Fast Converging Graded Mesh for Bodies of Revolution with Tip Singularities

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***Abstract*— In this paper, we study the plane wave scattering from perfectly electric conducting (PEC) bodies of revolution (BOR) with tip singularities. It is known that solutions to surface integral equations such as magnetic, electric, and combined field integral equations (MFIE, EFIE, and CFIE, respectively) are singular near the tips. Consequently, the convergence of method of moments (MoM) based on those surface integral equations is not optimal or guaranteed. By using appropriate graded meshes, one can retain the optimal convergence rate in MoM.**

***Keywords*—Tip Singularity, Graded Mesh, Optimal Convergence, Integral Equations, Method of Moments, Radar Cross Section.**

I. INTRODUCTION

Computational electromagnetics (CEM) technology has made tremendous progress in the last decade due largely to the advancement of fast solver and high performance computing technology. Consequently, CEM tools are being applied to ever more complex problems. Even though CEM tools still rely on radar cross section (RCS) measurements for validation, the measurement community is increasingly relying on CEM tools, especially those based on the method of moments (MoM), to validate their measured data to minimize measurement uncertainty. To provide prediction data for measurement validation, one typically needs to compute the RCS from 2 to 18 GHz using very fine grids to ensure solution convergence. This presents quite a challenge for MoM codes even for canonical targets of moderate sizes, especially if computation is required at every 10 MHz and every 0.1 degree. Instead of using a uniform mesh, one would like to use a non-uniform mesh that is denser near the singularities and coarser elsewhere to minimize the number of unknowns. In this paper, we investigate the choices of non-uniform mesh that give fast converging solutions for bodies of revolution (BOR). The theorem that defines the constraint for the graded mesh will be discussed and followed by numerical examples, particularly those of a 10-foot ogive and a 10-inch ogive with gap.

II. SOLUTIONS NEAR GEOMETRY SINGULARITIES

It is known that the solution of the scattering problem by the perfectly conducting ogive is singular due to the ogive tips. In fact, if r is the distance from the tip, the solution near the tip behaves as

$$|E| \sim r^{\mu_1 - 1}, \quad |H| \sim r^{\mu_2 - 1}, \quad 0 < \mu_1, \mu_2 < 1,$$

where μ_1 and μ_2 depend on the angle of the tip (see, for example, [1] and [2]). If piecewise polynomial basis functions defined on a uniform mesh are used to approximate the solutions, the convergence of MoM is not optimal due to the singular behavior of the solutions near the ogive tips. To retain the optimal convergence rate, one can either include the singular basis functions in the approximation or discretize the ogive with a graded mesh. It is easier to construct graded meshes and apply them to the existing MoM codes. Here we apply graded meshes to Cicero [1] which is a MoM computer code for bodies of resolution (BOR) (see [4], [5], [6], and references therein). We will need the following result in approximating the singular function $r^{-\alpha}$ on $[0,1]$, $0 < \alpha < 1$ (see [7]):

Theorem 1: *Let* p *be such that* $\alpha < 1/p$. *Define the following partition* τ_n^q *of* $[0,1]$

$$r_i = (i/n)^q, \quad i = 0,1,2,3,\ldots,n,$$

where $q = \frac{1+p}{1-\alpha p}$ *and is called the grading exponent of* τ_n. *Let* $S(\tau_n^q)$ *be the set of functions that are constant on each subinterval* $[r_i, r_{i+1}]$. *Then*

$$\inf_{\zeta \in S(\tau_n, q)} \left\| r^{-\alpha} - \zeta(r) \right\|_{L^p[0,1]} = O(1/n).$$

In fact, if $\zeta(r) \in S(\tau_n^q)$ is such that

$\zeta(r) = r_i^{-\alpha} \quad \text{in } [r_i, r_{i+1}], \quad i = 0,1,2,\ldots,n-1$, then

$$\left\| r^{-\alpha} - \zeta(r) \right\|_{L^p[0,1]} \le C/n.$$

The norm of $f \in L^p[0,1]$ is $\|f\|_{L^p[0,1]} = \sqrt[\frac{1}{p}]{\int_0^1 |f|^p dx}$. In engineering application, L^2-norm (i.e., $p=2$) is usually used. In this case, Theorem 1 states that, when $r^{-\alpha}$, for $0<r<1$, and $\alpha < 1/2$, is approximated by step functions, the best approximation in L^2-norm is obtained when these step functions are defined on a graded grid with grading exponent $q = \frac{3}{1-2\alpha}$. Note that the above theorem can be applied to singularities of arbitrary order such as vertices, edges, corners, etc. The singularity order α of field solutions near the ogive tips can be approximated by that of the field near the tip of the cone whose interior angle is the same as the angle of the ogive tip.

