Multi-Gird Technique for Solving Two Dimensional Quasi-Static Electromagnetic Structures

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Abstract This paper provides a memory and computational time analysis for four different techniques; used to solve quasi-static electromagnetic problems with an emphasis on the multi-grid algorithm. Multi-grid procedure is investigated here as one of the solutions that provides a huge memory saving relative to a direct matrix inversion solution, and a computational time improvement relative to other iterative solvers. This is due to the fact that the multi-grid algorithm uses a directly inversed solution (exact) solution at a coarser grid as an initial guess, in addition to the flexibility of using relaxation schemes that accelerates the rate of convergence. The analysis provided here is applied on a MEMS structure and a rectangular coaxial probe.

Introduction

Most researchers and engineers are concerned with the CPU time and memory usage when dealing with numerical solvers for large problems. Gauss-Elimination is considered one of the robust direct solvers for a matrix solution. Iterative solvers were also introduced to solve such problems, where they proved their efficient performance regarding memory and time, like Jacobi and Gauss-Seidel methods. Multi-grid technique is considered as an iterative solver that can be used to solve these problems. It was widely spread starting from early 1970's by Brandt [1], where it was introduced to perform a fast numerical simulation to solve boundary value problems. The multi-grid method is an efficient technique generally used for solving smooth partial differential equations (PDEs) [2–4]. Initial interest in the multi-grid method was to overcome the slow convergence rate of the classical iterative methods by updating blocks of grid points. Due to its superior performance multi-grid technique was involved in many applications like solving the problem of huge power grids involved in VLSI designs that are required to distribute large amounts of current [5]. They were also utilized in the computation of gravitational forces together with a local refining mesh strategy [6]. In addition to the previous applications, multi-grid technique was used to solve the basic flow through convergent-divergent geometries, which was impossible to be obtained analytically [7].

This paper presents a memory and computational time analysis for four different solvers, 1) direct matrix inversion solution based on Gauss-Elimination method, 2) Gauss-Seidel iterative solver (Lexicographical ordering), 3) Gauss-Seidel iterative solution (Chequer-board ordering), and 4) multi-grid technique. These solvers are used to solve the problem of a MEMS switch and a rectangular coaxial probe, simulated using finite difference method (FDM). Based on the number of operations required by each solver, the multi-grid technique is found to require the least storage requirements. For a solution of n equations the Gauss-Elimination method requires $2n^3/3 + O(n^2)$ operations, while the Gauss-Seidel requires $5n$ operations, and the multi-grid

technique comes in the lead with optimal $O(n)$ computational cost. It is clear that the multi-grid technique provides less memory storage relative to the other solvers. The time analysis of the proposed solvers will be illustrated in the sequel.

Multi-Grid Method

Starting from Poisson's equation for a 2-D problem in Cartesian coordinates system, such that

$$\frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} = -\frac{\rho}{\varepsilon} = -f(x,y) , \tag{1}$$

where the unknown potential $V(x,y)$ is determined due to the given source term $f(x,y)$ in a closed region. Applying the central finite difference approximation to equation (1) at any interior point, assuming a homogenous Dirichlet boundary conditions $V = 0$ on the boundaries and a non-uniform descritization, results in

$$V_{(i,j)} = A'_{(i,j)} V_{(i+1,j)} + B'_{(i,j)} V_{(i-1,j)} + D'_{(i,j)} V_{(i,j+1)} + E'_{(i,j)} V_{(i,j-1)} + f_{(i,j)} , \tag{2}$$

where (i, j) denotes the coordinates of a grid point in the 2-D domain, and the source term f could be in a medium where there is no free charge; i.e: $\rho = 0$ or a medium with a constant potential V_0. Based on a non-uniform descritization, which is used to analyze the provided structures to assure accurate numerical simulation, coefficients A', B', D', and E' are defined as

