Tesla Transformer and its Response with Square Wave and Sinusoidal Excitations

Eduard M. M. Costa

Colegiado de Engenharia Elétrica
Universidade Federal do Vale do São Francisco, Juazeiro, BA 48902-300, Brazil
eduard.montgomery@univasf.edu.br, edmontyc@gmail.com

Abstract **─** This paper analyses the output of Tesla transformer from input defined as square wave and sine wave and its transfer function obtained from its equivalent circuit. The transfer function presented is analyzed in terms of the time, and the output is obtained when applying an input defined as a step and a sine function. When using square wave, the high gain obtained is proved that is due to a sum of responses obtained through positive and negative step that forms the square wave. However, the gain obtained when the excitation is a sine wave is higher than square wave excitation. Thus, in this paper is shown that the Tesla transformer works with sinusoidal excitation too, and present higher gain than square wave excitation, both higher than expected in classical transformer theory (turns ratio). All analysis presented were based in simulating operation using MATLAB® comparing with experimental data.

Index Terms **─** Pulse transformers, resonance, Tesla transformers, transfer function, transformers.

I. INTRODUCTION

Tesla transformer [1,2] is a pulse transformer [3,4] that works at double resonance [5], used to generate high voltages, and so to generates high energy applied to systems as particle accelerators (particle colliders) [6], and others [7]. The theoretical foundation of Tesla transformer defines a double capacitive-inductive circuit, where the primary is excited by a square wave, which resonates with secondary, increasing the output voltage to kilovolts or megavolts.

Several works analyzing the Tesla transformer and the base of the high voltage that is an exception to the rule of the classical transformer theory (turns ratio) [8]. Example is the works of Costa [9-11], which analyzes the effects of induced electromotive force (EMF) at transformers built by planar and ring coils [12], effects of parasitic capacitances [13], going to the resonance with square wave excitation [14]. In these works, a transfer function is obtained, which is based in a specific response, whose result to square wave excitation defines a sum of responses that generates very high gains [1].

In the same way, using experimental results and other analysis of this response, an interesting result about Tesla transformer resonance is found in [15], where the excitation is a sine wave, whose response is higher than the found results of square wave excitation.

New analysis of this system are realized here, which is based in the transfer function of the Tesla transformer, whose parameters are referred to capacitive, inductive and resistive elements, instead of resonance frequencies shown at [1,10-12]. Thus, in this work are presented these results, which are simulated with MATLAB®, as well as experimental, showing that the Tesla transformer presents higher gain when excited with sinusoidal wave, than excited by square wave.

Thus, this paper is presented as follows: in Section II is presented the equivalent circuit and its transfer function formulation. In Section III is shown step response, as well as transient response to sinusoidal excitation. In Section IV are analyzed the responses at steady state for these excitation forms, showing a comparative between gains and, at Section V are presented the conclusions of this work.

II. EQUIVALENT CIRCUIT AND TRANSFER FUNCTION OF TESLA TRANSFORMER

Considering the circuit shown in Fig. 1 (a), that is the equivalent circuit of the Tesla transformer, which is shown based on impedances at Fig. 1 (b), where:

$$\begin{aligned} Z_1 &= \frac{1}{sC_1} // sL_1 = \frac{sL_1}{1+s^2L_1C_1}, \\ Z_2 &= sM, \\ Z_3 &= \frac{1}{sC_2} // sL_2 = \frac{sL_2}{1+s^2L_2C_2}, \\ Z_{R_1} &= R_1, \\ Z_{R_2} &= R_2, \end{aligned} \tag{1}$$

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then, the transfer function $G(s) = V_0(s)/V_i(s)$ is:

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{as}{fs^4 + ds^3 + es^2 + cs + b}, \tag{2}$$

where

$$\begin{aligned} a &= L_1 L_2, \\ b &= (L_1 + L_2 + M) R_1, \\ c &= (L_2 + M) L_1, \\ d &= C_2 L_1 L_2 M, \\ e &= ((C_1 + C_2) L_1 L_2 + (C_1 L_1 + C_2 L_2) M) R_1, \\ f &= C_1 C_2 L_1 L_2 M R_1, \end{aligned} \tag{3}$$

being R_1, R_2 the resistances, L_1, L_2 the inductances, C_1, C_2 the capacitances and M the mutual inductance, with index 1 and 2 being referenced to the primary and secondary, respectively.