III. GRADED MESH AND CONVERGENCE

Let S be the surface of the PEC ogive which is parameterized by arc-length

$$(l,\phi) \to (\rho(l),\phi,z(l)), \quad l \in [0,L],\ \phi \in [0,2\pi)$$

where L is the total arc-length of the generating curve and the axis of rotation is along the z direction. The tips occur at $l=0$ and $l=L$ with $\rho(0)=\rho(L)=0$. The components J_t and J_ϕ of the surface current $J = \hat{n} \times H$ behave like $l^{-\alpha}$ and $(L-l)^{-\alpha}$ near the tips, where $0<\alpha<1$. A graded mesh is constructed as follows. First, we divide $[0,L]$ into three subintervals $[0,\varepsilon]$, $[\varepsilon, L-\varepsilon]$, and $[L-\varepsilon, L]$ where $\varepsilon < L/4$ and is a rational. Let $\beta(x)$ be a twice differentiable function defined as

$$\beta(x) = \begin{cases} \varepsilon(x/\varepsilon)^q & \text{for } x \in [0,\varepsilon], \\ b(x) & \text{for } x \in [\varepsilon, L-\varepsilon], \\ L-\varepsilon\left(\dfrac{L-x}{\varepsilon}\right)^q & \text{for } x \in [L-\varepsilon, L], \end{cases}$$

where $q = \frac{1+p}{1-\alpha p}$, $0<p<1/\alpha$, and $b(x)$ is a "connecting function" which is monotonically increasing and has two continuous derivatives. (Such a function is constructed from a perfect spline in [8]). Then the nodes in the graded mesh τ_N^q is defined as

$$l_i = \beta(x_i), \quad x_i = L(i/N),\ i = 0,1,2,\ldots,N,$$

where x_i's represent a uniform mesh and l_i's are the mapped points in the graded mesh. Note that ε needs to be chosen so that ε and $L-\varepsilon$ coincide with one of the x_i's. It is common in MoM codes to approximate the fields with pulse functions (or piecewise constant), i.e. $S(\tau_N^q)$. Let $P_n : L^p([0,L]) \to S(\tau_N^q)$ be the orthogonal projection, that is,

$$\langle P_n u, v_n \rangle = \langle u, v_n \rangle, \quad \forall v_n \in S(\tau_n^q).$$

Then Galerkin approximation problem is to find $u_n \in S(\tau_N^q)$ such that

$$P_n A u_n = P_n f,$$

where A is the integral operator defined as in the combined field integral equation (CFIE) and f is the given incident field. We assume that the integral equation $Au = f$ has a unique solution. It can be shown that Galerkin approximation scheme is stable [9], that is,

$$\|P_n A v_n\|_{L^2(0,2\pi,L^p[0,L])} \ge C \|v_n\|_{L^2(0,2\pi,L^p[0,L])},$$

for all $v_n \in S(\tau_N^q)$ and some $C>0$ independent of v_n. Consequently, we obtain the error estimate [9]

$$\|u-u_n\|_{L^2(0,2\pi,L^p[0,L])} \le C \inf_{v_n \in S(\tau_N^q)} \|u-v_n\|_{L^2(0,2\pi,L^p[0,L])},$$

where u is the solution of the continuous problem and $u_n \in S(\tau_N^q)$ is the approximating solution. This implies that convergence rate for the graded mesh is optimal (for L^p-norm). In other words, the approximating solutions $u_n \in S\left(\tau_N^q\right)$ converges to u at the rate $\inf_{v_n \in S(\tau_N^q)} \|u-v_n\|_{L^2(0,2\pi,L^p[0,L])}$ which is the best possible for elements in $S\left(\tau_N^q\right)$. It is possible to show that the optimality holds for a more familiar weighted Sobolev space, $L_\alpha^2([0,L])$, whose norm is defined as