$$A_{(i,j)} = \left[\varepsilon_{(i,j-1)} \frac{(y_{(i,j)} - y_{(i,j-1)})}{(x_{(i+1,j)} - x_{(i,j)})} + \varepsilon_{(i,j)} \frac{(y_{(i,j+1)} - y_{(i,j)})}{(x_{(i+1,j)} - x_{(i,j)})} \right] \Big/ G ,$$

$$B_{(i,j)} = \left[\varepsilon_{(i-1,j)} \frac{(x_{(i,j)} - x_{(i-1,j)})}{(y_{(i,j+1)} - y_{(i,j)})} + \varepsilon_{(i,j)} \frac{(x_{(i+1,j)} - x_{(i,j)})}{(y_{(i,j+1)} - y_{(i,j)})} \right] \Big/ G ,$$

$$D_{(i,j)} = -\left[\varepsilon_{(i-1,j)} \frac{(y_{(i,j+1)} - y_{(i,j)})}{(x_{(i-1,j)} - x_{(i,j)})} + \varepsilon_{(i-1,j-1)} \frac{(y_{(i,j)} - y_{(i,j-1)})}{(x_{(i-1,j)} - x_{(i,j)})} \right] \Big/ G ,$$

$$E_{(i,j)} = -\left[\varepsilon_{(i-1,j-1)} \frac{(x_{(i,j)} - x_{(i-1,j)})}{(y_{(i,j-1)} - y_{(i,j)})} + \varepsilon_{(i,j-1)} \frac{(x_{(i+1,j)} - x_{(i,j)})}{(y_{(i,j-1)} - y_{(i,j)})} \right] \Big/ G ,$$

where G takes the form

$$G_{(i,j)} = \left[A_{(i,j)} + B_{(i,j)} + D_{(i,j)} + E_{(i,j)} \right] .$$

The number of different grid density used is known as the number of multi-grid levels L. The total number of points in both the x and y directions at each level L is then taken to be $N = 2^L + 1$ for a square domain. At each level, different descritization is used which is related to the descritization at the preceding finer grid level by $\Delta_{\text{finer}}/2$, for uniform meshing.

In the constructed multi-grid algorithm, the Gauss-Seidel solution with chequer-board ordering is used here as the relaxation or the smoothing scheme. The multi-grid algorithm can thus be clearly described by a block diagram as shown in Fig. 1.

START
K = L

r : Restrict the function to the coarser grid.
P : Prolongate the function to the finer grid.

L = 1 → $V_L = S^{-1}f$

$R_{L\to1} = r(f)$

V_1
(Initial guess): Direct inversion using R_1

K = L → V_L
(Desired Solution)

Nlevel = K
$V_K = P(V_{K-1})$

$V_K = S^{-1}f$

K = L

Pre-smoothing – level **K**
V_K (Gauss-Seidel)

K = K +1
$V_K = V_K + P(V_{K-1})$

Residual = $\nabla^2 V_K + f_K$
r(*Residual*)$_{K-1}$
K = K–1

K = 1

Post-smoothing – level **K**
(Gauss-Seidel)