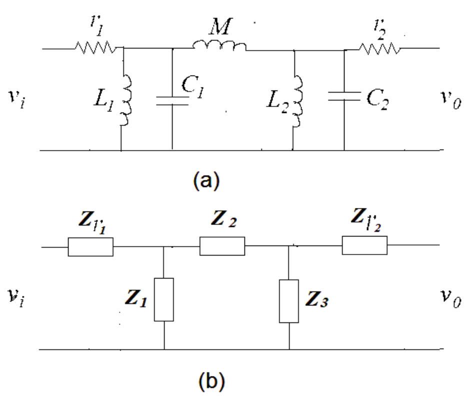


Fig. 1. (a) Equivalent circuit of Tesla transformer, and (b) equivalent circuit based on impedances.

Applying impulse to this transfer function and obtaining the inverse Laplace transform, we found:

$$f_{\uparrow}(t) = a\sum_k \left(ke^{kt} / \left(4fk^3 + 3dk^2 + 2ek + c\right)\right), \tag{4}$$

with k being the roots of the polynomial in w:

$$P(w) = fw^4 + dw^3 + ew^2 + cw + b, \tag{5}$$

with a, b, c, d, e and f defined as shown previously (Eq. (3)).

The impulse response of this function is shown in Fig. 2, where is observed that the transfer function present real and complex conjugate exponential terms (roots of the polynomial $P(w)$), what defines exponential and sinusoidal functions at response.

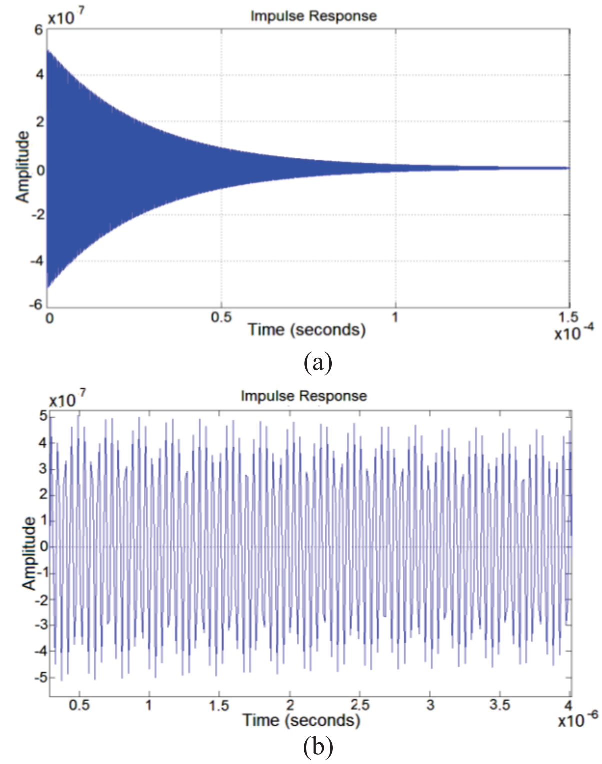


Fig. 2. (a) Impulse response of Tesla transformer, and (b) zoom showing double sinusoidal response.

III. RESPONSES OF TESLA TRANSFORMER TO SQUARE WAVE AND SINUSOIDAL EXCITATIONS

Based on transfer function $G(s)$ (Eq. (2)), applying unitary step ($V_i(s) = 1/s$), the response is:

$$V_0(s) = \frac{as}{fs^4 + ds^3 + es^2 + cs + b}\frac{1}{s}. \tag{6}$$

Making the inverse Laplace transform, then the result found is:

$$f(t) = a\sum_k \left(e^{kt} / \left(4fk^3 + 3dk^2 + 2ek + c\right)\right), \tag{7}$$

with k being the roots of the polynomial in w:

$$P(w) = fw^4 + dw^3 + ew^2 + cw + b, \tag{8}$$

which can be seen at Fig. 3, for the specific case where: $R_1 = 0.5\ \Omega$, $R_2 = 0.89\ \Omega$, $L_1 = 0.433\ \mu H$, $L_2 = 1.85\ \mu H$, $C_1 = 59.5$ pF, $C_2 = 52.5$ pF and $M = 2.801\ \mu H$, as data found in [9-11], referring to planar transformer with 10 turns at primary coil and 20 turns at secondary coil.