$$\|u\|_{L_\alpha^2([0,L])} = \left(\int_0^L |x^\alpha u|^2\, dx\right)^{1/2}.$$

Hence, if $J_{t,n}$ and $J_{\phi,n}$ are the approximate solutions of J_t and J_ϕ, respectively, then the following error estimates also hold

$$\|J_t - J_{t,n}\|_{L^2(0,2\pi,L_\alpha^2[0,L])} \le C/n,$$

$$\|J_\phi - J_{\phi,n}\|_{L^2(0,2\pi,L_\alpha^2[0,L])} \le C/n,$$

where α is the order of singularity.

IV. NUMERICAL RESULTS

In this section, we illustrate the benefits of using properly graded meshes with Cicero code in computing RCS of a 10-foot ogive and a 10-inch ogive with gap. Cicero is a MoM code with pulse basis and testing functions. In each of the following figures, the numbers in the legend are the number of sampling points per wavelength (ppw) used. For graded meshes, the "ppw" means that the total number of grid points are the same as that of the uniform mesh with the "ppw". All results are computed using CFIE.

A. *The 10-foot ogive*

The 10-foot ogive is 10-foot long from tip to tip and 1-foot wide at the waist. The tips of the ogive are at $z = \pm 1.524$ meters. The meshes are either uniform ($q = 1$) or graded with the grading exponents $q = 2$. We first compute the electric current components J_t and J_ϕ at 0.5 GHz. In Figure 1and Figure 2, we plot $|J_t|$, $|J_\phi|$ for θ-polarized incident field at $\theta^i = 0^\circ$ and $\theta^i = 20^\circ$, respectively.

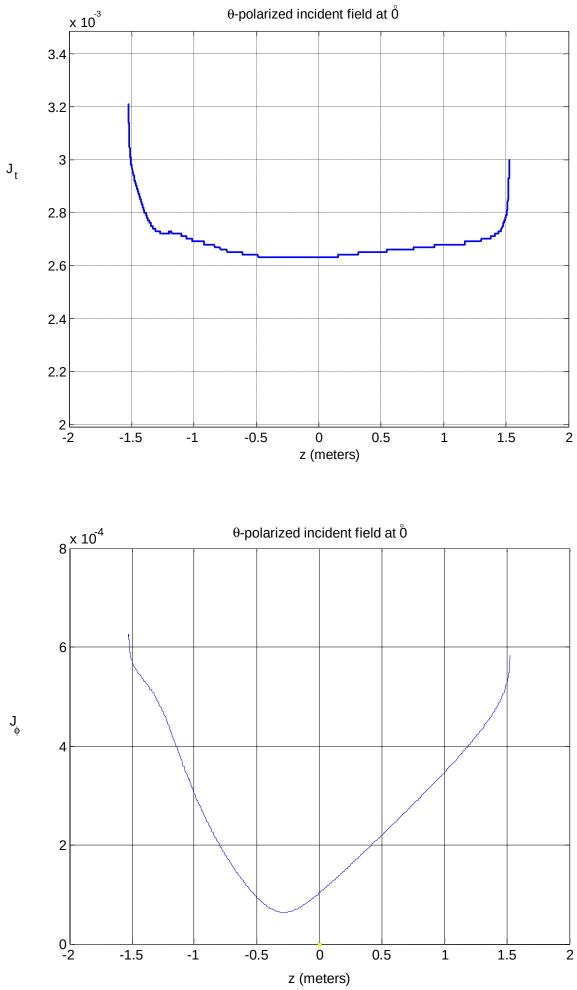


Figure 1: Magnitude J_t and J_ϕ for θ-polarized incidence field at $\theta = 0^o$. Tips of the 10-foot ogive are at z=1.524 m and z=-1.524 m.

Due to numerical limitations in Cicero such as piecewise approximations to the roof-top basis functions, we can only observe that J_t and J_ϕ and their derivatives tend to infinity at the tips instead of become infinity as expected in [2]. In any case, these singularities cause slow convergence in MoM using uniform meshes.