K = Nlevel
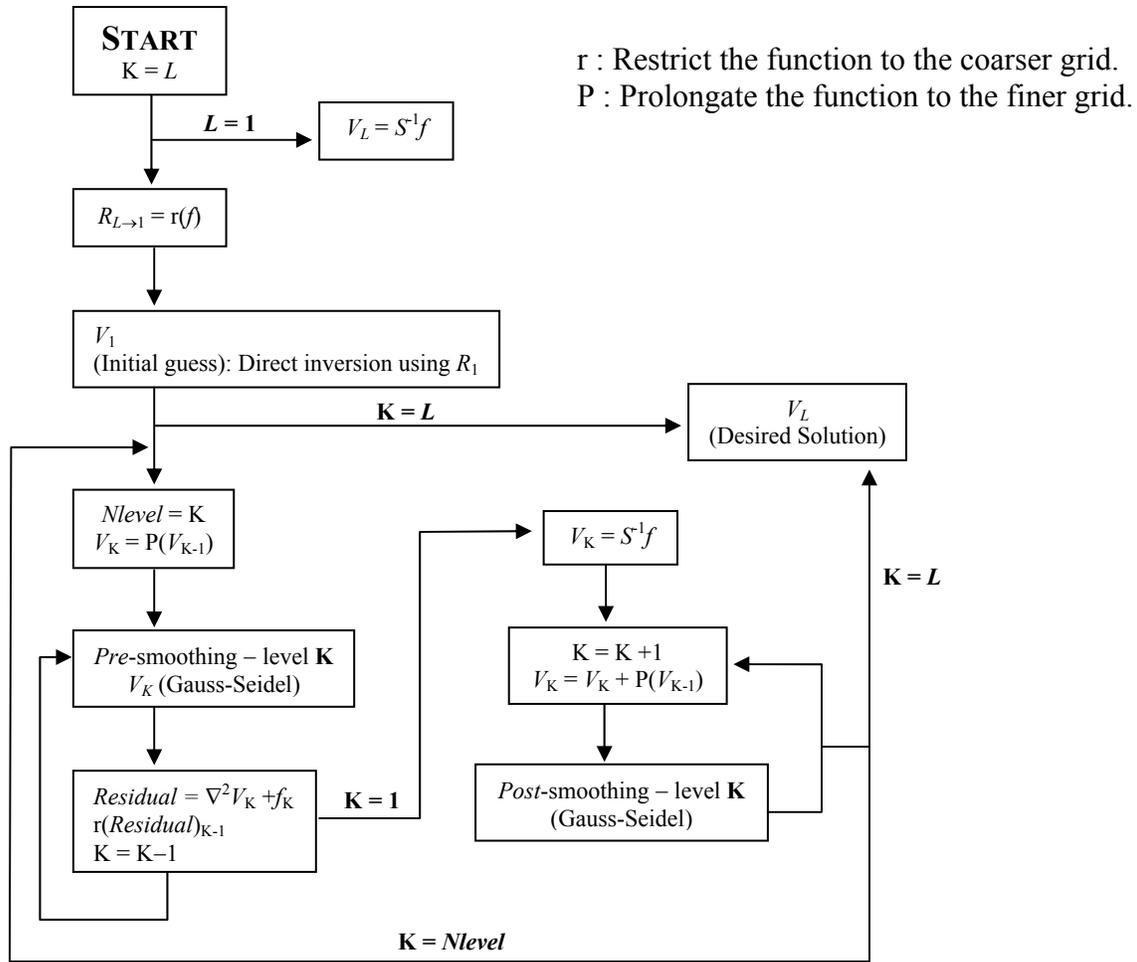
Fig. 1. Block diagram of *i* step of the multi-grid iteration.

The multi-grid technique thus starts with a fine grid, applying few pre-smoothing steps using a local algorithm like Gauss-Seidel. Computes the residual, restricts the residual to the coarser grid, improves the coarser grid correction recursively, and then prolongates the correction to the finer grid. Perform a few post-smoothing Gauss-Seidel iterations and returns the potential to the next finer grid.

In the provided structures 4 grid levels are used to extract the potential solution using the multi-grid solution.

Problem Description

a) MEMS structure

RF MEMS switches are constructed using thin metal membrane, which can be electrostatically actuated using dc-bias voltage. The presented switch is electrostatically actuated and is supported by double beams. A doubly supported or fixed-fixed beam RF MEMS switch usually consists of two parallel plates. One plate is fixed on the substrate, lower electrode, and the other is a movable membrane and is formed by a thin film metal that has good mechanical