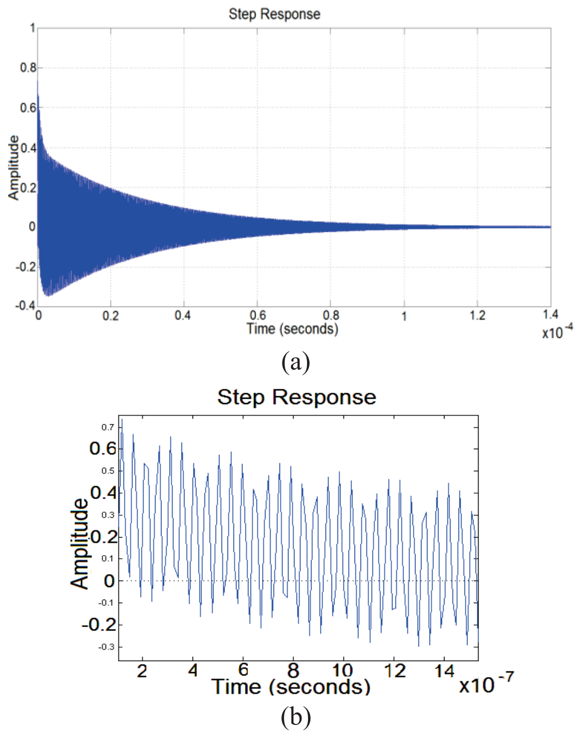


Fig. 3. (a) Step response of the Tesla transformer, and (b) zoom showing double sinusoidal oscillation.

Thus, when applied an inverted unitary step (negative sign), the result for this function is only the inverted sign of the function $f(t)$.

By other side, considering the application of sinusoidal excitation to $G(s)$, or:

$$V_0(s)=\frac{as}{fs^4+ds^3+es^2+cs+b}\frac{\omega}{s^2+\omega^2}, \tag{9}$$

and applying the inverse Laplace transform, the result found is:

$$\begin{aligned}v_0(t)=&\frac{a\omega\left(\left(b-e\omega^2+f\omega^4\right)\cos(\omega t)+\omega\left(c-d\omega^2\right)\sin(\omega t)\right)}{bx+\omega^2 y+f^2\omega^8}\\&-a\omega\Bigg[f^2\omega^4\sum_k\left(\frac{k^3e^{kt}}{q(k)}\right)-e^2\omega^2\sum_k\left(\frac{ke^{kt}}{q(k)}\right)-f^2\omega^6\sum_k\left(\frac{ke^{kt}}{q(k)}\right)\\&-d^2\omega^4\sum_k\left(\frac{ke^{kt}}{q(k)}\right)-bd\sum_k\left(\frac{k^2e^{kt}}{q(k)}\right)+cd\omega^2\sum_k\left(\frac{ke^{kt}}{q(k)}\right)\\&+bf\sum_k\left(\frac{k^3e^{kt}}{q(k)}\right)-bf\omega^2\sum_k\left(\frac{ke^{kt}}{q(k)}\right)+2ef\omega^4\sum_k\left(\frac{ke^{kt}}{q(k)}\right)\\&+bc\sum_k\left(\frac{e^{kt}}{q(k)}\right)-cf\omega^2\sum_k\left(\frac{k^2e^{kt}}{q(k)}\right)-de\omega^2\sum_k\left(\frac{k^2e^{kt}}{q(k)}\right)\\&-ef\omega^2\sum_k\left(\frac{k^3e^{kt}}{q(k)}\right)-bd\omega^2\sum_k\left(\frac{e^{kt}}{q(k)}\right)+be\sum_k\left(\frac{ke^{kt}}{q(k)}\right)\Bigg]/\\&\left(bx+\omega^2y+f^2\omega^8\right),\end{aligned} \tag{10}$$

where $x=b-2e\omega^2+2f\omega^4$ and $y=c^2-(2cd+e^2)\omega^2+(d^2-2ef)\omega^4$ and,

$$q(k)=4fk^3+3dk^2+2ek+c. \tag{11}$$

For this case, using the same parameters previously applied to step excitation example ($R_1=0.5\ \Omega$, $R_2=0.89\ \Omega$, $L_1=0.433\ \mu H$, $L_2=1.85\ \mu H$, $C_1=59.5$ pF, $C_2=52.5$ pF and $M=2.801\ \mu H$), the transient response is shown at Fig. 4, considering an excitation at low frequencies.