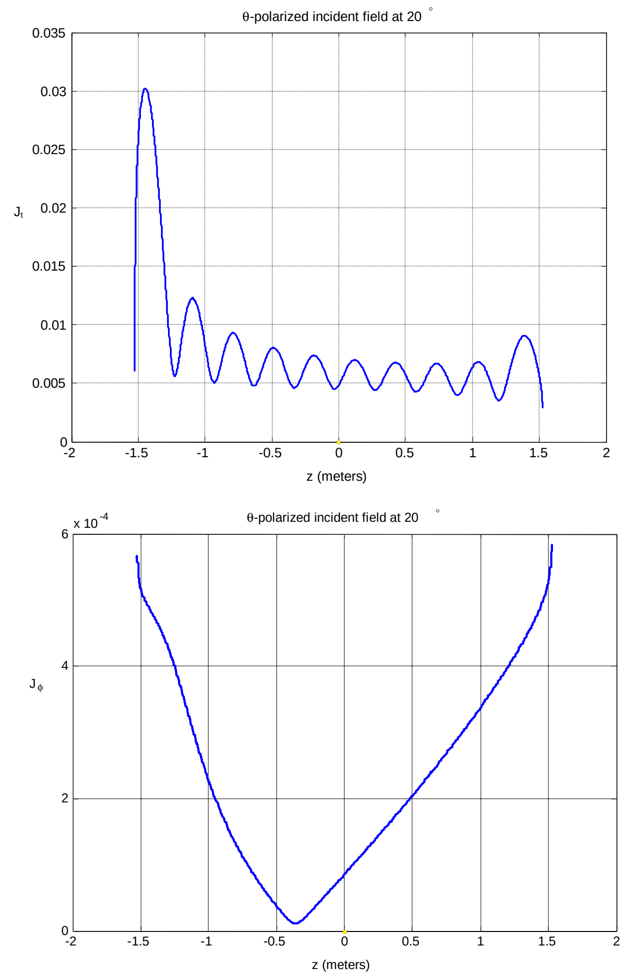
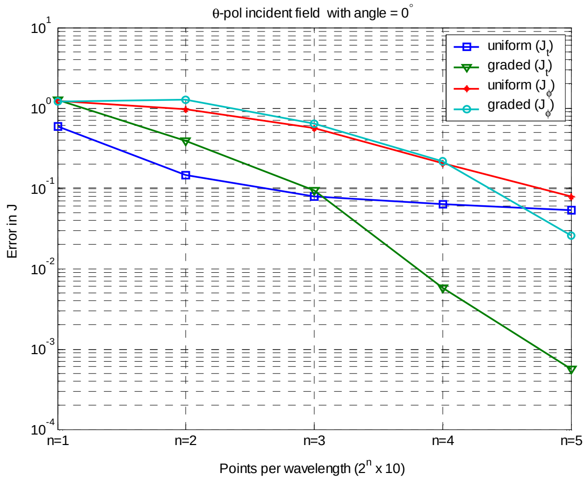


Figure 2: Magnitude J_t and J_ϕ for θ-polarized incidence field at $\theta = 20^o$.

We assume that the electric currents computed with 640 points per wavelength are "exact" and plot the relative errors e_{h_n} in Figure 3 and Figure 4

$$e_{h_n} = \frac{\left\| J_\nu^{h_n} - J_\nu^{h_{min}} \right\|_{L^{1/2}(0,L)}}{\left\| J_\nu^{h_n} \right\|_{L^{1/2}(0,L)}}, \quad n = 1, 2, \ldots, 5$$

where ν is either t or ϕ, h_n is the mesh size corresponding to the number of points per wavelength pww = $2^n \times 10$, and h_{min} is the mesh size for $n = 6$ (or 640 points per wavelength). It is observed that the errors in graded meshes are smaller than their uniform counterparts for sufficiently large ppw and decrease at a faster rate.



