a beam with Gaussian amplitude distribution at the transmitting aperture $z = 0$, with a waist radius w_o and a phase front with radius of curvature R_o, an expression for MCF within the second order Rytov approximation was derived and evaluated for a beam wave propagating through a turbulent atmosphere [1-3]:

$$\Gamma_2(\boldsymbol{\rho}_c, \boldsymbol{\rho}_d, z) = \frac{w_o^2}{w^2} exp(g_1) exp(g_2),$$

$$g_1 = -\frac{k}{2}\left(\frac{w_o^2}{w^2}\right)(g_3 - j2g_4),$$

$$g_2 = -4.352kC_n^2 \int_0^{L_z} \left(\gamma_I(z)\frac{L_z - z}{k}\right)^{5/6} {}_1\mathrm{F}_1\left(-\frac{5}{6}, 1; g_5\right) dz,$$

$$g_5 = -\frac{k|\gamma_R \boldsymbol{\rho}_d - j2\gamma_I \boldsymbol{\rho}_c|^2}{4\gamma_I(z)(L_z - z)},$$

$$g_3 = 2\alpha_1\left(\rho_c^2 + \frac{\rho_d^2}{4}\right),$$

$$g_4 = [\alpha_2 - (\alpha_1^2 + \alpha_2^2)L_z](\boldsymbol{\rho}_c \cdot \boldsymbol{\rho}_d),$$

$$w^2 = w_o^2\left[(1 - \alpha_2 L_z)^2 + \alpha_1^2 {L_z}^2\right],$$

$$\boldsymbol{\rho}_c = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2},$$

$$\boldsymbol{\rho}_d = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1,$$

$$\alpha = \alpha_1 + j\alpha_2 = \frac{2}{kw_o^2} + j\frac{1}{R_o},$$

$$\gamma(z, L_z) = \frac{1 + j\alpha z}{1 + j\alpha L_z} = \gamma_R - j\gamma_I, \qquad (3)$$

where ${}_1F_1$ is the confluent hypergeometric fuction, C_n^2 is the refractive index structure constant, L_z is the distance from the output aperture. Expanding the unknown MCF in Eq. (3) into its power series about an arbitrary structure constant C_{no}^2, as:

$$\Gamma_2 = \sum_{i=0}^{\infty} a_i (C_n^2 - C_{no}^2)^i, \qquad (4)$$

where

$$a_i = \frac{1}{i!}\frac{\partial \Gamma_2}{\partial C_n^2}\bigg|_{C_n^2 = C_{no}^2}$$

$$= \frac{1}{i!}\frac{w_o^2}{w^2}\left(\frac{1}{C_n^2} g_2\right)^i exp(g_1) exp(g_2)\bigg|_{C_n^2 = C_{no}^2}. \qquad (5)$$

$[L/M]$ Pade' approximants are obtained by truncating the power series at N, then matching to a rational:

$$\sum_{i=0}^{N} a_i (C_n^2 - C_{no}^2)^i \approx \frac{\sum_{l=0}^{L} p_l (C_n^2 - C_{no}^2)^l}{1 + \sum_{m=1}^{M} q_m (C_n^2 - C_{no}^2)^m}. \qquad (6)$$

$N + 1$ equations reached from expansion of Eq. (6). The $q's$ are attained from the last M of these equations:

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} = -\begin{bmatrix} a_L & a_{L-1} & \cdots & a_{L-M+1} \\ a_{L+1} & a_L & \cdots & a_{L-M} \\ \vdots & \vdots & & \vdots \\ a_{L+M-1} & a_{L+M-2} & \cdots & a_L \end{bmatrix}^{-1} \begin{bmatrix} a_{L+1} \\ a_{L+2} \\ \vdots \\ a_{L+M} \end{bmatrix}, \qquad (7)$$

whereas the $p's$ are found from the rest of equations:

$$\begin{bmatrix} a_o & 0 & \cdots & 0 \\ a_1 & a_o & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_L & a_{L-1} & \cdots & a_o \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_L \end{bmatrix} = \begin{bmatrix} p_o \\ p_1 \\ \vdots \\ p_L \end{bmatrix}. \qquad (8)$$

III. NUMERICAL APPLICATIONS AND VALIDATION

With the derivation of asymptotic expansion coefficients accomplished, an investigation must now be made with regard to their applicability and validity. To demonstrate the efficiency of the technique, three simulations will be carried out on a $1.6\, GHz$ personal computer. In all simulations, 41 structure constant values and 9 transverse space points are implemented for direct numerical solution. As a first check, power series expansion around $C_{n0}^2 = 10^{-16}$ with $[2/5]$ Pade' approximants used, and both expansions compared with direct numerical solution as shown in Fig. 1. The propagation parameters given by $w_0 = 0.05\, m$, $L_z = 2.5\, km$, $\lambda = 630\, nm$. It requires 145.9531s to obtain the solution with direct solution, though, it only takes 4.3281s for single point expansion. It may be inferred that the power series approximates well for small values of C_n^2, but drops sharply for larger values, while Pade' expansions shows very good approximation even deep in lager C_n^2 values region.

