpha_\theta(t=0) \\ \frac{s}{\beta}\frac{\partial^2\tilde{\psi}^\beta}{\partial\theta^2} + \frac{\partial\tilde{H}^\beta_\theta(t=0)}{\partial\theta} \\ \frac{s}{\beta}\frac{\partial\tilde{\psi}^\beta}{\partial\theta} + \tilde{H}^\beta_\theta(t=0) \end{bmatrix} = -\begin{bmatrix} \frac{s\partial\tilde{\psi}^\alpha}{\partial r} + \tilde{H}^\alpha_r(t=0) \\ \frac{s\partial\tilde{\psi}^\beta}{\partial r} + \tilde{H}^\beta_r(t=0) \end{bmatrix}$$

where

$$DeT = f_n(\beta,\alpha)f_n(\alpha,\beta) - g_n(\beta,\alpha)g_n(\alpha,\beta)$$

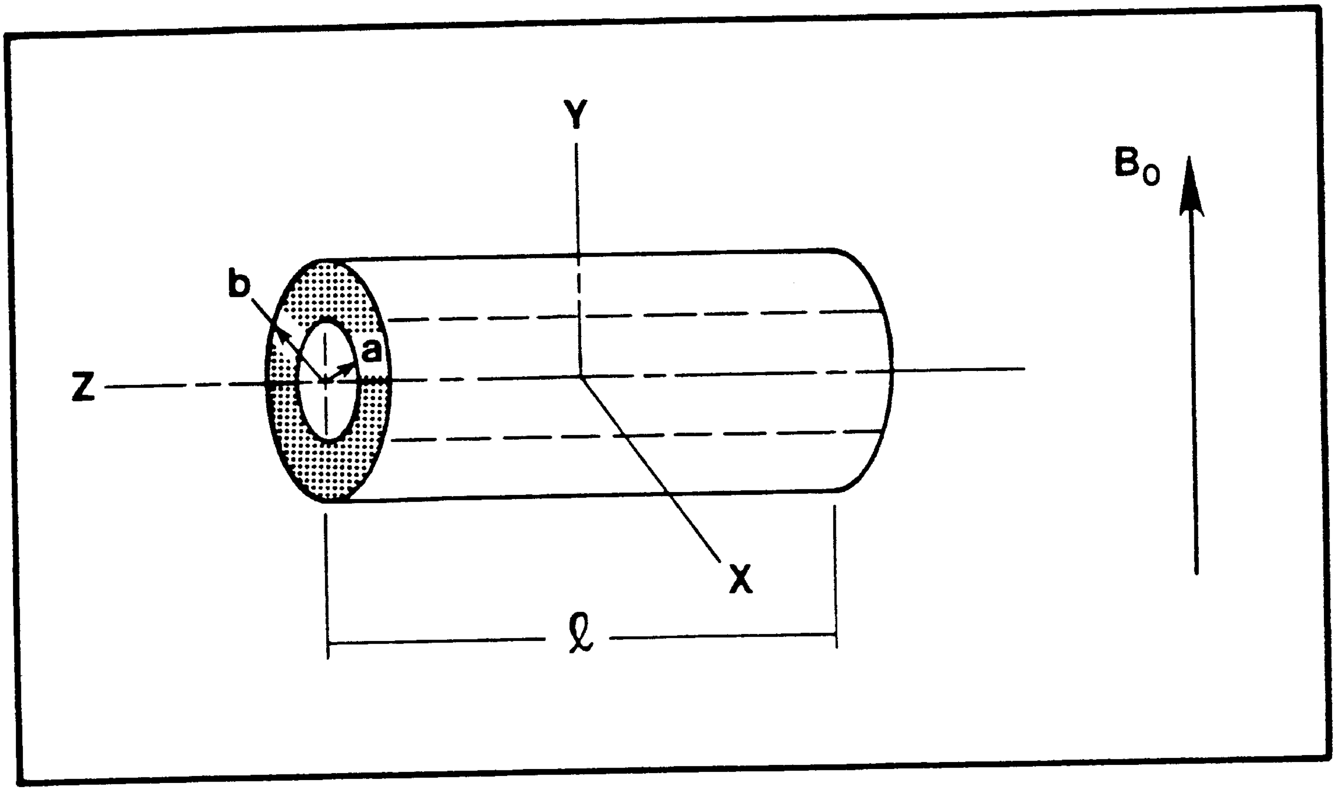


Figure 1. Felix Cylinder stressed by a vertical B field; inner radius a, outer radius b