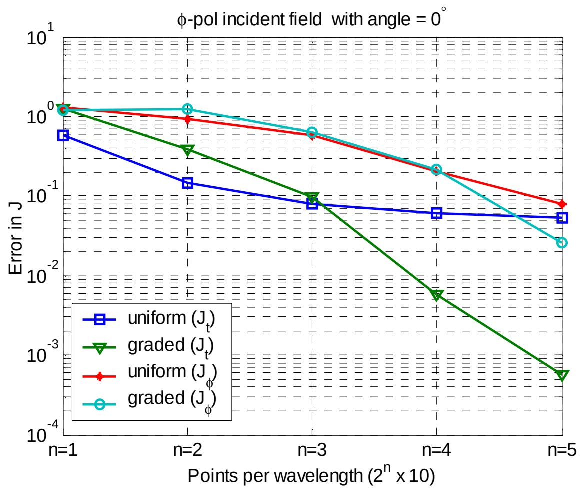
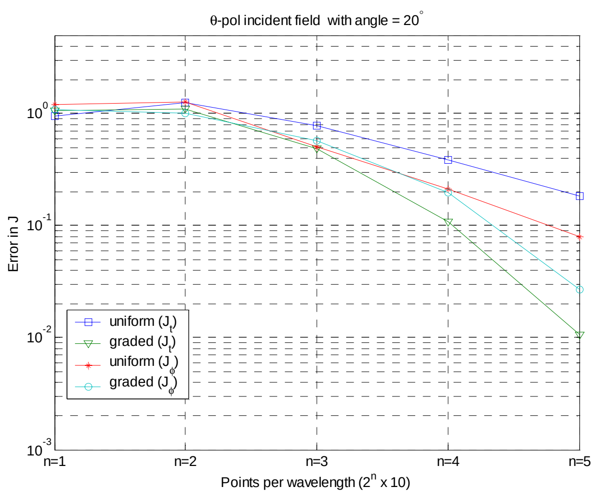


Figure 3**:** Logarithmic relative error for $\left(J_t, J_\phi\right)$ at $\theta = 0^0$ in graded meshes are smaller and decrease faster than those in uniform meshes.



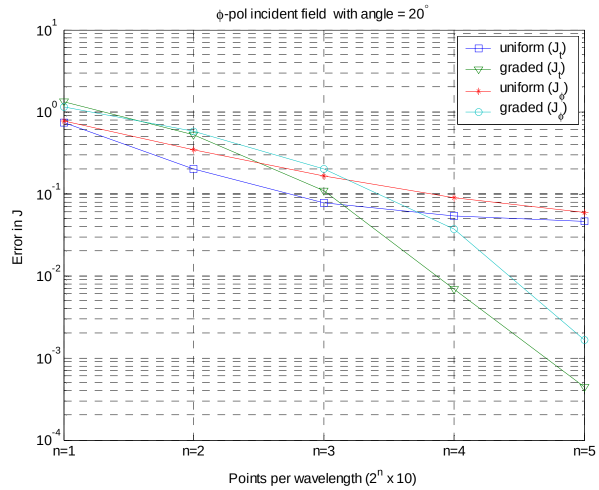


Figure 4**:** Logarithmic relative error for $\left(J_t, J_\phi\right)$ at $\theta = 20^0$ in graded meshes are smaller and decrease faster than those in uniform meshes.

Next we compare the RCS results for different meshes. It can be seen from the below figures that the graded mesh performs better than the uniform one with the same number of unknowns, that is, RCS results of the graded meshes converge at a faster rate. In Figure 5, RCS at 0.5 GHz for $\theta\theta$ polarization computed with the 640 ppw-uniform mesh is viewed as the "exact" result. The zero-degree angle corresponds to the tip scattering direction while the 90-degree angle corresponds to the broadside scattering. We observe that the result computed with the graded mesh at 60 ppw already overlaps with the "exact" solution while the RCS curve computed with the uniform mesh at 60 ppw has not converged, especially near $\theta = 20^o$. In fact, it requires at least 160 ppw in a uniform mesh to yield the same accuracy as the 60 ppw in a graded mesh.