This same output can be seen as:

$$V_0(s)=\frac{p(s)}{q(s)}\frac{\omega}{s^2+\omega^2}, \tag{12}$$

where the polynomial $q(s)=fs^4+ds^3+es^2+cs+b$ can be seen as:

$$q(s)=(s+q_1)(s+q_2)\cdots(s+q_4), \tag{13}$$

whose roots are real and complex conjugate, may present all roots distinct, as well as with multiplicity, with known results of the linear systems. In both cases, the steady state response is:

$$v_{0ss}(t)=Y\left|G(j\omega)\right|\sin(\omega t+\varphi), \tag{14}$$

with ϕ being an angle lag and Y the peak at output.

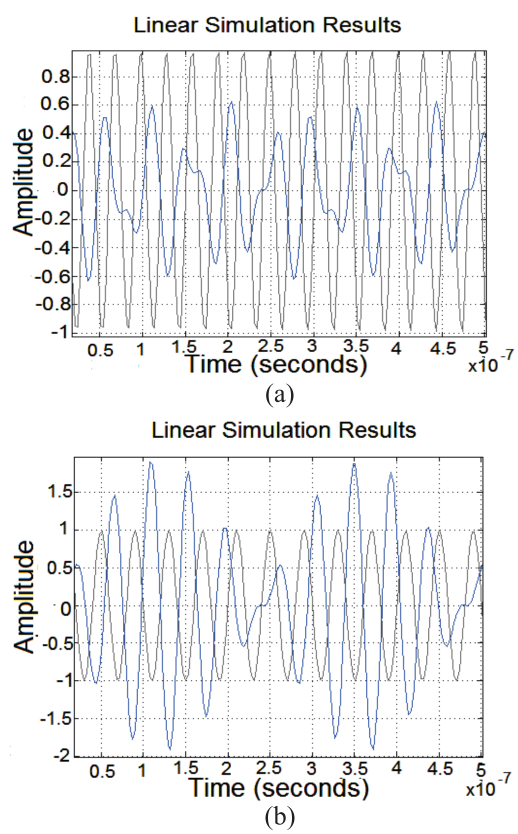
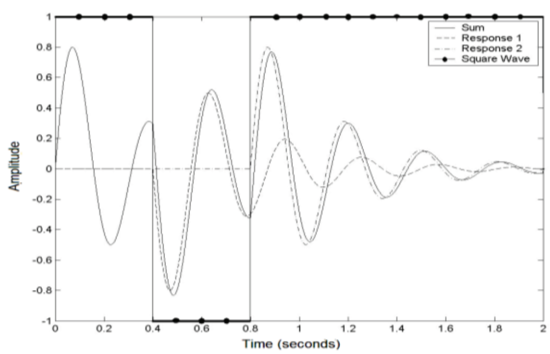


Fig. 4. Responses of Tesla transformer at low frequencies to sinusoidal excitation, where all low frequencies occur an overlap of responses as shown in [1].

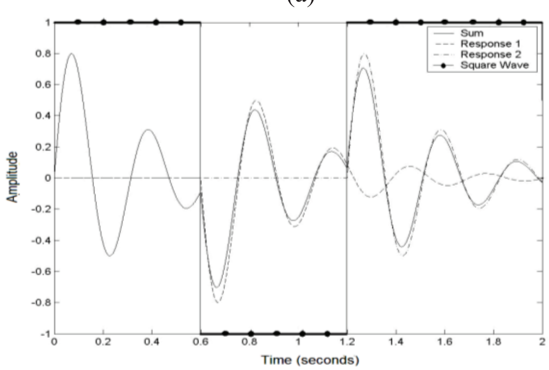
IV. RESPONSES OF TESLA TRANSFORMER FOR STEADY STATE

Considering steady state, the Tesla transformer shows an output that can present low or high amplitude, depending on the excitation frequency at input. Both in the case of the square wave excitation as the sinusoidal excitation, according to excitation frequency tends to resonance, this gain increases rapidly, reaching high values, which do not follow classical transformer theory (turn ratio).

Considering first the input excitation as a square wave, we have that the step response is given by Eq. (7). Consequently, the excitation with an inverted step (step with negative sign) generates this same response with negative sign. How a square wave can be seen as a sequence of steps up and down, accordingly shown previously, then the response of the Tesla transformer for low frequencies (without resonance) is a series of responses up and down over each rise and fall of the square wave. However, as the square wave frequency increases, these responses overlap, summing up when the responses peak have same sign and subtracting when have inverted signs. These results can be seen at Fig. 5.