properties like Au or Cu prepared by electroplating process as described in [8]. A schematic diagram of a fixed-fixed beam shunt-capacitive RF MEMS switch is shown in Fig. 2. For the shunt-capacitive RF MEMS switch given in [9], where L (bridge length) = 300 μm, t (membrane thickness) = 2 μm, g_o (initial gap height) = 1.5 μm, W (lower electrode width) = 100 μm, t_m (lower electrode thickness) = 0.8 μm, t_{ox} (oxide layer thickness) = 0.4 μm, t_d (dielectric layer thickness) = 0.15 μm, and silicon-nitride Si_3N_4 and silicon-oxide SiO_2 dielectric layers having relative permittivity of 7.6 and 3.9, respectively, the potential distribution after solving the static problem in the computation region, based on a direct matrix inversion, is shown in Fig. 3.

Table 1, and Table 2 shows the computational time processed by each solution for a 5 % and 1 % maximum error, respectively, for a domain of 65×129 grid points. The computed error is relative to that of the solution generated using Gauss-Elimination method. It can be clearly seen from Table 1 that the direct matrix inversion based on Gauss-Elimination provides less computational time relative to the other iterative solvers, on the other hand it is the most memory-consuming algorithm. The multi-grid technique proofs its efficiency regarding the time saving; as it is tremendously faster than the Gauss-Seidel iterative solver with lexicographical ordering and more than 2.5 times faster than the Gauss-Seidel iterative solver with chequer-board ordering. A sparse matrix is used for the Gauss-Elimination solution to purge the zero elements and thus provides the ability of computing the solution. If a non sparse matrix for the 65×129 domain size was used, a matrix size of 8385×8385 having both zero and non-zero elements will be required to be stored in memory; and thus the solver will go out of memory. Thus for practical structures simulated by millions of cells, and even by using the functionality of the sparse matrix, the direct inversion solution will be impossible to be computed; making the multi-grid algorithm the only fastest, with respect to the four provided solvers, and accurate way to provide a solution for the problem.
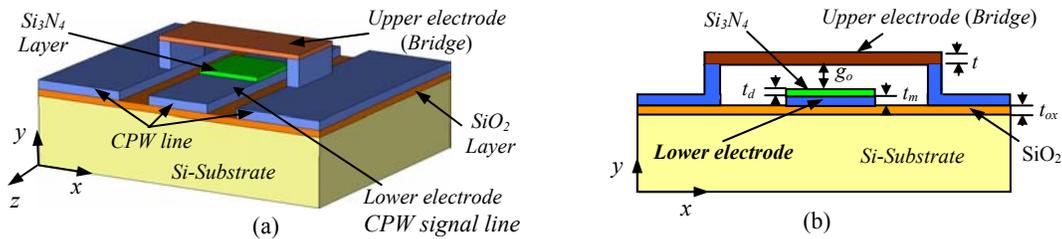
Fig. 2. Schematic diagram of fixed-fixed beam RF MEMS switch, (a) 3-D structure, (b) 2-D structure, x- y plane section.
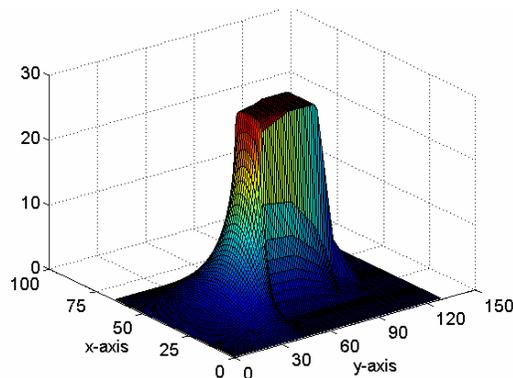
Fig. 3. Potential distribution in the computational domain with dc bias voltage of 30 volts.

Table 1. Solution time for four solvers for the problem described in Fig. 2.

	Time (secs)	Max. Percentage error (%)
Matrix Inversion (Reference Solution)	0.6719	
Gauss-Seidel (Lexicographical ordering)	109.5469	5
Gauss-Seidel (Chequer-board ordering)	54.6875	5
Multi-Grid	22.679	5

Table 2. Solution time for four solvers for the problem described in Fig. 2.