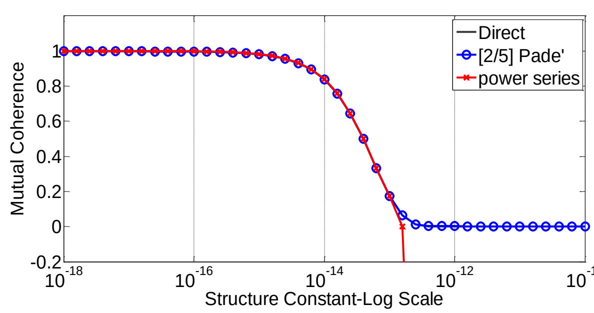

Fig. 1. Normalized mutual coherence function $\Gamma_2(0,0,L_z)$ versus C_n^2.

In Fig. 2, simulation parameters are $C_{n0}^2 = 10^{-16}$ $w_0 = 0.005\, m$, $L_z = 2.5\, km$, $\lambda = 630\, nm$. To study the effect of numerator and denominator degrees on quality of Pade' expansions, three different values are taken, namely, $[L/M] = [3/3]$, $[L/M] = [2/4]$ and $[L/M] = [4/2]$. Simulations show that when the denominator degree is larger than numerator degree, Pade' expansions approximated the solution very well. In the contrary, other orders did not for larger values of C_n^2, the approximation skyrockets when numerator degree is larger, and drops steeply when numerator and denominator degrees are equal. The difference in time between the three cases is a fraction of a second and can be neglected. However, it should be noted that as the number of power series coefficients is increased to larger integers, their values become very prohibitive and Pade' matrix becomes close to singular. This is expected since their values increase as powers of wave number.

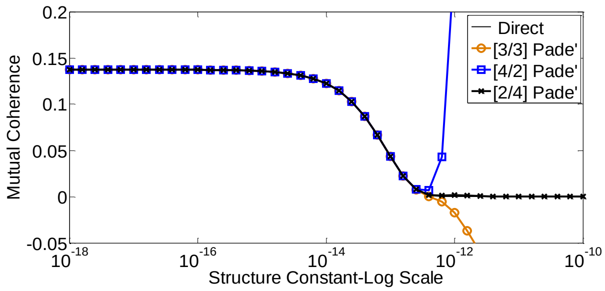

Fig. 2. Normalized mutual coherence function $\Gamma_2(\rho_c = 0.1,0,L_z)/\Gamma_2(0,0,L_z)$ versus structure constant C_n^2 for different Pade' orders.

Turning into another case, and shifting the expansion point toward larger fluctuations, namely, $C_{n0}^2 = 10^{-14}$, as shown in Fig. 3. An obvious observation is that power series does not approximate in larger fluctuations region, and has an almost constant error in smaller fluctuations region, which is expected, since the solution has almost constant slopes for smaller fluctuations. Amazingly enough, Pade' expansions agrees very well with direct solution for both small and large fluctuations. Simulation parameters given by $w_0 = 0.05\ m$, $L_z = 2.5\ km$, $\lambda = 630\ nm$, $[L/M] = [2/5]$.

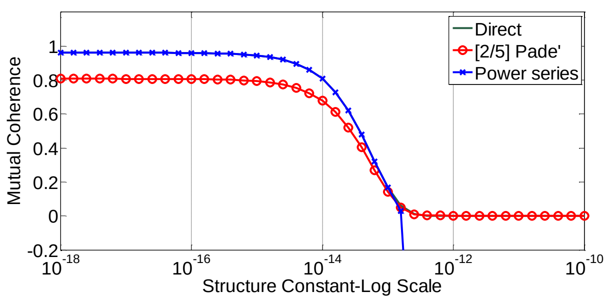

Fig. 3. Mutual coherence function $\Gamma_2(0,0,L_z)$ versus structure constant C_n^2 for a different expansion point $C_{n0}^2 = 10^{-14}$.

IV. CONCLUSION

Numerical methods offer a strong technique to compute the mutual coherence function of electromagnetic beam waves scattered by atmosphere. The efficiency is extremely increased when combined with Pade' approximation. It is shown that direct solution takes about 32 times the required time for Pade' expansion in presented simulations. In this research, it is also shown that a rational expansion with denominator degree larger than numerator gives much more accurate solution. Finally, rational asymptotes work very fine for small fluctuations region as well as for larger fluctuations.

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