(a)



(b)

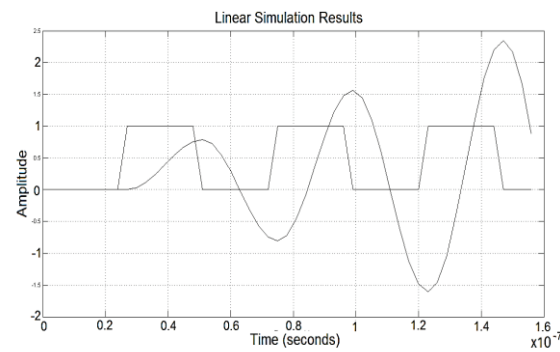
Fig. 5. Response of Tesla transformers to square wave at rise and fall: (a) sum and (b) subtraction.

When the square wave frequency approaches the frequency of the system response, this sum of responses increases output voltage, as shown at Fig. 6.

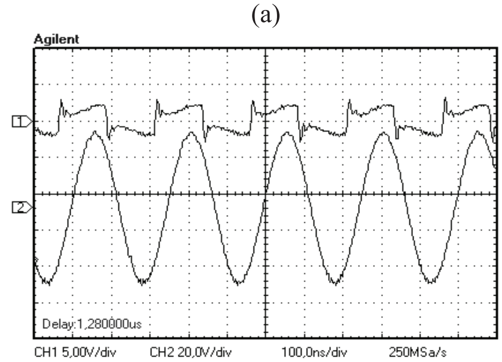
Also, when the square wave frequency tends to frequency of the response, the system reaches resonance, giving an output with high gain, defined by:

$$v_0(t) = 2.023\text{x}10^5/\exp(3.361\text{x}10^{10}t) - 5.021\text{x}10^5/\exp(1.26\text{x}10^6 t) + 0.29\cos(1.307\text{x}10^8 t)/\exp(3.55\text{x}10^4 t) + 0.52\sin(1.3075\text{x}10^8 t)/\exp(3.55\text{x}10^4 t), \quad (15)$$

shown at Fig. 7, that is the result for the specific case to values of inductances, capacitances and resistances found in [9-11] for the case of the planar transformer with 10 turns at primary and 20 turns at secondary, which presents in simulation result the gain $G = v_0/v_i = 465$.

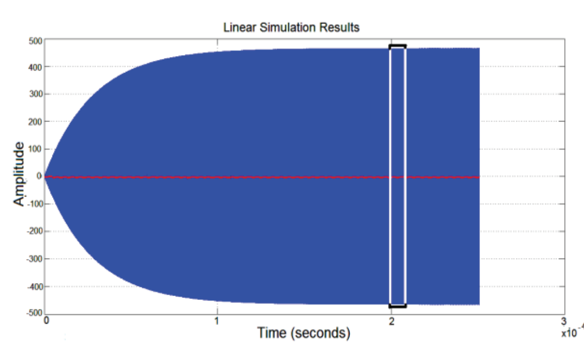


(a)

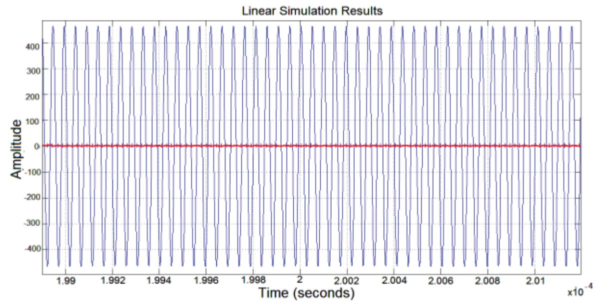


(b)

Fig. 6. Resonance of Tesla coil; frequency of square wave approximately equal to frequency of step response: (a) simulated response for transient response (sum of responses), and (b) experimental result.



(a)



(b)

Fig. 7. Steady state of Tesla transformer excited with square wave in resonance: (a) general response and (b) zoomed.

By other side, considering sinusoidal excitation, on experimental results found in [15], we see that the output at resonance is greater than square wave excitation. Taking the results of the Eq. (9), we have at low frequencies the result shown in Fig. 4. As the sinusoidal excitation frequency increases, approaching to resonance frequency, the system responds with a higher growth than sum of responses of the square wave excitation.