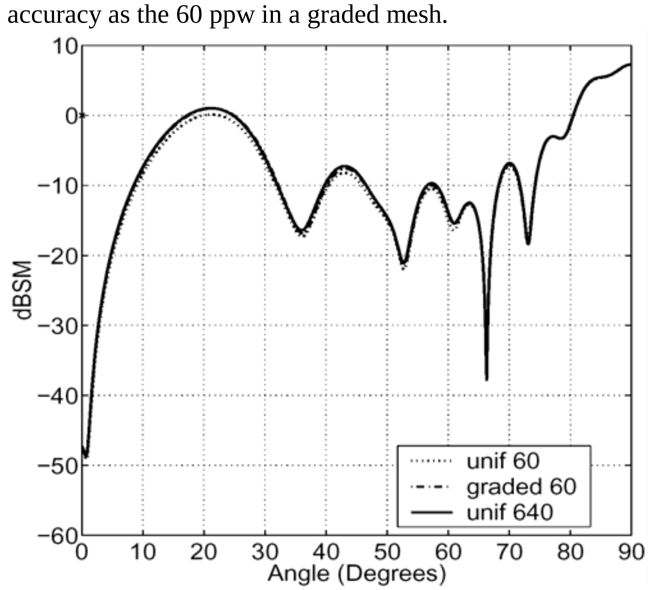


Figure 5: RCS of the 10-foot ogive at 0.5 GHz ($\theta\theta$ polarization). The "exact" solution is represented by the 640 ppw-uniform mesh (solid line). At 60 ppw, the uniform-mesh solution has not converged (dotted line) while the graded-mesh solution (dash-dotted line) overlaps with the "exact" solution.

At higher frequencies, tip singularity becomes more problematic. In Figure 6, RCS at 5 GHz for the 10-foot ogive is computed near the nose-on (grazing) angular region by using uniform meshes of different grid densities. The curves oscillate near -69.5 dBsm.

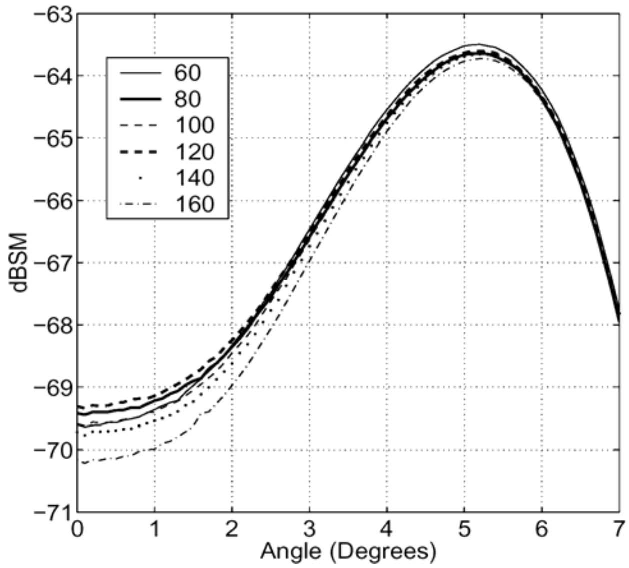


Figure 6: Uniform mesh - RCS of the 10-foot ogive at 5 GHz near the grazing angle ($\phi\phi$ polarization).

However, RCS at 5 GHz for the 10-foot ogive computed with graded meshes approaches monotonically to a converged solution as seen in Figure 7.

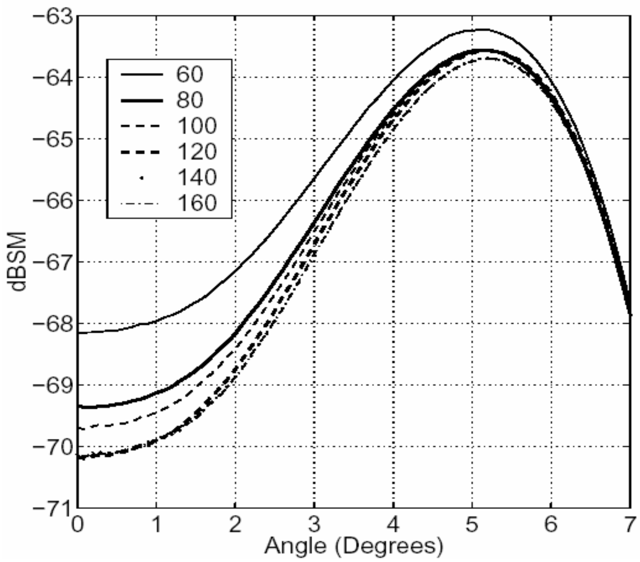


Figure 7: Graded mesh - RCS of the 10-foot ogive at 5 GHz near the grazing angle ($\phi\phi$ polarization).