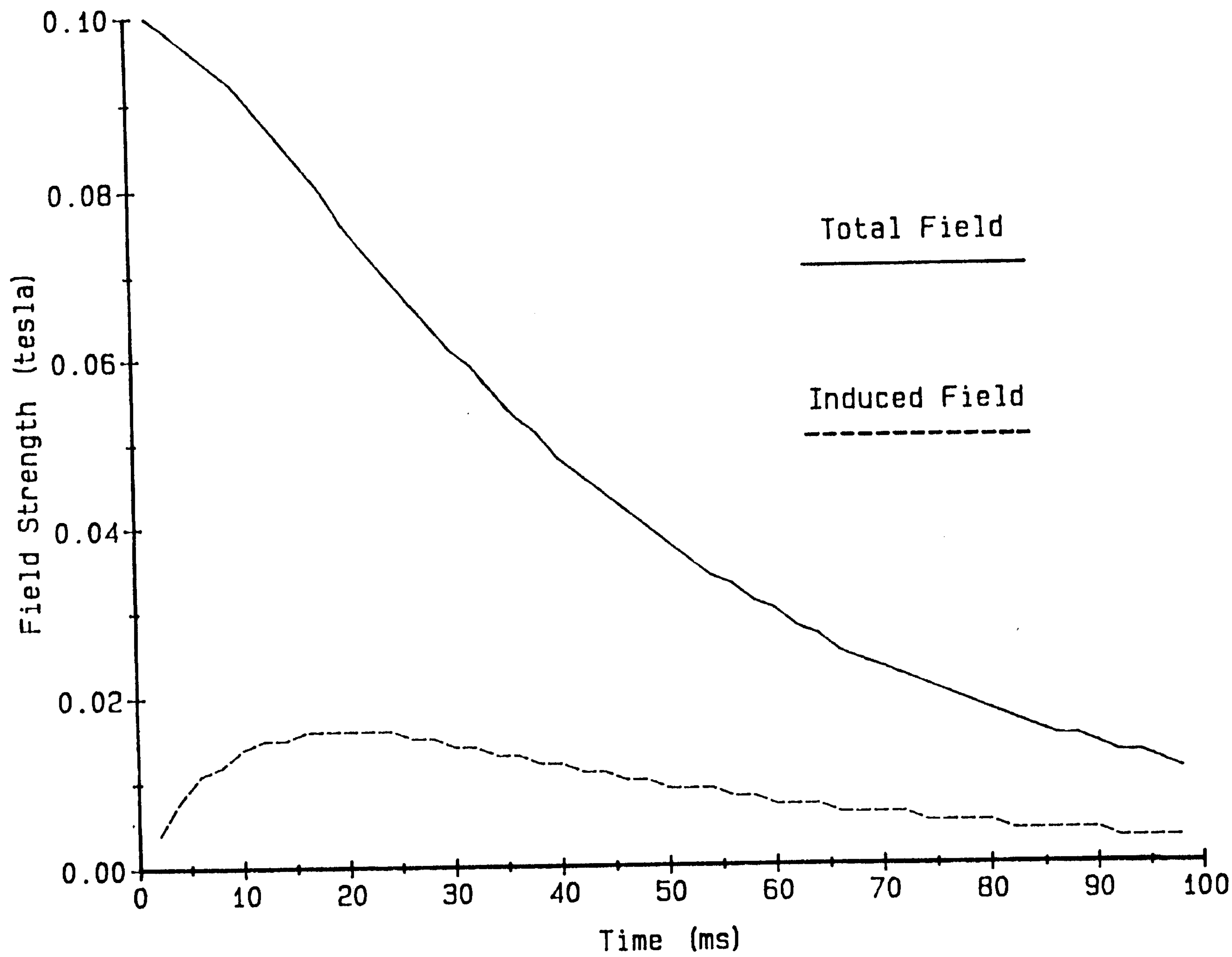


Fig. 2 Predicted Transient Field for the cylinder in Fig. 1. This field differs from the analytic solution to less than 0.01%.