	Time (secs)	Max. Percentage error (%)
Matrix Inversion (Reference Solution)	0.6719	
Gauss-Seidel (Lexicographical ordering)	165.1875	1
Gauss-Seidel (Chequer-board ordering)	74.0469	1
Multi-Grid	41.9375	1

b) Rectangular coaxial probe

Figure 4 presents the geometry description of a rectangular coaxial probe of width and length equals to 0.5 mm. A constant potential of 100 volts is assigned to the conductor probe, where the potential distribution in the computational domain, generated from the Gauss-Elimination solution, is shown in Fig. 5. Table 3 presents a computational time comparison between the four solvers, through which one can notice the outstanding performance of the multi-grid algorithm over the other proposed solvers for a maximum percentage error of 2 %. The explanation for this performance could be because of simulating a conductor structure, where the coefficients at the conductor are forced to be zero in the multi-grid solution and thus accelerating the convergence rate.
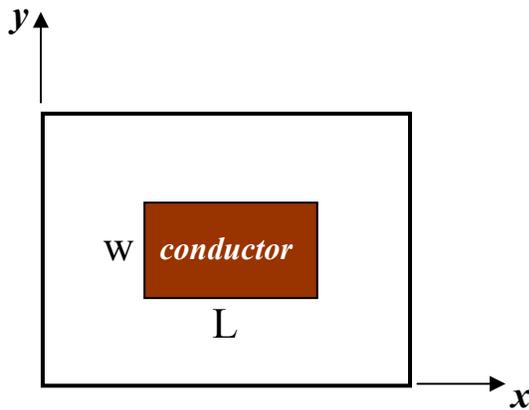
Fig. 4. Rectangular coaxial probe simulated in a domain size of 129×129.
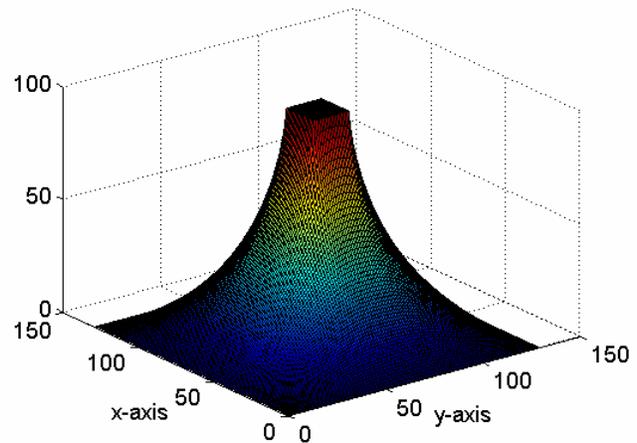
Fig. 5. Potential distribution in the computational domain with a voltage of 100 volts.

Table 3. Solution time for four solvers, for the problem described in Fig. 4.

	Time (secs)	Max. Percentage error (%)
Matrix Inversion (Reference Solution)	2.3281	
Gauss-Seidel (Lexicographical ordering)	28.875	2
Gauss-Seidel (Chequer-board ordering)	21.5469	2
Multi-Grid	0.7134	2

All calculations were performed with Matlab version 7, Release 14 on a 3.2 GHz P4 personal computer with a 2 GB RAM.

Conclusions

This paper presents an analysis, regarding the time and memory consumption, of four solvers applied to two quasi-static electromagnetic problems. It is found that the multi-grid technique as an iterative solver is superior regarding the time saving over Gauss-Seidel iterative solvers of different ordering. In addition to the time saving relative to the presented iterative solvers, the multi-grid technique also provides a huge memory saving when compared to the direct matrix inversion even when a sparse matrix is being used. For large electromagnetic quasi-static problems, the direct inversion is expected to fail because of the memory shortage even when using a sparse matrix, which allows the multi-grid solution to take the lead in solving such application.

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