B. *The 10-inch ogive with gap*

To examine further the advantage of graded meshes, we also consider the ogive with gap (see Figure 8), one of the test targets proposed by Electromagnetic Code Consortium [10], [11]. The ogive is 10 inches long, subtending a half angle of 22.62 degree, and with a maximum radius of 1 inch at the middle. The generating curve for this ogive is part of a circular arc with a 13 inch radius. A small rectangular groove cut out around the middle of the ogive. The circumferential groove is 0.25 inches wide by 0.25 inches deep. The bottom of the groove forms a ring 0.75 inches in radius.

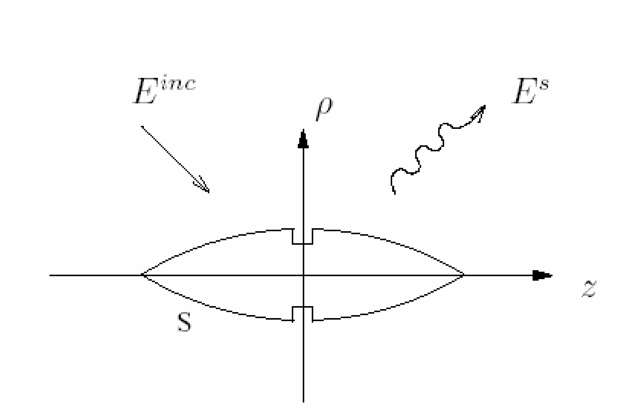


Figure 8: Ogive with gap.

Here, the solutions to the integral equations have both edge and tip singularities due to the groove and the ogive tips. As in the case of the 10-foot ogive, we design appropriate graded meshes to improve the convergence rate in MoM solutions. There are three corners in the generating arc and each has different angles. Thus, we construct a graded mesh with three grading exponents (q_1, q_2, q_3). For example, the distribution of points in a graded mesh with $(q_1, q_2, q_3) = (2, 2, 2)$ is plotted in Figure 9.

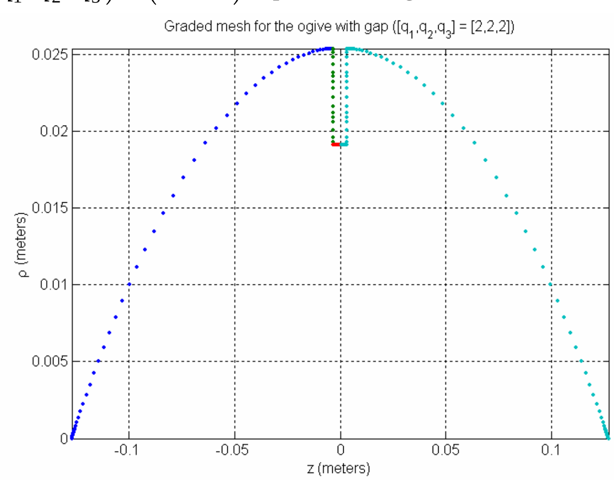


Figure 9: Point distribution of a graded mesh on the generating arc of ogive with gap.

In Figure 10 and Figure 11, RCS curves at 2 GHz are plotted. Uniform-mesh solutions converge much slower than graded-mesh solutions, especially near the grazing angular region. We see that at 80 ppw, RCS of the graded mesh already converges while that of uniform mesh does not.

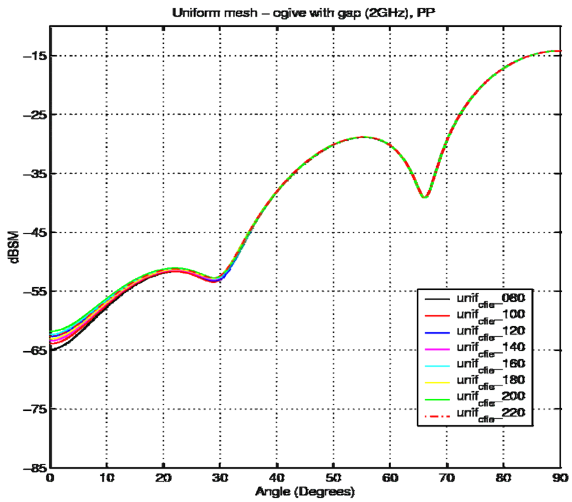


Figure 10: Uniform mesh - RCS of ogive with gap at 2 GHz ($\phi\phi$ polarization).