The resonance frequency of Tesla transformer is reached when:

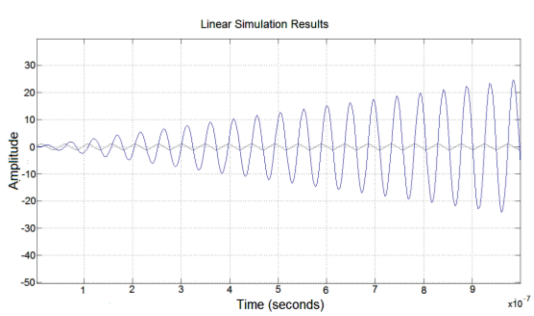
$$\omega = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}. \quad (16)$$

As this value of ω is a pole of Eq. (9), the system tends to present an output with infinite growth (making $s = j\omega$). In this way, as seen mathematically, the system output rapidly increases, as the case of the shown example of the Eq. (15), what can be seen at Fig. 8, where the gain at steady state is $G = v_0/v_i = 719.1$, since that the input is unitary.

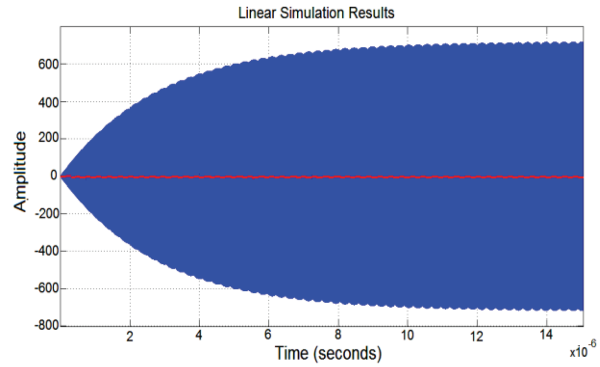
Comparing the gain values obtained with square wave excitation and sinusoidal excitation at their maximum peaks, we find:

$$G_{\sin} / G_{sq} = 719.1/465 = 1.546. \quad (17)$$

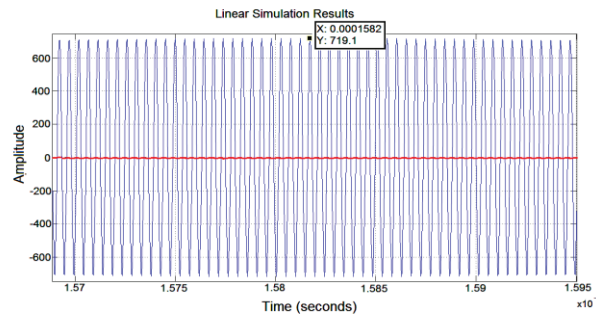
This result, when compared with experimental results as example shown in Fig. 6, we see that the gain is lesser due to limitations of the used equipment, but does not invalidate the obtained results.



(a)



(b)



(c)

Fig. 8. Resonant response of Tesla transformer to sinusoidal excitation: (a) transient response, (b) general response, and (c) zoomed response of steady state.

V. CONCLUSION

In this paper was seen the Tesla transformer response at square wave and sinusoidal excitation, where was obtained the transfer function and the responses of these inputs in time domain. It is noted both mathematically as experimentally the effects of high gain due to these inputs, proving that the square wave excitation defines an output, whose response is a sum of step responses (positive and negative), and with sinusoidal excitation, the gain is higher than square wave at steady state. Thus, it is shown that the Tesla transformer operates with sinusoidal excitation and the obtained gains for this case are higher than obtained gains referring to square wave excitation, as shown both simulation as experimental, for the presented specific case, as shown in [15].

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Eduard M. M. Costa was born in Esperança, PB, Brazil, in 1970. He received his M.Sc. and D.Sc. degrees in Electrical Engineering from Universidade Federal da Paraíba, Brazil in 1997 and 2001 respectively. He worked on projects of automation and electromagnetism. He has published several papers and books in the related areas, being their main areas of research.

Costa works in Universidade Federal do Vale do São Francisco (UNIVASF), Juazeiro, BA, Brazil, where he develops researches about electromagnetism, automation, and others. He's a Member of Brazilian Electromagnetism Society (SBMag) and IEEE.