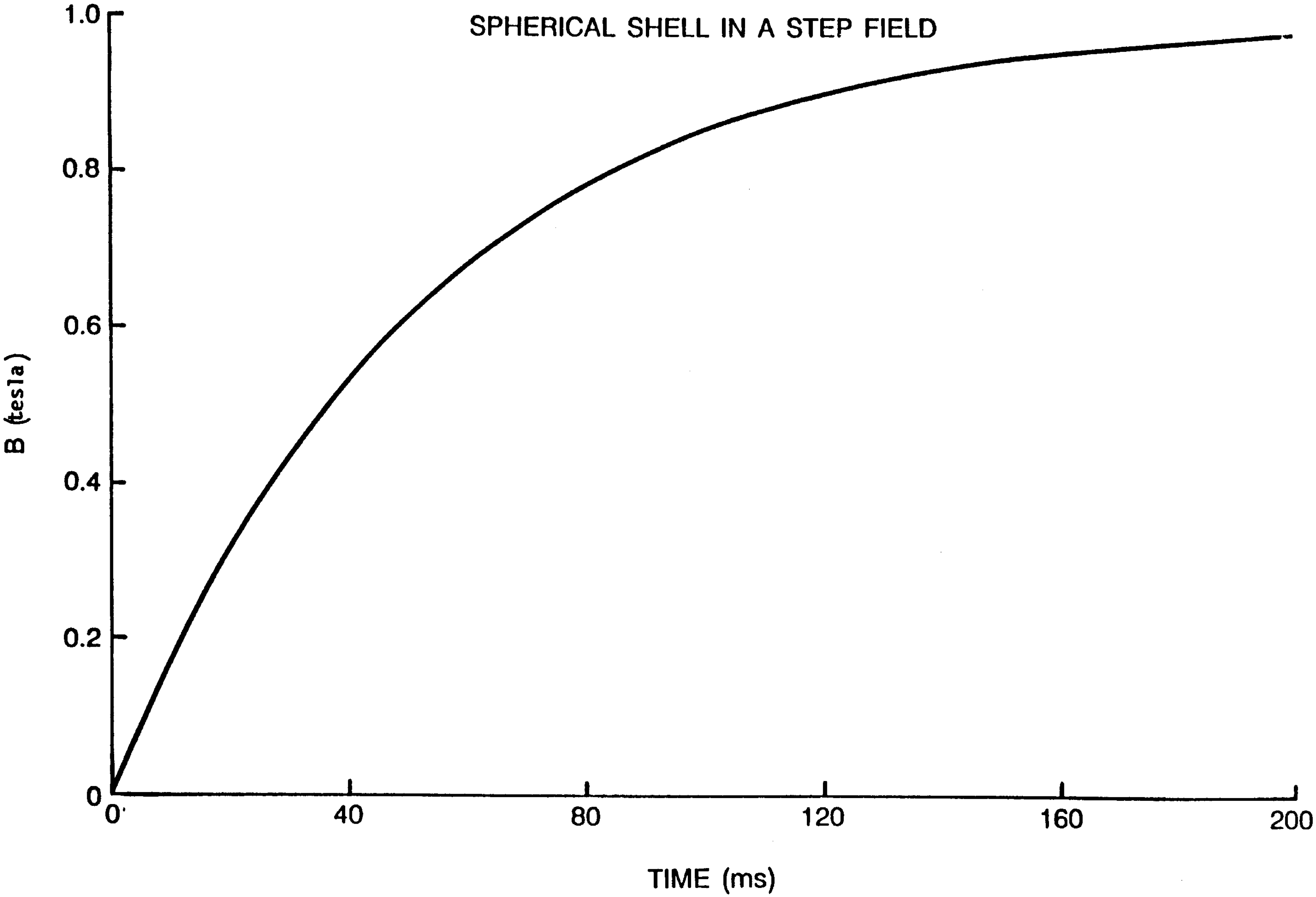


Fig. 4 Predicted Induced Field internal to the sphere when an external field is turned on at t = 0; exact and predicted field differ by less than 0.01%.