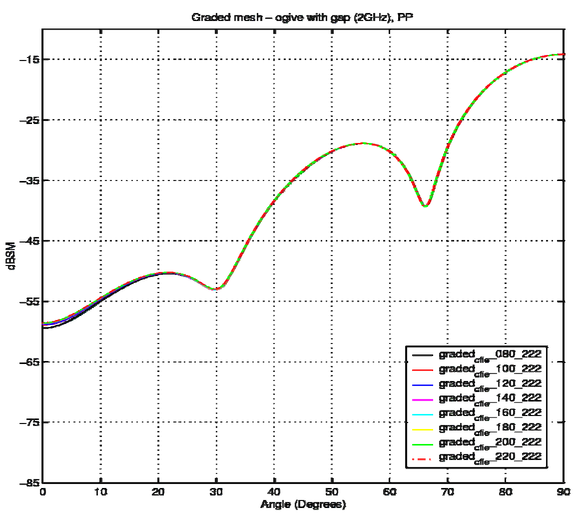


Figure 11: Graded mesh - RCS of ogive with gap at 2 GHz ($\phi\phi$ polarization).

Furthermore, the RCS differences at $\phi = 0^o$ (the grazing angle) decrease for graded meshes while oscillate for the uniform meshes as seen in Figure 12 and Figure 13.

V. CONCLUSIONS

In this paper, we present a construction of graded meshes that enable a faster convergence using MoM for BOR targets with tip singularities. Numerical results are given for the PEC ogive with and without gap using CFIE. These preliminary results show that faster convergence can be achieved if one chooses a graded mesh using the technique outlined in this paper. We have also observed the similar improvement in convergence for other types of integral equations. The technique can be easily generalized to non-BOR targets which have tip and edge singularities. This will be reported in the future.

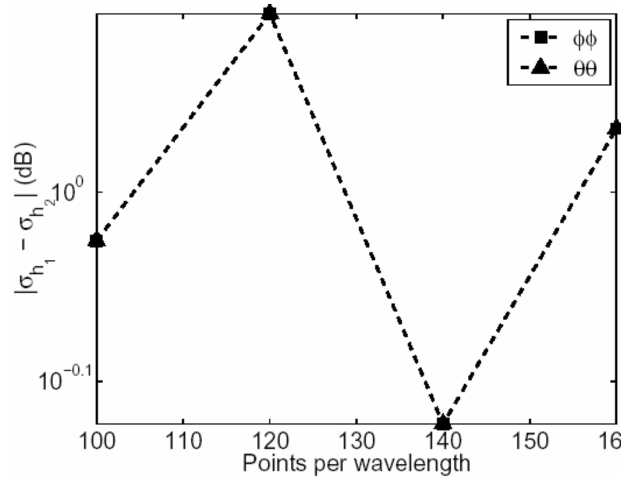


Figure 12: Uniform mesh at 2 GHz – RCS errors in ogive with gap at $\theta = 0^o$.

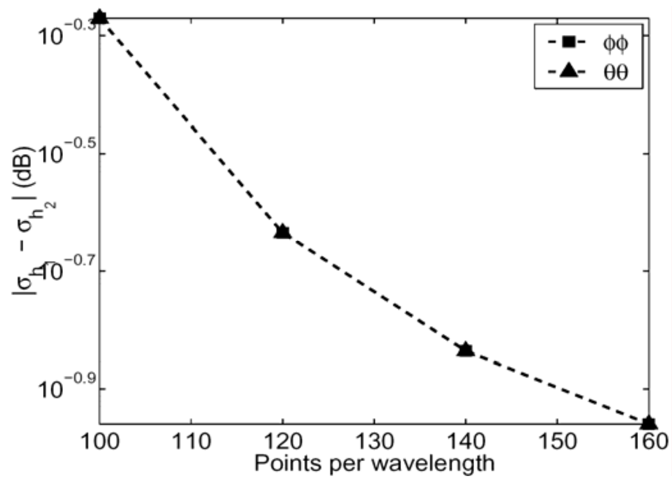


Figure 13: Graded mesh at 2 GHz – RCS errors in ogive with gap at $\theta = 0^o$.

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